

POSITIVE DIFFERENT ORDERS FRACTIONAL 2D LINEAR SYSTEMS

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Abstract : A new class of positive different orders fractional 2D linear systems is introduced. A notion of (α, β) orders difference of 2D function is proposed. Fractional 2D state equations of linear systems are given and their solutions are derived using 2D \mathcal{Z} -transform. The classical Cayley-Hamilton theorem is extended to the 2D fractional linear systems. Necessary and sufficient conditions for the positivity, reachability and controllability to zero of the fractional 2D linear systems are established.

1. INTRODUCTION

The most popular models of two-dimensional (2D) linear systems are the models introduced by Roesser (1975), Fornasini-Marchesini (1976, 1978) and Kurek (1985).

The models have been extended for positive systems in Kaczorek (1996, 2002, 2005) and Valcher (1987). An overview of 2D linear system theory is given in Bose (1982, 1985), Gałkowski (1977, 2001), Kaczorek (1985) and some recent result in positive systems has been given in the monographs Farina and Marchesini (2000), Kaczorek (2002) and in paper Valcher (1977). Reachability and minimum energy control of positive 2D systems with one delay in states have been considered in Kaczorek (2005). The notion of internally positive 2D system (model) with delay in states and in inputs has been introduced and necessary and sufficient conditions for the internal positivity, reachability, controllability, observability and the minimum energy control problem have been established in Kaczorek (2005). The notions of positive fractional discrete-time and continuous-time linear systems have been introduced in Kaczorek (2003, 2007). The notion for 2D positive fractional hybrid linear systems has been extended in Kaczorek (2008). The realization problem for positive 1D and 2D linear systems has been considered in Kaczorek (2003, 2005), Kaczorek and Busłowicz (2004) and Kaczorek (2007). Recently, a new class of fractional 2D linear systems has been introduced in Kaczorek (2008).

In this paper a new class of positive fractional 2D linear systems will be introduced. A notion of (α, β) orders difference of 2D function will be proposed. Solution to the fractional 2D state equations of the linear systems will be derived using the 2D \mathcal{Z} -transform. The classical Cayley-Hamilton theorem will be extended to the 2D fractional linear systems. Necessary and sufficient conditions for the positivity, reachability and controllability of the 2D linear fractional systems are established.

To the best knowledge of the author the positive fractional 2D linear systems have not been considered yet.

2. FRACTIONAL 2D STATE EQUATIONS AND THEIR SOLUTIONS

Let $\mathfrak{R}_+^{n \times m}$ be the set of nonnegative real $n \times m$ matrices and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n \times n$ identity matrix will be denoted by I_n .

Definition 1.

The (α, β) orders fractional difference of an 2D function x_{ij} is defined by the formula

$$\Delta^{\alpha, \beta} x_{ij} = \sum_{k=0}^i \sum_{l=0}^j c_{\alpha\beta}(k, l) x_{i-k, j-l}, \quad (1a)$$

$$n-1 < \alpha < n, \quad n-1 < \beta < n; \quad n \in N = \{1, 2, \dots\}$$

where $\Delta^{\alpha, \beta} x_{ij} = \Delta_i^\alpha \Delta_j^\beta x_{ij}$ and

$$c_{\alpha, \beta}(k, l) = \begin{cases} 1 & \text{for } k=0 \text{ or/and } l=0 \\ (-1)^{k+l} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)\beta(\beta-1)\dots(\beta-l+1)}{k!l!} & \text{for } k+l > 0 \end{cases} \quad (1b)$$

The justification of Definition 1 is given in Appendix A. Consider the (α, β) order fractional 2D linear system, described by the state equations

$$\Delta^{\alpha, \beta} x_{i+1, j+1} = A_0 x_{ij} + A_1 x_{i+1, j} + A_2 x_{i, j+1} + B_0 u_{ij} + B_1 u_{i+1, j} + B_2 u_{i, j+1} \quad (2a)$$

$$y_{ij} = Cx_{ij} + Du_{ij} \quad (2b)$$

where $x_{ij} \in \mathfrak{R}^n$, $u_{ij} \in \mathfrak{R}^m$, $y_{ij} \in \mathfrak{R}^p$ are the state, input and output vectors and $A_k \in \mathfrak{R}^{n \times n}$, $B_k \in \mathfrak{R}^{n \times m}$, $k = 0, 1, 2$, $C \in \mathfrak{R}^{p \times n}$, $D \in \mathfrak{R}^{p \times m}$.

Using Definition 1 we may write the equation (2a) in the form

$$x_{i+1,j+1} = \bar{A}_0 x_{ij} + \bar{A}_1 x_{i+1,j} + \bar{A}_2 x_{i,j+1} - \sum_{\substack{k=0 \\ k+l > 0}}^{i+1} \sum_{l=0}^{j+1} c_{\alpha\beta}(k,l) x_{i-k+1,j-l+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \quad (3)$$

where $\bar{A}_0 = A_0 - I_n \alpha \beta$, $\bar{A}_1 = A_1 + I_n \beta$, $\bar{A}_2 = A_2 + I_n \alpha$.

From (1b) it follows that the coefficients $c_{\alpha\beta}(k,l)$ in (1a) strongly decrease when k and l increase. Therefore, in practical problems it is assumed that i and j are bounded by some natural numbers L_1 and L_2 . In this case the equation (3) takes the form

$$x_{ij} = \sum_{p=1}^i T_{i-p,j-1} (\bar{A}_1 x_{p0} + B_1 u_{p0}) + \sum_{q=1}^j T_{i-1,j-q} (\bar{A}_2 x_{0q} + B_2 u_{0q}) + \sum_{p=1}^{i-1} T_{i-p-1,j-1} \bar{A}_0 x_{p0} + \sum_{q=1}^{j-1} T_{i-1,j-q-1} \bar{A}_0 x_{0q} + T_{i-1,j-1} \bar{A}_0 u_{00} + \sum_{p=0}^{i-1} \sum_{q=0}^{j-1} T_{i-p-1,j-q-1} B_0 u_{pq} + \sum_{p=0}^i \sum_{q=0}^j (T_{i-p-1,j-q-1} B_1 + T_{i-p,j-q-1} B_2) u_{pq} \quad (5)$$

where the transition matrices T_{pq} are defined by the formula

$$T_{pq} = \begin{cases} I_n & \text{for } p=q=0 \\ \bar{A}_0 T_{p-1,q-1} + \bar{A}_1 T_{p,q-1} + \bar{A}_2 T_{p-1,q} - \sum_{\substack{k=0 \\ k+l < p+q-2}}^{p-1} \sum_{l=0}^{q-1} c_{\alpha\beta}(p-k,q-l) T_{kl} & \text{for } p+q > 0 \\ 0 \text{ (zero matrix)} & \text{for } p < 0 \text{ or/and } q < 0 \end{cases} \quad (6)$$

Proof.

Let $X(z_1, z_2)$ be the 2D Z -transform of x_{ij} defined by

$$X(z_1, z_2) = \mathcal{Z}[x_{ij}] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{ij} z_1^{-i} z_2^{-j} \quad (7)$$

Taking into account that

$$\begin{aligned} \mathcal{Z}[x_{i+1,j+1}] &= z_1 z_2 [X(z_1, z_2) - X(z_1, 0) - X(0, z_2) + x_{00}] \\ \mathcal{Z}[x_{i+1,j}] &= z_1 [X(z_1, z_2) - X(0, z_2)], \quad X(0, z_2) = \sum_{j=0}^{\infty} x_{0j} z_2^{-j} \\ \mathcal{Z}[x_{i,j+1}] &= z_2 [X(z_1, z_2) - X(z_1, 0)], \quad X(z_1, 0) = \sum_{i=0}^{\infty} x_{i0} z_1^{-i} \\ \mathcal{Z}[x_{i-k,j-l}] &= z_1^{-k} z_2^{-l} X(z_1, z_2) \end{aligned} \quad (8)$$

$$x_{i+1,j+1} = \bar{A}_0 x_{ij} + \bar{A}_1 x_{i+1,j} + \bar{A}_2 x_{i,j+1} - \sum_{k=0}^{L_1+1} \sum_{l=0}^{L_2+1} c_{\alpha\beta}(k,l) x_{i-k+1,j-l+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \quad (3a)$$

Note that the fractional systems are 2D linear systems with delays increasing with i and j .

The boundary conditions for the equation (3) (and (3a)) are given in the form

$$x_{i0}, i \in Z_+ \quad \text{and} \quad x_{0j}, j \in Z_+ \quad (4)$$

Theorem 1.

The solution of equation (3) with boundary conditions (4) is given by

then from (3) with (4) we obtain

$$\begin{aligned} X(z_1, z_2) &= G^{-1}(z_1, z_2) \{ (B_0 + B_1 z_1 + B_2 z_2) U(z_1, z_2) - \\ & z_1 [\bar{A}_1 B_1] \begin{bmatrix} X(0, z_2) \\ U(0, z_2) \end{bmatrix} - z_2 [\bar{A}_2 B_2] \begin{bmatrix} X(z_1, 0) \\ U(z_1, 0) \end{bmatrix} + \\ & z_1 z_2 [X(z_1, 0) + X(0, z_2) - x_{00}] \} \end{aligned} \quad (9)$$

where

$$G(z_1, z_2) = \begin{bmatrix} I_n z_1 z_2 + \sum_{k=0}^{j+1} \sum_{l=0}^{j+1} I_n c_{\alpha\beta}(k,l) z_1^{-(k-1)} z_2^{-(l-1)} - \bar{A}_0 - \bar{A}_1 z_1 - \bar{A}_2 z_2 \end{bmatrix}, \quad (10)$$

and $U(z_1, z_2) = \mathcal{Z}[u_{ij}]$

Let

$$G^{-1}(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)} \quad (11)$$

From the equality

$$G^{-1}(z_1, z_2) G(z_1, z_2) = G(z_1, z_2) G^{-1}(z_1, z_2) = I_n$$

it follows that

$$\left(\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)} \right) G(z_1, z_2) = \tag{12}$$

$$G(z_1, z_2) \left(\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)} \right) = I_n$$

Comparison of the coefficients at the same powers of z_1 and z_2 of (12) yields the formula (6). Substituting (11) into (9) we obtain

$$X(z_1, z_2) = \left(\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)} \right) \times \left\{ (B_0 + B_1 z_1 + B_2 z_2) U(z_1, z_2) - z_1 [\bar{A}_1 B_1] \begin{bmatrix} X(0, z_2) \\ U(0, z_2) \end{bmatrix} - z_2 [\bar{A}_2 B_2] \begin{bmatrix} X(z_1, 0) \\ U(z_1, 0) \end{bmatrix} + z_1 z_2 [X(z_1, 0) + X(0, z_2) - x_{00}] \right\} \tag{13}$$

Using the 2D inverse Z transform to (13) we obtain the desired formula (5).

3. EXTENSION OF THE CAYLEY-HAMILTON THEOREM

From (10) we have

$$G(z_1, z_2) = z_1 z_2 \bar{G}(z_1, z_2) \tag{14}$$

where

$$\bar{G}(z_1, z_2) = I_n + \sum_{k=0}^{L_1+1} \sum_{l=0}^{L_2+1} I_n c_{\alpha\beta}(k, l) z_1^{-k} z_2^{-l} - \bar{A}_0 z_1^{-1} z_2^{-1} - \bar{A}_q z_2^{-1} - \bar{A}_2 z_1^{-1} \tag{15}$$

Let

$$\det \bar{G}(z_1, z_2) = \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} a_{N_1-k, N_2-l} z_1^{-k} z_2^{-l} \tag{16}$$

It is assumed that i and j are bounded by some natural numbers L_1, L_2 which determine the degrees N_1, N_2 . From (14) and (11) it follows that

$$G^{-1}(z_1, z_2) = z_1^{-1} z_2^{-1} \bar{G}^{-1}(z_1, z_2) = z_1^{-1} z_2^{-1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q} \tag{17}$$

and

$$\bar{G}^{-1}(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q} \tag{18}$$

where T_{pq} are defined by (6).

Theorem 2.

Let (16) be the characteristic polynomial of the system (2). Then the matrices T_{kl} satisfy the equation

$$\sum_{k=0}^{N_1} \sum_{l=0}^{N_2} a_{kl} T_{kl} = 0 \tag{19}$$

Proof.

From the definition of inverse matrix and (16), (18) we have

$$\text{Adj } \bar{G}(z_1, z_2) = \left(\sum_{k=0}^{N_1} \sum_{l=0}^{N_2} a_{N_1-k, N_2-l} z_1^{-k} z_2^{-l} \right) \left(\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q} \right) \tag{20}$$

where $\text{Adj } \bar{G}(z_1, z_2)$ is the adjoint matrix of $\bar{G}(z_1, z_2)$.

Comparison of the coefficients at the same power $z_1^{-N_1} z_2^{-N_2}$ of the equality (20) yields (19) since the degrees of $\text{Adj } \bar{G}(z_1, z_2)$ are less than N_1 and N_2 .

Theorem 2 is an extension of the well-known classical Cayley-Hamilton theorem for the 2D fractional system (2).

4. POSITIVITY OF THE FRACTIONAL 2D LINEAR SYSTEMS

Lemma 1.

a) If $0 < \alpha < 1$ and $1 < \beta < 2$ then

$$c_{\alpha\beta}(k, l) < 0 \text{ for } k = 1, 2, \dots; l = 2, 3, \dots \tag{21a}$$

b) If $1 < \alpha < 2$ and $0 < \beta < 1$ then

$$c_{\alpha\beta}(k, l) < 0 \text{ for } k = 2, 3, \dots; l = 1, 2, \dots \tag{21b}$$

Proof.

The proof will be accomplished by induction.

The hypothesis is true for $k = 1$ and $l = 2$ in (21a) since

$$c_{\alpha\beta}(1, 2) = (-1)^3 \frac{\alpha\beta(\beta-1)}{2} < 0$$

Assuming that the hypothesis is true for the pair (k, l) , $k+l \geq 3$ we shall show that it is also valid for the pairs $(k+1, l)$, $(k, l+1)$ and $(k+1, l+1)$.

From (1b) we have

$$c_{\alpha\beta}(k+1, l) = c_{\alpha\beta}(k, l) \frac{l-\alpha}{k+1} < 0$$

since $c_{\alpha\beta}(k, l) < 0$ for $k = 1, 2, \dots; l = 2, 3, \dots$

Similarly

$$c_{\alpha\beta}(k, l+1) = c_{\alpha\beta}(k, l) \frac{k-\alpha}{l+1} < 0$$

since $c_{\alpha\beta}(k, l) < 0$ for $k=1,2,\dots; l=2,3,\dots$

and

$$c_{\alpha\beta}(k+1, l+1) = c_{\alpha\beta}(k, l) \frac{(\alpha-\beta)(l-\beta)}{(k+1)(l+1)} < 0$$

since $c_{\alpha\beta}(k, l) < 0$ for $k=1,2,\dots; l=2,3,\dots$

The proof of (21b) is similar.

Lemma 2.

If (21) is met and

$$\bar{A}_k \in \mathfrak{R}_+^{n \times n} \text{ for } k=0,1,2 \quad (22)$$

then

$$T_{pq} \in \mathfrak{R}_+^{n \times n} \text{ for } p, q \in Z_+ \quad (23)$$

Proof.

If the conditions (21) and (22) are satisfied then from (6) we have (23).

Definition 2.

The system (2) is called the (internally) positive fractional 2D system if and only if $x_{ij} \in \mathfrak{R}_+^n$ and $y_{ij} \in \mathfrak{R}_+^p$, $i, j \in Z_+$ for any boundary conditions $x_{i0} \in \mathfrak{R}_+^n$, $i \in Z_+$, $x_{0j} \in \mathfrak{R}_+^n$, $j \in Z_+$ and all input sequences $u_{ij} \in \mathfrak{R}_+^m$, $i, j \in Z_+$.

Theorem 3.

The fractional 2D system (2) for $0 < \alpha < 1$ and $1 < \beta < 2$ (or $1 < \alpha < 2$ and $0 < \beta < 1$) is positive if and only if

$$\begin{aligned} \bar{A}_k \in \mathfrak{R}_+^{n \times n}, B_k \in \mathfrak{R}_+^{n \times m}, k=0,1,2, \\ C \in \mathfrak{R}_+^{p \times n}, D \in \mathfrak{R}_+^{p \times m} \end{aligned} \quad (24)$$

Proof. Necessity.

Let us assume that the system is positive and $x_{00}=e_{ni}$, $i=1,\dots,n$ (e_{ni} is the i th column of I_n) $x_{01}=x_{10}=0$ $u_{ij}=0$, $i,j \in Z_+$. Then from (3) for $i=j=0$ and $u_{ij}=0$ $i,j \in Z_+$

we obtain $x_{11} = \bar{A}_0 e_{ni} = \bar{A}_{0i} \in \mathfrak{R}_+^n$, where \bar{A}_{0i} is the i th column of the matrix \bar{A}_0 . This implies $\bar{A}_0 \in \mathfrak{R}_+^{n \times n}$ since $i=1,\dots,n$. If we assume that $x_{10}=e_{ni}$, $x_{00}=x_{01}=0$ and $u_{ij}=0$, $i,j \in Z_+$ then from (3) for $i=j=0$ we obtain

$$R_{hk} = [M_0, M_1^1, \dots, M_h^1, M_1^2, \dots, M_k^2, M_{11}, \dots, M_{1k}, M_{21}, \dots, M_{hk-1}] \quad (26)$$

$x_{11} = \bar{A}_1 e_{ni} = \bar{A}_{1i} \in \mathfrak{R}_+^n$ and this implies $\bar{A}_1 \in \mathfrak{R}_+^{n \times n}$.

In a similar way we may prove that $\bar{A}_2 \in \mathfrak{R}_+^{n \times n}$. Assuming $u_{00}=e_{ni}$, $u_{ij}=0$ $i,j \in Z_+$ $i+j>0$ and $x_{00}=x_{10}=x_{01}=0$ from (3) for $i=j=0$ we obtain $x_{11} = B_0 e_{ni} = B_{0i} \in \mathfrak{R}_+^m$ for $i=1,\dots,m$ and this implies $B_0 \in \mathfrak{R}_+^{n \times m}$. In a similar way it can be shown that $B_k \in \mathfrak{R}_+^{n \times m}$ for $k=1,2$ and $C \in \mathfrak{R}_+^{p \times n}$, $D \in \mathfrak{R}_+^{p \times m}$.

Sufficiency.

If the conditions (24) are met then by Lemma 2 $T_{pq} \in \mathfrak{R}_+^{n \times n}$ and from (5) we have $x_{ij} \in \mathfrak{R}_+^n$ for $i, j \in Z_+$ since $x_{i0} \in \mathfrak{R}_+^n$, $x_{0j} \in \mathfrak{R}_+^n$ and $u_{ij} \in \mathfrak{R}_+^m$ for $i, j \in Z_+$.

From (2b) we have $y_{ij} \in \mathfrak{R}_+^p$ since $C \in \mathfrak{R}_+^{p \times n}$, $D \in \mathfrak{R}_+^{p \times m}$ and $x_{ij} \in \mathfrak{R}_+^n$, $u_{ij} \in \mathfrak{R}_+^m$ for $i, j \in Z_+$.

Remark.

From (1b) and (3) it follows that if $\alpha = \beta$, $0 < \alpha < 1$ then $c_{\alpha\beta}(k, l) > 0$ for $k, l = 1, 2, \dots$ and the fractional 2D system (2) is not positive.

5. REACHABILITY AND CONTROLLABILITY TO ZERO

Definition 3.

The positive fractional 2D system (2) is called reachable at the point $(h, k) \in Z_+ \times Z_+$ if and only if for zero boundary conditions (4) ($x_{i0}=0$, $i \in Z_+$, $x_{0j}=0$, $j \in Z_+$) and every vector $x_f \in \mathfrak{R}_+^n$ there exists a sequence of inputs $u_{ij} \in \mathfrak{R}_+^m$ for

$$\begin{aligned} (i, j) \in D_{hk} = \\ \{(i, j) \in Z_+ \times Z_+ : 0 \leq i \leq h, 0 \leq j \leq k, i+j \neq h+k\} \end{aligned} \quad (25)$$

such that $x_{hk} = x_f$.

A vector is called monomial if and only if its one component is positive and the remaining components are zero.

Theorem 4.

The positive 2D fractional system (2) is reachable at the point (h, k) if and only if the reachability matrix

$$\begin{aligned}
 M_0 &= T_{h-1,k-1}B_0, \quad M_i^1 = T_{h-i,k-1}B_1 + T_{h-i-1,k-1}B_0, \quad i=1,\dots,h \\
 M_j^2 &= T_{h-1,k-1}B_2 + T_{h-1,k-j-1}B_0, \quad j=1,\dots,k \\
 M_{ij} &= T_{h-i-1,k-j-1}B_0 + T_{h-i,k-j-1}B_1 + T_{h-i-1,k-1}B_2, \quad i=1,\dots,h, \quad j=1,\dots,k
 \end{aligned} \tag{27}$$

contains n linearly independent monomial columns.

Proof.

Using the solution (5) for $i = h, j = k$ and zero boundary conditions we obtain

$$x_f = R_{hk}u(h, k) \tag{28}$$

$$u(h, k) = [u_{00}^T, u_{10}^T, \dots, u_{h0}^T, u_{01}^T, \dots, u_{0k}^T, u_{11}^T, \dots, u_{1k}^T, u_{21}^T, \dots, u_{h,k-1}^T]^T \tag{29}$$

and T denotes the transpose.

For the positive fractional 2D system (2) from (27) and (26) we have $M_0 \in \mathfrak{R}_+^{n \times m}$, $M_i^1 \in \mathfrak{R}_+^{n \times m}$, $M_j^2 \in \mathfrak{R}_+^{n \times m}$, $M_{ij} \in \mathfrak{R}_+^{n \times m}, i=1,\dots,h, j=1,\dots,k$ and $R_{hk} \in \mathfrak{R}_+^{n \times [(h+1)(k+1)-1]m}$.

From (28) it follows that there exists a sequence $u_{ij} \in \mathfrak{R}_+^m$ for $(i, j) \in D_{hk}$ for every $x_f \in \mathfrak{R}_+^n$ if and only if the matrix (26) contains n linearly independent monomial columns.

The following theorem gives sufficient conditions for the reachability of the positive fractional 2D system (2).

Theorem 5.

The positive fractional 2D system (2) is reachable at the point (h, k) if $\text{rank } R_{hk} = n$ and the right inverse R_{hk}^r of the matrix (26) has nonnegative entries

$$R_{hk}^r = R_{hk}^T [R_{hk} R_{hk}^T]^{-1} \in \mathfrak{R}_+^{[(h+1)(k+1)-1]m \times n} \tag{30}$$

Proof.

If $\text{rank } R_{hk} = n$ then there exists the right inverse R_{hk}^r of the matrix R_{hk} . If the condition (30) is met then from (28) we obtain

$$u(h, k) = R_{hk}^r x_f \in \mathfrak{R}_+^{[(h+1)(k+1)-1]m}$$

for every $x_f \in \mathfrak{R}_+^n$.

Example 1.

Consider the positive fractional 2D system (2) with

where

$$\begin{aligned}
 \bar{A}_0 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \\
 B_0 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
 \end{aligned} \tag{31}$$

To check the reachability at the point $(h, k) = (1, 1)$ of the system we use Theorem 4. From (27) and (26) we obtain

$$\begin{aligned}
 M_0 &= B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad M_1^1 = B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
 M_1^2 &= B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M_{ij} = 0
 \end{aligned}$$

for $i \geq 1, j \geq 1$

$$R_{11} = [M_0, M_1^1, M_1^2] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{32}$$

The first two columns of (32) are linearly independent monomial columns and by Theorem 4 the positive fractional 2D system (2) with (31) is reachable at the point (1,1). The sequence of inputs steering the state of the system from zero boundary conditions to an arbitrary state

$$x_f \in \mathfrak{R}_+^2 \text{ at the point } (1, 1) \text{ is given by } \begin{bmatrix} u_{00} \\ u_{10} \end{bmatrix} = x_f$$

and $u_{01} = 0$.

Using (30) and (32) we obtain

$$\begin{aligned}
 R_{11}^r &= R_{11}^T [R_{11} R_{11}^T]^{-1} = \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}
 \end{aligned} \tag{33}$$

From (33) it follows that the condition (30) is not satisfied in spite of the fact that the system is reachable at the point (1,1). Note that the system is reachable at the point (1,1) for any fractional orders (α, β) $0 < \alpha < 1$, $1 < \beta < 2$ (or $1 < \alpha < 2$, $0 < \beta < 1$) and any matrices \bar{A}_k , $k = 0, 1, 2$.

Lemma 3.

If the positive fractional 2D system (2) is with finite memory then

$$x_{bc}(i, j) = \sum_{p=1}^i (T_{i-p, j-1} \bar{A}_1 + T_{i-p-1, j-1} \bar{A}_0) x_{p0} + \sum_{q=1}^j (T_{i-1, j-q} \bar{A}_2 + T_{i-1, j-q-1} \bar{A}_0) x_{0q} + T_{i-1, j-1} \bar{A}_0 x_{00} = 0 \quad (35)$$

for $i \geq n_1, j \geq n_2$ and any nonzero boundary conditions (4).

Proof.

Using the expansion (11) and (34) we obtain $T_{ij} = 0$ for $i \geq n_1, j \geq n_2$ and the equality (35) for any nonzero boundary conditions (4).

Definition 5.

The positive fractional 2D system (2) is called reachable for the nonzero boundary conditions (NBC)

$$x_{i0} \in \mathfrak{R}_+^n, i \in Z_+ \text{ and } x_{0j} \in \mathfrak{R}_+^n, j \in Z_+ \quad (36)$$

at the point $(h, k) \in Z_+ \times Z_+$ if for every vector $x_f \in \mathfrak{R}_+^n$ there exists a sequence of inputs $u_{ij} \in \mathfrak{R}_+^m$ for $(i, j) \in D_{hk}$ such that $x_{hk} = x_f$.

Theorem 6.

The positive fractional 2D system (2) is reachable for NBC at the point (h, k) ($h \geq n_1, k \geq n_2$) if and only if the system is with finite memory and the reachability matrix (26) contains n linearly independent monomial columns.

Proof.

Using the solution (5) for $i = h, j = k$ and taking into account that $x_{hk} = x_f$ we obtain

$$x_f - x_{bc}(h, k) = R_{hk} u(h, k) \quad (37)$$

where R_{hk} and $x_{bc}(h, k)$ are defined by (26) and (35) respectively.

If the positive fractional 2D system (2) is with finite memory then by Lemma 3 there exists a point (h, k)

Definition 4.

The positive fractional 2D system (2) is called the system with finite memory if its characteristic polynomial has the form

$$\det G(z_1, z_2) = cz_1^{n_1} z_2^{n_2} \quad (34)$$

where c is a constant coefficient.

($h \geq n_1, k \geq n_2$) such that (35) holds and $x_f = R_{hk} u(h, k)$. In this case by Theorem 4 there exists a sequence of inputs $u_{ij} \in \mathfrak{R}_+^m$ for $(i, j) \in D_{hk}$ satisfying the equality (28). If it is not the case then $x_f - x_{bc}(h, k) \notin R_{hk} u(h, k)$ since by assumption the NBC (36) are arbitrary and the vector $x_f \in \mathfrak{R}_+^n$ is also arbitrary. In this case there does not exist a sequence of inputs $u_{ij} \in \mathfrak{R}_+^m$ for $(i, j) \in D_{hk}$ satisfying (37).

Definition 6.

The positive fractional 2D system (2) is called controllable to zero at the point (h, k) ($h \geq n_1, k \geq n_2$) if and only if for any NBC (36) there exists a sequence of inputs $u_{ij} \in \mathfrak{R}_+^m$ for $(i, j) \in D_{hk}$ such that $x_{hk} = 0$.

Theorem 7.

The positive fractional 2D system (2) is controllable to zero at the point (h, k) ($h \geq n_1, k \geq n_2$) if and only if the system is with finite memory.

Proof.

If the system is with finite memory then by Lemma 3 (35) holds for $h \geq n_1$ and $k \geq n_2$. For $x_f = 0$ from (37) we have

$$x_{bc}(h, k) + R_{hk} u(h, k) = 0 \quad (38)$$

The equation (38) is satisfied for $u(h, k) = 0$.

If the condition (35) is not satisfied then does not exist $u(h, k) \in \mathfrak{R}_+^{[(h+1)(k+1)-1]m}$ satisfying (38) since for the positive system $R_{hk} \in \mathfrak{R}_+^{n \times [(h+1)(k+1)-1]m}$ and

$$x_{bc}(h, k) \in \mathfrak{R}_+^n.$$

6. CONCLUDING REMARKS

A new class of 2D fractional linear systems has been introduced. The notion of (α, β) orders $0 < \alpha < 1, 1 < \beta < 2$ or $1 < \alpha < 2, 0 < \beta < 1$ fractional 2D difference has been proposed. The fractional 2D state equations of linear systems have been given and their solutions have been derived using the 2D Z transform. The classical Cayley-Hamilton theorem has been extended for the fractional 2D systems. Necessary and sufficient conditions have been established for the positivity, reachability and controllability to zero of the fractional 2D linear systems. It has been shown that the fractional 2D system (2) is positive if $0 < \alpha < 1, 1 < \beta < 2$ or $1 < \alpha < 2, 0 < \beta < 1$. The fractional 2D system is not positive if $\alpha = \beta$. The considerations can be easily extended for fractional 2D linear systems with delays. An extension of these considerations for fractional 2D continuous-time linear systems is an open problem.

Using (A.1) for an 2D discrete function x_{ij} we obtain

$$\Delta_i^{n_1} \Delta_j^{n_2} x_{ij} = \Delta_j^{n_2} \Delta_i^{n_1} x_{ij} = \sum_{k=0}^i (-1)^k \binom{n_1}{k} \Delta_j^{n_2} x_{i-k,j} = \sum_{k=0}^i (-1)^k \binom{n_1}{k} \sum_{l=0}^j (-1)^l \binom{n_2}{l} x_{i-k,j-l} = \sum_{l=0}^j (-1)^l \binom{n_2}{l} \sum_{k=0}^i (-1)^k \binom{n_1}{k} x_{i-k,j-l} = \sum_{k=0}^i \sum_{l=0}^j (-1)^{k+l} \binom{n_1}{k} \binom{n_2}{l} x_{i-k,j-l} \tag{A.3}$$

for $n_1, n_2 \in N$ and $i, j \in Z_+$

Note that

$$\binom{n_1}{k} \binom{n_2}{l} = \begin{cases} 1 & \text{for } k=0 \text{ or/and } l=0 \\ \frac{n_1(n_1-1)\dots(n_1-k+1)n_2(n_2-1)\dots(n_2-l+1)}{k!l!} & \\ \text{for } k+l > 0 \end{cases} \tag{A.4}$$

is also well defined for $n_1 = \alpha$ and $n_2 = \beta$, where α and β are any real numbers. Thus (A.4) can be used for defining the α, β orders of an 2D function x_{ij} .

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Appendix. Justification of the definition 1.

It is well-known that for a discrete function x_i the n -order difference is given by

$$\Delta_i^n x_i = \Delta^{n-1} x_i - \Delta^{n-1} x_{i-1} = \sum_{k=0}^i (-1)^k \binom{n}{k} x_{i-k} \tag{A.1}$$

$n \in N = \{1,2,\dots\}, \quad i \in Z_+ \{0,1,\dots\}$

where

$$\binom{n}{k} = \begin{cases} 1 & \text{for } k=0 \\ \frac{n!}{k!(n-k)!} & \end{cases} \tag{A.2}$$

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