SOLVABILITY OF 2D HYBRID LINEAR SYSTEMS – COMPARSION OF THREE DIFFERENT METHODS

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Abstract: A class of positive hybrid linear systems is introduced. Three different methods for computation of solutions of the hybrid system are proposed. The considerations are illustrated by numerical example. Simulations of solution have been shown for the methods.

1. INTRODUCTION

In positive systems, inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones, not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs Benvenuti and Farina (2004), Kaczorek (2001). Recent developments in positive systems theory and some new results are given in Kaczorek (2003). The realization problem for positive discrete-time and continuos-time systems without and with delays was considered in Benvenuti and Farina (2004), Farina and Rinaldi (2000), Kaczorek (2001, 2004, 2005, 2006), Kaczorek and Busłowicz (2004).

The main purpose of this paper is presentation and comparison of three methods for computation of solution of positive 2D hybrid systems. Three different solutions of the hybrid linear systems will be derived. The considered methods will be illustrated by numerical example. Using Matlab/Simulink there will be performed comparison simulations of the methods.

2. EQUATIONS OF THE HYBRID SYSTEMS

Let $R^{n\times m}$ be the set of $n\times m$ matrices with entries form the field of real number R and Z_+ be the set of nonnegative integers. The $n\times n$ identity matrix will be denoted by I_n.

Equations of the 2D hybrid linear system have the form

$$\dot{x}_1(t,i) = A_{11}x_1(t,i) + A_{12}x_2(t,i) + B_1u(t,i)\,,\ t\in R_+ \tag{1a}$$

$$x_2(t,i+1) = A_{21}x_1(t,i) + A_{22}x_2(t,i) + B_2u(t,i)\,,\ i\in Z_+ \tag{1b}$$

$$y(t,i) = C_1x_1(t,i) + C_2x_2(t,i) + Du(t,i) \tag{1c}$$

where $\dot{x}_1(t,i) = \dfrac{\partial x_1(t,i)}{\partial t}$, $x_1(t,i)\in R^{n_1}$, $x_2(t,i)\in R^{n_2}$, $u(t,i)\in R^m$, $y(t,i)\in R^P$ and A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D are real matrices with appropriate dimensions.

Boundary conditions for (1a) and (1b) have the form

$$x_1(0,i) = x_1(i)\,,\ i\in Z_+ \ \text{and}\ x_2(t,0) = x_2(t)\,,\ t\in R_+ \tag{2}$$

Note that the hybrid system (1) has a similar structure as the Roesser model (Kaczorek, 2001; Klamka, 1991; Roesser, 1975).

Let $R_+^{n\times m}$ be the set of $n\times m$ real matrices with nonnegative entries and R_+^n, $R_+^{n\times 1}$.

Definition 1.

The hybrid system (1) is called internally positive if $x_1(t,i)\in R_+^{n_1}$, $x_2(t,i)\in R_+^{n_2}$, and $y(t,i)\in R_+^p$, $t\in R_+$, $i\in Z_+$ for arbitrary boundary conditions $x_1(i)\in R_+^{n_1}$, $i\in Z_+$, $x_2(t)\in R_+^{n_2}$, $t\in R_+$ and inputs $u(t,i)\in R_+^m$, $t\in R_+$, $i\in Z_+$.

Let M_n be the set of $n\times m$ Metzler matrices (real matrices with nonnegative off-diagonal entries).

Theorem 1.

(Kaczorek, 2001) The hybrid system (1) is internally positive if and only if

$$A_{11}\in M_{n_1}, A_{12}\in R_+^{n_1\times n_2}, A_{21}\in R_+^{n_2\times n_1}, A_{22}\in R_+^{n_2\times n_2},$$
$$B_1\in R_+^{n_1\times m}, B_2\in R_+^{n_2\times m}, C_1\in R_+^{p\times n_1}, C_2\in R_+^{p\times n_2}, D\in R_+^{p\times m}$$

3. COMPUTATION OF SOLUTIONS

Method 1.

Along with equations (1a), (1b), consider the following determining equations

$$X_{k+1,i}^1 = A_{11}X_{k,i}^1 + A_{12}X_{k,i}^2 + B_1U_{k,i} \tag{3a}$$

$$X_{k,i+1}^2 = A_{21}X_{k,i}^1 + A_{22}X_{k,i}^2 + B_2U_{k,i} \tag{3b}$$

with initial conditions of the form

$$X_{0,i}^1 = 0 \text{ for i} = 0,1,... \tag{4a}$$

$$X_{k,0}^2 = 0 \text{ for } k = 0,1,... \tag{4b}$$

$$U_{k,i} = \begin{cases} I_m, & k = i = 0 \\ 0, & k^2 + i^2 \neq 0 \end{cases} \tag{4c}$$

Lemma 1.

The following conditions hold:
for $k = 1,2,...$

$$(A_{11} + A_{12}w(I_{n_2} - A_{22}w)^{-1}A_{21})^{k-1} \times$$
$$(B_1 + A_{12}w(I_{n_2} - A_{22}w)^{-1}B_2) \equiv \sum_{j=0}^{\infty} X_{k,j}^1 w^j$$
$$(I_{n_2} - A_{22}w)^{-1}A_{21}w(A_{11} + A_{12}w(I_{n_2} - A_{22}w)^{-1}A_{21})^{k-1} \times$$
$$(B_1 + A_{12}w(I_{n_2} - A_{22}w)^{-1}B_2) \equiv \sum_{j=0}^{\infty} X_{k,j}^2 w^j$$

for $j = 1,2,...$

$$(A_{22} + A_{21}w(I_{n_1} - A_{11}w)^{-1}A_{12})^{j-1} \times$$
$$(B_2 + A_{21}w(I_{n_1} - A_{11}w)^{-1}B_1) \equiv \sum_{k=0}^{\infty} X_{k,j}^2 w^k$$
$$(I_{n_1} - A_{11}w)^{-1}A_{12}w(A_{22} + A_{21}w(I_{n_1} - A_{11}w)^{-1}A_{12})^{j-1} \times$$
$$(B_2 + A_{21}w(I_{n_1} - A_{11}w)^{-1}B_1) \equiv \sum_{j=0}^{\infty} X_{k,j}^1 w^k$$

and

$$(I_{n_2} - A_{22}w)^{-1}B_2w \equiv \sum_{j=0}^{\infty} X_{0,j}^2 w^j$$

$$(I_{n_1} - A_{11}w)^{-1}B_1w \equiv \sum_{k=0}^{\infty} X_{k,0}^1 w^k$$

Where $|w| < w_1$, $w \in C$ and w_1 is a sufficiently small positive number. Proof by induction is given in (Marchenko and Poddubnaya (2005), Marchenko at al (2005).
Applying the Laplace transformation with respect to t and the Z-transformation with respect to i, we write the equations (1a), (1b) in the form

$$\begin{bmatrix} I_{n_1}s - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z - A_{22} \end{bmatrix}\begin{bmatrix} X_1(s,z) \\ X_2(s,z) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}U(s,z) + \begin{bmatrix} X_1(0,z) \\ zX_2(s,0) \end{bmatrix} \tag{5}$$

where $X_k(s,z) = Z[L(x_k(t,i))]$, $k = 1,2$
$X_1(0,z) = Z[x_1(0,i))]$, $X_2(s,0) = L[x_2(t,0)]$.

The equations (5) can be rewritten as

$$\begin{bmatrix} I_{n_1}s - A_{11} - A_{12}(I_{n_2}z - A_{22})^{-1}A_{21} & 0 \\ -A_{21} & I_{n_2}z - A_{22} \end{bmatrix}\begin{bmatrix} X_1(s,z) \\ X_2(s,z) \end{bmatrix} =$$
$$\begin{bmatrix} B_1 + A_{12}(I_{n_2}z - A_{22})^{-1}B_2 \\ B_2 \end{bmatrix}U(s,z) + \tag{6a}$$
$$\begin{bmatrix} X_1(0,z) + A_{12}(I_{n_2}z - A_{22})^{-1}zX_2(s,0) \\ zX_2(s,0) \end{bmatrix}$$

and

$$\begin{bmatrix} I_{n_1}s - A_{11} & -A_{12} \\ 0 & I_{n_2}z - A_{22} - A_{21}(I_{n_1}s - A_{11})^{-1}A_{12} \end{bmatrix}\begin{bmatrix} X_1(s,z) \\ X_2(s,z) \end{bmatrix} =$$
$$\begin{bmatrix} B_1 \\ B_2 + A_{21}(I_{n_1}s - A_{11})^{-1}B_1 \end{bmatrix}U(s,z) + \tag{6b}$$
$$\begin{bmatrix} X_1(0,z) \\ zX_2(s,0) + A_{21}(I_{n_1}s - A_{11})^{-1}X_1(0,z) \end{bmatrix}$$

It follows from (6) and Lemma 1 given in (Marchenko and Poddubnaya (2005), Marchenko at al (2005), that

$$X_1(s,z) = \sum_{k=0}^{\infty}\frac{1}{s^{k+1}}(A_{11} + A_{12}(I_{n_2}z - A_{22})^{-1}A_{21})^k \times$$
$$\{(B_1 + A_{12}(I_{n_2}z - A_{22})^{-1}B_2)U(s,z) +$$
$$X_1(0,z) + A_{12}(I_{n_2}z - A_{22})^{-1}zX_2(s,0)\} =$$
$$\sum_{k=1}^{\infty}\frac{1}{s^k}(A_{11} + A_{12}z^{-1}(I_{n_2} - A_{22}z^{-1})^{-1}A_{21})^{k-1} \times$$
$$\{(B_1 + A_{12}z^{-1}(I_{n_2} - A_{22}z^{-1})^{-1}B_2)U(s,z) +$$
$$X_1(0,z) + A_{12}(I_{n_2}z - A_{22})^{-1}zX_2(s,0)\} =$$
$$\sum_{k=1}^{\infty}\frac{1}{s^k}\sum_{j=0}^{\infty}X_{k,j}^1\frac{1}{z^j}U(s,z) +$$
$$\sum_{k=1}^{\infty}\frac{1}{s^k}(A_{11} + A_{12}z^{-1}(I_{n_2} - A_{22}z^{-1})^{-1}A_{21})^{k-1}X_1(0,z) +$$
$$\sum_{k=1}^{\infty}\frac{1}{s^k}(A_{11} + A_{12}z^{-1}(I_{n_2} - A_{22}z^{-1})^{-1}A_{21})^{k-1} \times$$
$$A_{12}z^{-1}(I_{n_2} - A_{22}z^{-1})^{-1}zX_2(s,0) \tag{7a}$$

Similarly, we obtain

$$
\begin{aligned}
&X_2(s,z)=\sum_{j=1}^{\infty}\frac{1}{z^j}(A_{22}+A_{21}s^{-1}(I_{n_1}-A_{11}s^{-1})^{-1}A_{12})^{j-1}\times\\
&\{(B_2+A_{21}s^{-1}(I_{n_1}-A_{11}s^{-1})^{-1}B_1)U(s,z)+\\
&zX_2(s,0)+A_{21}s^{-1}(I_{n_1}-A_{11}s^{-1})^{-1}X_1(0,z)\}=\\
&\sum_{j=1}^{\infty}\frac{1}{z^j}\sum_{k=0}^{\infty}X_{k,j}^2\frac{1}{s^k}U(s,z)+\\
&\sum_{j=1}^{\infty}\frac{1}{z^j}(A_{22}+A_{21}s^{-1}(I_{n_1}-A_{11}s^{-1})^{-1}A_{12})^{j-1}zX_2(s,0)+\\
&\sum_{j=1}^{\infty}\frac{1}{z^j}(A_{22}+A_{21}s^{-1}(I_{n_1}-A_{11}s^{-1})^{-1}A_{12})^{j-1}\times\\
&A_{21}s^{-1}(I_{n_1}-A_{11}s^{-1})^{-1}X_1(0,z)
\end{aligned}
\tag{7b}
$$

Let $X_{k,i}^1=X_{k,i}^{11}$, $X_{k,i}^2=X_{k,i}^{21}$ be the solution of (3a), (3b) with $B_1=I_{n_1}$, $B_2=0$ and $X_{k,i}^1=X_{k,i}^{12}$, $X_{k,i}^2=X_{k,i}^{22}$ with $B_1=0$, $B_2=I_{n_2}$.

Then we have $X_{k,0}^{12}=A_{11}{}^{k-1}B_1=0$, $X_{0,k}^{21}=A_{22}{}^{k-1}B_2=0$, $k=1,2,...$ and

$$
\begin{aligned}
&X_1(s,z)=\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{1}{z^j}\frac{1}{s^k}X_{k,j}^1U(s,z)+\\
&\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{1}{z^j}\frac{1}{s^k}X_{k,j}^{11}X_1(0,z)+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{z^{j-1}}\frac{1}{s^k}X_{k,j}^{12}X_2(s,0)
\end{aligned}
\tag{8a}
$$

$$
\begin{aligned}
&X_2(s,z)=\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{z^j}\frac{1}{s^k}X_{k,j}^2U(s,z)+\\
&\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{z^{j-1}}\frac{1}{s^k}X_{k,j}^{22}X_2(s,0)+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{z^j}\frac{1}{s^k}X_{k,j}^{21}X_1(0,z)
\end{aligned}
\tag{8b}
$$

Using inverse transforms to (8), we obtain the solution (8b)of hybrid linear system (1) in the form

$$
\begin{aligned}
&x_1(t,i)=\sum_{k=1}^{\infty}\sum_{j=0}^{i}X_{k,j}^1\int_0^t\frac{(t-\tau)^{k-1}}{(k-1)!}u(\tau,i-j)d\tau\ +\\
&\sum_{k=1}^{\infty}\sum_{j=0}^{i}X_{k,j}^{11}\frac{t^{k-1}}{(k-1)!}x_1(0,i-j)+\\
&\sum_{k=1}^{\infty}X_{k,i+1}^{12}\int_0^t\frac{(t-\tau)^{k-1}}{(k-1)!}x_2(\tau,0)d\tau
\end{aligned}
\tag{9a}
$$

$$
\begin{aligned}
&x_2(t,i)=\sum_{k=1}^{\infty}\sum_{j=1}^{i}X_{k,j}^2\int_0^t\frac{(t-\tau)^{k-1}}{(k-1)!}u(\tau,i-j)d\tau+\\
&\sum_{k=1}^{\infty}\sum_{j=1}^{i}X_{k,j}^{21}\frac{t^{k-1}}{(k-1)!}x_1(0,i-j)+\\
&\sum_{j=1}^{i}X_{0,j}^2u(t,i-j)+X_{0,i+1}^{22}x_2(t,0)+\\
&\sum_{k=1}^{\infty}X_{k,i+1}^{22}\int_0^t\frac{(t-\tau)^{k-1}}{(k-1)!}x_2(\tau,0)d\tau
\end{aligned}
\tag{9b}
$$

Method 2.

Applying the Laplace transformation with respect to t and the Z-transformation with respect to i, we write the equations (1a), (1b) in the form

$$
\begin{bmatrix} I_{n_1}-s^{-1}A_{11} & -s^{-1}A_{12}\\ -z^{-1}A_{21} & I_{n_2}-z^{-1}A_{22}\end{bmatrix}\begin{bmatrix}X_1(s,z)\\ X_2(s,z)\end{bmatrix}=
\begin{bmatrix}s^{-1}B_1\\ z^{-1}B_2\end{bmatrix}U(s,z)+\begin{bmatrix}s^{-1}X_1(0,z)\\ X_2(s,0)\end{bmatrix}
\tag{10}
$$

and

$$
\begin{bmatrix}X_1(s,z)\\ X_2(s,z)\end{bmatrix}=\begin{bmatrix} I_{n_1}-s^{-1}A_{11} & -s^{-1}A_{12}\\ -z^{-1}A_{21} & I_{n_2}-z^{-1}A_{22}\end{bmatrix}^{-1}\times
\left(\begin{bmatrix}s^{-1}B_1\\ z^{-1}B_2\end{bmatrix}U(s,z)+\begin{bmatrix}s^{-1}X_1(0,z)\\ X_2(s,0)\end{bmatrix}\right)
\tag{11}
$$

where $X_k(s,z)=Z[L(x_k(t,i))]$, $k=1,2$
$X_1(0,z)=Z[x_1(0,i))]$, $X_2(s,0)=L[x_2(t,0)]$.

Let

$$
T_{1,0}=\begin{bmatrix}A_{11} & A_{12}\\ 0 & 0\end{bmatrix},\ T_{0,1}=\begin{bmatrix}0 & 0\\ A_{21} & A_{22}\end{bmatrix}
\tag{12}
$$

and

$$
\begin{bmatrix} I_{n_1}-s^{-1}A_{11} & -s^{-1}A_{12}\\ -z^{-1}A_{21} & I_{n_2}-z^{-1}A_{22}\end{bmatrix}^{-1}=
[I_{n_1+n_2}-T_{1,0}s^{-1}-T_{0,1}z^{-1}]^{-1}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}T_{i,j}s^{-i}z^{-j}
\tag{13}
$$

where

$$
T_{i,j}=\begin{cases} I_{n_1+n_2} & \text{for}\quad i=j=0\\ T_{1,0}T_{i-1,j}+T_{0,1}T_{i,j-1} & \text{for } i,j=0,1,...\ \ i+j>0\\ 0 & \text{for}\ \ i<0\ \text{or/and}\ j<0\end{cases}
\tag{14}
$$

From definition of inverse matrix and (13), we have

$$[I_{n_1+n_2} - T_{1,0}s^{-1} - T_{0,1}z^{-1}]\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}T_{i,j}s^{-i}z^{-j}\right) = I_{n_1+n_2} \quad (15)$$

Comparison of the coefficients at the same powers of s and z of the equality (15) yields (14).

Substituting (13) into (11), we obtain

$$\begin{bmatrix} X_1(s,z) \\ X_2(s,z) \end{bmatrix} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}T_{i,j}s^{-i}z^{-j}\left(\begin{bmatrix} s^{-1}B_1 \\ z^{-1}B_2 \end{bmatrix}U(s,z) + \begin{bmatrix} s^{-1}X_1(0,z) \\ X_2(s,0) \end{bmatrix}\right) =$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(T_{i,j}s^{-(i+1)}z^{-j}B_{10} + T_{i,j}s^{-i}z^{-(j+1)}B_{01})U(s,z) +$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(T_{i,j}s^{-(i+1)}z^{-j}\begin{bmatrix} X_1(0,z) \\ 0 \end{bmatrix} + T_{i,j}s^{-i}z^{-j}\begin{bmatrix} 0 \\ X_2(s,0) \end{bmatrix}) \quad (16)$$

where $B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$, $B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$.

Applying the inverse transforms to (16), we obtain

$$\begin{bmatrix} x_1(t,i) \\ x_2(t,i) \end{bmatrix} = \sum_{k=0}^{\infty}\sum_{l=0}^{i}T_{k,i-l}B_{10}\int_0^t \frac{(t-\tau)^k}{k!}u(\tau,l)d\tau +$$

$$\sum_{k=0}^{\infty}\sum_{l=0}^{i}T_{k,i-l-1}B_{01}\int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!}u(\tau,l)d\tau +$$

$$\sum_{k=0}^{\infty}\sum_{l=0}^{i}T_{k,i-l}\frac{t^k}{k!}\begin{bmatrix} x_1(0,l) \\ 0 \end{bmatrix} +$$

$$\sum_{k=0}^{\infty}T_{k,i}\int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!}\begin{bmatrix} 0 \\ x_2(\tau,0) \end{bmatrix}d\tau \quad (17)$$

Method 3.

From definition, solution of the differential equation (1a) has the form

$$x_1(t,i) = e^{A_{11}t}x_1(0,i) + \int_0^t e^{A_{11}(t-\tau)}\left(A_{12}x_2(\tau,i) + B_1u(\tau,i)\right)d\tau \quad (18)$$

and solution of the difference equation (1b) is given by

$$x_2(t,i) = A_{22}{}^i x_2(t,0) + \sum_{k=0}^{i-1}A_{22}{}^{i-k-1}\left(A_{21}x_1(t,k) + B_2u(t,k)\right) \quad (19)$$

Substituting (19) into (18), we obtain

$$x_1(t,i) = e^{A_{11}t}x_1(0,i) + \int_0^t e^{A_{11}(t-\tau)}B_1u(\tau,i)d\tau +$$

$$\int_0^t e^{A_{11}(t-\tau)}A_{12}A_{22}^i x_2(\tau,0)d\tau +$$

$$\sum_{k=0}^{i-1}\int_0^t e^{A_{11}(t-\tau)}A_{12}A_{22}{}^{i-k-1}B_2u(\tau,k)d\tau + \quad (20)$$

$$\sum_{k=0}^{i-1}\int_0^t e^{A_{11}(t-\tau)}A_{12}A_{22}{}^{i-k-1}A_{21}x_1(\tau,k)d\tau =$$

$$\overline{x}_1(t,i) + \sum_{k=0}^{i-1}P_{i-k-1}x_1(t,k)$$

where

$$\overline{x}_1(t,i) = e^{A_{11}t}x_1(0,i) +$$

$$\int_0^t e^{A_{11}(t-\tau)}[A_{12}A_{22}^i x_2(\tau,0) + B_1u(\tau,i)]d\tau +$$

$$\sum_{k=0}^{i-1}\int_0^t e^{A_{11}(t-\tau)}A_{12}A_{22}{}^{i-k-1}B_2u(\tau,k)d\tau \quad (21)$$

$$P_j f(t) = \int_0^t e^{A_{11}(t-\tau)}A_{12}A_{22}{}^j A_{21}f(\tau)d\tau, \quad j \in Z_+$$

Substituting (20) into (19), we obtain

$$x_2(t,i) = A_{22}{}^i x_2(t,0) + \sum_{k=0}^{i-1}A_{22}{}^{i-k-1}B_2u(t,k) +$$

$$\sum_{k=0}^{i-1}A_{22}{}^{i-k-1}A_{21}e^{A_{11}t}x_1(0,k) +$$

$$\sum_{k=0}^{i-1}\int_0^t A_{22}{}^{i-k-1}A_{21}e^{A_{11}(t-\tau)}B_1u(\tau,k)d\tau +$$

$$\sum_{k=0}^{i-1}\int_0^t A_{22}{}^{i-k-1}A_{21}e^{A_{11}(t-\tau)}A_{12}A_{22}^k x_2(\tau,0)d\tau + \quad (22)$$

$$\sum_{k=0}^{i-1}\sum_{l=0}^{k}\int_0^t A_{22}{}^{i-k-1}A_{21}e^{A_{11}(t-\tau)}A_{12}A_{22}{}^{k-l-1}B_2u(\tau,l)d\tau +$$

$$\sum_{k=0}^{i-1}\sum_{l=0}^{k}\int_0^t A_{22}{}^{i-k-1}A_{21}e^{A_{11}(t-\tau)}A_{12}A_{22}{}^{k-l-1}A_{21}x_1(\tau,l)d\tau$$

Solutions of hybrid linear system (1) have the form (20) and (22).

4. NUMERICAL EXAMPLE

Transfer function of the hybrid system is given by

$$T(s,z)=\frac{2sz+s+3z+2}{sz-0.1s+0.9z-0.1} \tag{23}$$

and its realization has the form (n = 1, m = 1)

$$A_{11}=[-0.9],\ A_{12}=[1\ \ 0],\ A_{21}=\begin{bmatrix}0.01\\1.1\end{bmatrix},\ A_{22}=\begin{bmatrix}0.1&0\\1&0\end{bmatrix},$$
$$B_1=[1],\ B_2=\begin{bmatrix}0.1\\1\end{bmatrix},\ C_1=[1.2],\ C_2=[2\ \ 1],\ D=[2] \tag{24}$$

Let the initial conditions be given by $x_1(0,0)=0$, $x_2(0,0)=\begin{bmatrix}0\\0\end{bmatrix}$, $x_1(0,i)=1$, $x_2(t,0)=\begin{bmatrix}1\\1\end{bmatrix}$ for i = 1,2,…, $t\in[1,\infty)$ and input $u(t,i)=1$ for $t\ge 0$ and $i\ge 0$.
Find $x_1(1,1), x_2(1,1)$.

Using method 1 we obtain:

$$\begin{aligned}x_1(1,1)&=\sum_{k=1}^{\infty}\sum_{j=0}^{1}X_{k,j}^{1}\frac{1^k}{k!}u(0,1-j)+\\&\sum_{k=1}^{\infty}\sum_{j=0}^{1}X_{k,j}^{11}\frac{1^{k-1}}{(k-1)!}x_1(0,1-j)+\\&\sum_{k=1}^{\infty}X_{k,2}^{12}\frac{1^k}{k!}x_2(0,0)=\\&\sum_{k=1}^{\infty}\frac{1^k}{k!}(X_{k,0}^{1}u(0,1)+X_{k,1}^{1}u(0,0))+\\&\sum_{k=1}^{\infty}\frac{1^{k-1}}{(k-1)!}(X_{k,0}^{11}x_1(0,1)+X_{k,1}^{11}x_1(0,0))+\\&\sum_{k=1}^{\infty}X_{k,2}^{12}\frac{1^k}{k!}x_2(0,0)\end{aligned} \tag{25a}$$

$$\begin{aligned}x_2(1,1)&=\sum_{k=1}^{\infty}\sum_{j=1}^{1}X_{k,j}^{2}\frac{1^k}{k!}u(0,1-j)+\\&\sum_{k=1}^{\infty}\sum_{j=1}^{1}X_{k,j}^{21}\frac{1^{k-1}}{(k-1)!}x_1(0,1-j)+\\&\sum_{j=1}^{1}X_{0,j}^{2}u(1,1-j)+\sum_{k=1}^{\infty}X_{k,2}^{22}\frac{1^k}{k!}x_2(0,0)+X_{0,2}^{22}x_2(1,0)=\\&\sum_{k=1}^{\infty}\frac{1^k}{k!}X_{k,1}^{2}u(0,0)+\sum_{k=1}^{\infty}\frac{1^{k-1}}{(k-1)!}X_{k,1}^{21}x_1(0,0)+X_{0,1}^{2}u(1,0)+\\&\sum_{k=1}^{\infty}X_{k,2}^{22}\frac{1^k}{k!}x_2(0,0)+X_{0,2}^{22}x_2(1,0)\end{aligned} \tag{25b}$$

Taking into account the initial conditions and the input we obtain

$$\begin{aligned}x_1(1,1)&=\sum_{k=1}^{\infty}\frac{1}{k!}(X_{k,0}^{1}+X_{k,1}^{1})+\sum_{k=1}^{\infty}\frac{1}{(k-1)!}X_{k,0}^{11}\\x_2(1,1)&=\sum_{k=1}^{\infty}\frac{1}{k!}X_{k,1}^{2}+X_{0,1}^{2}+X_{0,2}^{22}\begin{bmatrix}1\\1\end{bmatrix}\end{aligned} \tag{26}$$

If we make three iterations, then the solution takes the form

$$\begin{aligned}x_1(1,1)&=\sum_{k=1}^{3}\frac{1}{k!}(X_{k,0}^{1}+X_{k,1}^{1})+\sum_{k=1}^{3}\frac{1}{(k-1)!}X_{k,0}^{11}=\\&(X_{1,0}^{1}+X_{1,1}^{1})+\frac{1}{2}(X_{2,0}^{1}+X_{2,1}^{1})+\frac{1}{6}(X_{3,0}^{1}+X_{3,1}^{1})+\\&X_{1,0}^{11}+X_{2,0}^{11}+\frac{1}{2}X_{3,0}^{11}=B_1+A_{12}B_2+\\&\frac{1}{2}(A_{11}B_1+A_{11}A_{12}B_2+A_{12}A_{21}B_1)+\\&\frac{1}{6}(A_{11}{}^2B_1+A_{11}{}^2A_{12}B_2+A_{11}A_{12}A_{21}B_1+\\&A_{12}A_{21}A_{11}B_1)+1+A_{11}+\frac{1}{2}A_{11}{}^2\\x_2(1,1)&=\sum_{k=1}^{3}\frac{1}{k!}X_{k,1}^{2}+X_{0,1}^{2}+X_{0,2}^{22}\begin{bmatrix}1\\1\end{bmatrix}=\\&X_{1,1}^{2}+\frac{1}{2}X_{2,1}^{2}+\frac{1}{6}X_{3,1}^{2}+X_{0,1}^{2}+X_{0,2}^{22}\begin{bmatrix}1\\1\end{bmatrix}=\\&A_{21}B_1+\frac{1}{2}A_{21}A_{11}B_1+\frac{1}{6}A_{21}A_{11}{}^2B_1+B_2+A_{22}\begin{bmatrix}1\\1\end{bmatrix}\end{aligned} \tag{27}$$

Substituting (24) into (27), we obtain final value

$$x_1(1,1)=1{,}261$$
$$x_2(1,1)=\begin{bmatrix}0{,}207\\2.752\end{bmatrix} \tag{28}$$

Using method 2, we obtain:

$$\begin{aligned}\begin{bmatrix}x_1(1,1)\\x_2(1,1)\end{bmatrix}&=\sum_{k=0}^{\infty}\sum_{l=0}^{1}T_{k,1-l}B_{10}\frac{1^{k+1}}{(k+1)!}u(0,l)+\\&\sum_{k=0}^{\infty}\sum_{l=0}^{1}T_{k,1-l-1}B_{01}\frac{1^k}{k!}u(0,l)+\\&\sum_{k=0}^{\infty}\sum_{l=0}^{1}T_{k,1-l}\frac{1^k}{k!}\begin{bmatrix}x_1(0,l)\\0\end{bmatrix}+\sum_{k=0}^{\infty}T_{k,1}\frac{1^k}{k!}\begin{bmatrix}0\\x_2(0,0)\end{bmatrix}\end{aligned} \tag{29}$$

Taking into account the initial conditions and the input we obtain

$$\begin{aligned}\begin{bmatrix}x_1(1,1)\\x_2(1,1)\end{bmatrix}&=\sum_{k=0}^{\infty}T_{k,1}B_{10}\frac{1}{(k+1)!}+\\&\sum_{k=0}^{\infty}T_{k,0}B_{10}\frac{1}{(k+1)!}+\sum_{k=0}^{\infty}T_{k,0}B_{01}\frac{1}{k!}+\sum_{k=0}^{\infty}T_{k,0}\frac{1}{k!}\begin{bmatrix}1\\0\end{bmatrix}\end{aligned} \tag{30}$$

If we make three iterations then the solution takes the form

$$\begin{bmatrix} x_1(1,1) \\ x_2(1,1) \end{bmatrix} = T_{0,1}B_{10} + B_{10} + B_{01} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + T_{1,1}B_{10}\frac{1}{2} +$$
$$T_{1,0}B_{10}\frac{1}{2} + T_{1,0}B_{01} + T_{1,0}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + T_{2,1}B_{10}\frac{1}{6} +$$
$$T_{2,0}B_{10}\frac{1}{6} + T_{2,0}B_{01}\frac{1}{2} + T_{2,0}\frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$
$$\begin{bmatrix} 0 \\ A_{21}B_1 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} A_{12}A_{21}B_1 \\ A_{21}A_{11}B_1 \end{bmatrix} + \tag{31}$$
$$\frac{1}{2}\begin{bmatrix} A_{11}B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} A_{12}B_2 \\ 0 \end{bmatrix} + \begin{bmatrix} A_{11} \\ 0 \end{bmatrix} +$$
$$\frac{1}{6}\begin{bmatrix} (A_{11}A_{12}A_{21} + A_{12}A_{21}A_{11})B_1 \\ (A_{21}A_{11}A_{11})B_1 \end{bmatrix} +$$
$$\frac{1}{6}\begin{bmatrix} A_{11}A_{11}B_1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} A_{11}A_{12}B_2 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} A_{11}A_{11} \\ 0 \end{bmatrix}$$

and

$$x_1(1,1) = 1{,}247$$
$$x_2(1,1) = \begin{bmatrix} 0{,}107 \\ 1{,}753 \end{bmatrix} \tag{32}$$

Using method 3, we obtain:

For $i = 0$, we have

$$x_1(1,0) = e^{A_{11}}x_1(0,0) + \int_0^1 e^{A_{11}(1-\tau)}A_{12}x_2(\tau,0)d\tau +$$
$$\int_0^1 e^{A_{11}(1-\tau)}B_1u(\tau,0)d\tau =$$
$$e^{A_{11}}x_1(0,0) + e^{A_{11}}A_{11}^{-1}A_{12}x_2(0,0) - A_{11}^{-1}A_{12}x_2(1,0) + \tag{33}$$
$$e^{A_{11}}A_{11}^{-1}B_1u(0,0) - A_{11}^{-1}B_1u(1,0)$$
$$x_2(1,1) = A_{21}x_1(1,0) + A_{22}x_2(1,0) + B_2u(1,0)$$

Substituting the initial conditions and the input, we have

$$x_1(1,0) = -A_{11}^{-1}A_{12}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{A_{11}}A_{11}^{-1}B_1 - A_{11}^{-1}B_1 = 1{,}1771$$
$$x_2(1,1) = A_{21}(-A_{11}^{-1}A_{12}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{A_{11}}A_{11}^{-1}B_1 - A_{11}^{-1}B_1) + \tag{34}$$
$$A_{22}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + B_2 = \begin{bmatrix} 0{,}218 \\ 3{,}948 \end{bmatrix}$$

and, for $i = 1$

$$x_1(1,1) = e^{A_{11}}x_1(0,1) + e^{A_{11}}A_{11}^{-1}A_{12}x_2(0,1)$$
$$- A_{11}^{-1}A_{12}x_2(1,1) + e^{A_{11}}A_{11}^{-1}B_1u(0,1) - A_{11}^{-1}B_1u(1,1) \tag{35}$$

where

$$x_2(0,1) = A_{21}x_1(0,0) + A_{22}x_2(0,0) + B_2u(0,0) = B_2 \tag{36}$$

Substituting the given data, we obtain

$$x_1(1,1) = e^{A_{11}} + e^{A_{11}}A_{11}^{-1}A_{12}B_2 - A_{11}^{-1}A_{12}\begin{bmatrix} 0.218 \\ 3.948 \end{bmatrix}$$
$$+ e^{A_{11}}A_{11}^{-1}B_1 - A_{11}^{-1}B_1 = 1{,}263 \tag{37}$$

Final value

$$x_1(1,1) = 1{,}263$$
$$x_2(1,1) = \begin{bmatrix} 0{,}218 \\ 3{,}948 \end{bmatrix} \tag{38}$$

Remark 1.

Obtained results for $x_1(1,1)$, $x_2(1,1)$ are different for different method (Tab. 1). To obtain some valid results more computations for i = 2, 3,… need to be performed. The number of iteration k in (27) and (31) need to be also increased.

Tab. 1. Final values for $x_1(1,1)$, $x_2(1,1)$ (for k = 3)

State variable	Method 1	Method 2	Method 3
$x_1(1,1)$	1,261	1,247	1,263
$x_{21}(1,1)$	0,207	0,107	0,218
$x_{22}(1,1)$	2,752	1,753	3,948

5. MATLAB/SIMULINK SIMULATIONS

Using Simulink toolbox we can model given transfer function (25) in the form

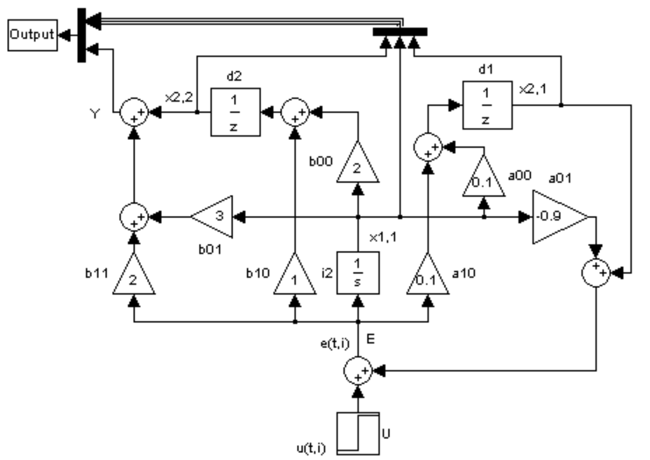

Fig. 1. Matlab/Simulink state variable diagram for transfer function (25)

Simulating i from 0 to 10 with sample time equal one, we obtain ending values of the simulation:

$$x_1 = 1{,}249$$
$$x_2 = \begin{bmatrix} 0{,}125 \\ 2{,}499 \end{bmatrix} \quad (41)$$
$$y = 6{,}249$$

Next step is implementation of considered methods in Matlab.
For simulations we use given initial conditions and input, also the number of iterations is increased (in (29) and (33)). After performing some simulations, we obtain the following results
Table 2 contains the final values from simulations for three methods. Those results are the state vectors $x_1(t,i)$, $x_2(t,i)$ for $t = 1$ and $i = 6$ with $k = 30$.
Figure 2 shows the diagram generated by Matlab. Diagram shows changes of the values of state vectors with the number i of steps.

Tab. 2. Final values

State variable	Method 1 (dash dot line)	Method 2 (dash dash line)	Method 3 (solid line)	Simulink response
x_1	1,143	1,147	1,148	1,249
x_{21}	0,123	0,124	0,124	0,125
x_{22}	2,376	2,386	2,387	2,499
Execute time [s]	21,744	18,251	0,032	

For $t = 1$, $i = 12$ and $k = 30$ we obtain

Tab. 3. Final values

State variable	Method 1 (dash dot line)	Method 2 (dash dash line)	Method 3 (solid line)
x_1	1,143	1,147	1,148
x_{21}	0,123	0,124	0,124
x_{22}	2,376	2,386	2,387
Execute time [s]	78,266	64,172	0,031

For $t = 10$, $i = 6$ and $k = 30$ we obtain

Tab. 4. Final values

State variable	Method 1 (dash dot line)	Method 2 (dash dash line)	Method 3 (solid line)
x_1	1,250	1,250	1,250
x_{21}	0,125	0,125	0,125
x_{22}	2,500	2,500	2,500
Execute time [s]	21,844	18,251	0,016

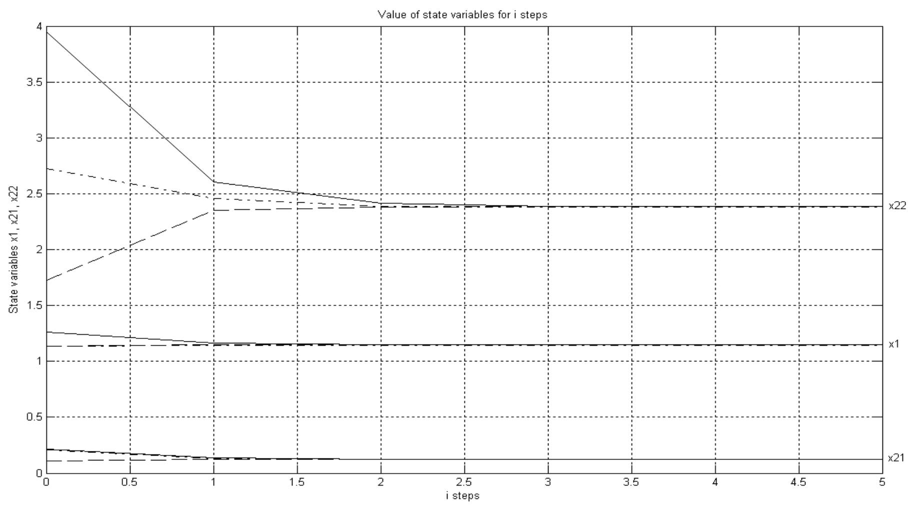

Fig. 2. Computational results for $t = 1$ and $i = 6$ with $k = 30$.

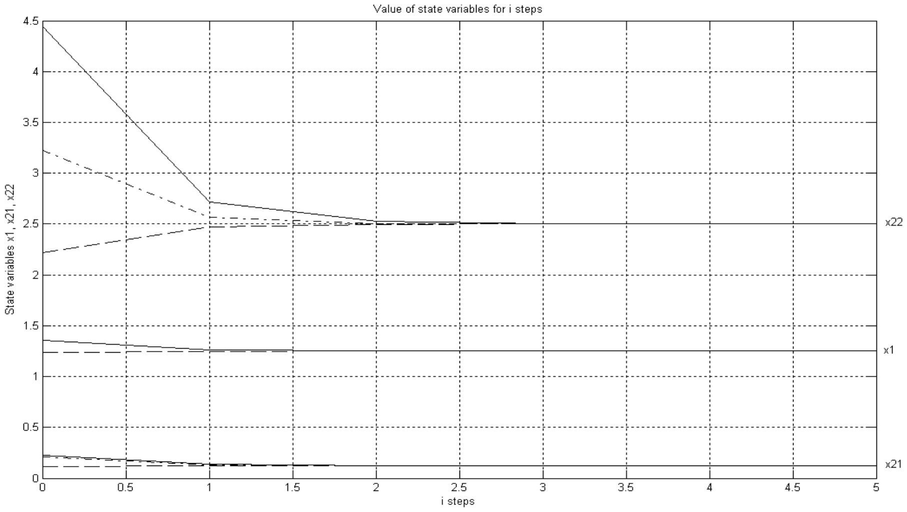

Fig. 3. Computational results for $t = 10$, $i = 6$ and $k = 30$

6. CONCLUDING REMARKS

General conclusion is that all three methods gives the same final results.

The first two methods are similar. To compute the solution $x(t,i)$ using those methods we do not need to know the values of the solution in the previous steps but we have to compute in the first method the matrices $X_{k,i}^{1}, X_{k,i}^{2}$ using the determining equations (3) or the matrices $T_{i,j}$ defined by (14) in the second method. In the third method the solution $x(t,i)$ is computed recursively using the initial conditions.

From the simulations it follows that the three methods give similar results after at least three steps.

The calculations have been performed on the Pentium M – 1,7GHz processor with 1GB RAM.

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This work was supported by the State Committee for Scientific Research of Poland under the grant NN 514 1939 33.