THE ELASTICITY PROBLEM FOR A STRATIFIED SEMI-INFINITE MEDIUM CONTAINING A PENNY-SHAPED CRACK FILLED WITH A GAS

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Abstract: This paper deals with a periodic two-layered elastic half-space weakened by an interface penny-shaped crack filled with a gas. The study is based on the approximate treatment by using the linear elasticity with microlocal parameters in the axisymmetric case. Applying the Hankel integral transforms, we obtain a system of dual integral equations. It is reduced to a set of two integral equations which are solved numerically. Some results concerning the variation of the internal gas pressure and the stress intensity factors of mode I and mode II are illustrated graphically.

1. INTRODUCTION

The mechanical behavior of solids containing gas- or liquid-filled cracks was a subject of the investigation of many authors. Interest in such problems is stimulated by their wide applications in mining engineering, gas- and oil-producing industries. Zazovskiy (1979a, 1979b) studied the possibility of propagation of a plane crack in rocks while it is pumped with a fluid or taking into account a filtration of the liquid substance into the solid structure. Sulym and Yevtushenko (1980) solved a plane problem for a crack filled with a compressible barotropic liquid. They suggested to simulate the crack's filler by a constant pressure dependent on the crack opening and determined from the equation of state of the fluid. Baluyeva and Dashevsky (1994, 1995) considered the problems of gas-filled cracks in an infinite elastic medium. Based on the concept of stress intensity factors criteria in fracture mechanics they obtained the estimations for crack growth while the mass of the gas in the crack monotonically increases. A combined thermal and mechanical influence of the heat-conducting ideal gas filling a crack on the stress-and-strain state was studied by Matczyński et al. (1999).

The present contribution is a sequel to some our earlier investigations (Kaczyński and Monastyrskyy, 2004; 2005) involving the problems for a periodic stratified space and an isotropic half-space containing a liquid-filled penny-shaped crack. It is devoted to examine the integrated effect of an ideal gas, filling a penny-shaped interface crack in a periodic two-layered semi-infinite medium, on the variation of stress intensity factors under the external both tensile and compressive load.

2. STATEMENT OF THE PROBLEM

2.1. Description of the problem

Let us consider a periodic stratified half-space, in which every repeated unit lamina of thickness l consists of two perfectly bonded layers of thicknesses l_1 and l_2 ($l = l_1+l_2$) with different Lame's constants λ_1, μ_1 and λ_2, μ_2. The direction of the layering is perpendicular to the boundary of the body.

The composite is weakened by a penny-shaped interface crack of radius a, which is located on a plane parallel to the boundary at the distance h. The crack is filled with a fixed amount of gas.

Let refer the body to an axially symmetric co-ordinate system (r, θ, z), introducing it in the manner that the z-axis is perpendicular to the layering, directing towards the boundary, and the coordinate basic origin coincides with the center of the crack (see Fig. 1). So the crack occupies the region $\{(r, \theta, z)\colon r \le a, 0 \le \theta < 2\pi, z = 0\}$.

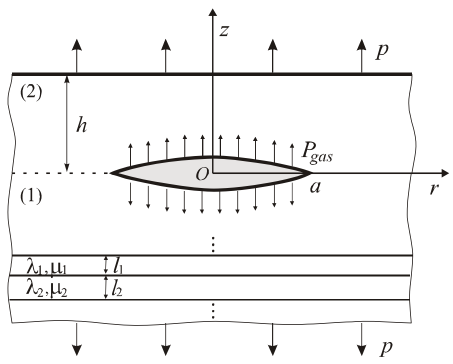

Fig. 1. Periodic stratified half-space having a gas-filled crack

The body is subjected to a uniform normal pressure p applied at infinity and at the boundary. Moreover, the surfaces of the crack are under the internal pressure of the gas P_{gas}. Notice that this parameter does not remain constant during the interaction and is unknown *a priori* because
the change of the external load leads to the change of the crack's volume, which consequently induces the magnitude of the internal pressure of the gas.

The problem under study lies in the determination of the stress-and-strain state of the composite being considered, paying much attention on the distribution of stress in the neighbourhood of the defect. Especially, the stress intensity factors as the local important parameters controlling the fracture instability are of prime interest.

2.2. Governing equations

In the description of the macroscopic behavior of solids with a periodic microheterogeneous structure Woźniak and Matysiak (1987) developed the homogenized theory of linear elasticity with microlocal parameters. We base on this approach that leads in the static axisymmetric case to the governing equations and constitutive relations of certain homogenized model of the treated body, given in terms of the unknown macro-displacements $u_z(r, z)$ and $u_r(r, z)$ as follows (Pusz, 1988):

$$C\frac{\partial}{\partial r}\left[r\frac{\partial u_z}{\partial r}\right]+\left(B+C\right)\frac{\partial^2\left[r u_r\right]}{\partial r\partial z}+A_1 r\frac{\partial^2 u_z}{\partial z^2}=0,$$
$$A_2\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\left(r u_r\right)}{\partial r}\right]+\left(B+C\right)\frac{\partial^2 u_z}{\partial r\partial z}+C\frac{\partial^2 u_r}{\partial z^2}=0, \tag{1}$$
$$\sigma_{zz}=A_1\frac{\partial u_z}{\partial z}+B\frac{1}{r}\frac{\partial\left(r u_r\right)}{\partial r},\ \sigma_{rz}=C\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right),$$

where

$$A_1=\frac{\left(\lambda_1+2\mu_1\right)\left(\lambda_2+2\mu_2\right)}{\left(1-\eta\right)\left(\lambda_1+2\mu_1\right)+\eta\left(\lambda_2+2\mu_2\right)}>0,$$
$$A_2=A_1+\frac{4\eta\left(1-\eta\right)\left(\mu_1-\mu_2\right)\left(\lambda_1-\lambda_2+\mu_1-\mu_2\right)}{\left(1-\eta\right)\left(\lambda_1+2\mu_1\right)+\eta\left(\lambda_2+2\mu_2\right)}>0,$$
$$B=\frac{\left(1-\eta\right)\lambda_2\left(\lambda_1+2\mu_1\right)+\eta\lambda_1\left(\lambda_2+2\mu_2\right)}{\left(1-\eta\right)\left(\lambda_1+2\mu_1\right)+\eta\left(\lambda_2+2\mu_2\right)}>0, \tag{2}$$
$$C=\frac{\mu_1\mu_2}{\left(1-\eta\right)\mu_1+\eta\mu_2}>0;\ \eta=\frac{l_1}{l}.$$

2.3. Boundary conditions

Within the scope of the above homogenized model it is convenient to pose the considered problem for the half-space $\{(r, \theta, z)$: $0\le r<\infty,\ 0\le\theta<2\pi,\ -\infty<z\le 0\}$ and the layer $\{(r, \theta, z)$: $0\le r<\infty,\ 0\le\theta<2\pi,\ \ 0<z\le h\}$. Hereinafter we use the superscripts $^{(1)}$ and $^{(2)}$ to refer quantities to the half-space and to the layer, respectively (see Fig. 1).

Following the classical approach based on the superposition principle, the problem is separated into two parts. For the first trivial part, the uncracked stratified medium is assumed to be loaded uniformly at the boundaries. Next we pass to the second part involving perturbations caused by the gas-filled crack. The boundary conditions of this perturbed crack problem can be written as

(a) on the bonding interfacial plane $z = 0$:

$$\begin{aligned}
&\sigma_{zz}^{(1)}(r,0)=\sigma_{zz}^{(2)}(r,0)=-P_{gas}-p, && 0\le r\le a,\\
&\sigma_{rz}^{(1)}(r,0)=\sigma_{rz}^{(2)}(r,0)=0, && 0\le r\le a,\\
&\sigma_{rz}^{(1)}(r,0)=\sigma_{rz}^{(2)}(r,0), && a\le r<\infty,\\
&\sigma_{zz}^{(1)}(r,0)=\sigma_{zz}^{(2)}(r,0), && a\le r<\infty,\\
&u_z^{(1)}(r,0)=u_z^{(2)}(r,0), && a\le r<\infty,\\
&u_r^{(1)}(r,0)=u_r^{(2)}(r,0), && a\le r<\infty,
\end{aligned} \tag{3}$$

(b) on the bounding surface of the semi-infinite medium and at infinity:

$$\begin{aligned}
&\sigma_{zz}^{(2)}(r,h)=\sigma_{rz}^{(2)}(r,h)=0,\\
&\sigma_{zz}^{(1)}(r,-\infty)=\sigma_{rz}^{(1)}(r,-\infty)=0.
\end{aligned} \tag{4}$$

Moreover, in order to find the unknown gas pressure P_{gas}, we use the well-known Mendeleyev–Clapeyron equation

$$P_{gas}V=g_0=const\,, \tag{5}$$

where V stands for the volume of a gas, which is equal to the volume of the crack, and g_0 – the constant, depending on mass and molar mass of gas and temperature.

3. METHOD OF SOLUTION

To solve the above-mentioned problem, we use the similar technique as that developed by Monastyrskyy and Kaczyński (2005). Because of the complexity of appearing expressions only a brief description of the proposed method will be outlined.

The proper integral representation of stresses and displacements within the every domain 1 and 2 can be constructed by solving the auxiliary boundary value problem. Instead of the boundary conditions $(3)_{1\text{-}2}$ and $(3)_{5\text{-}6}$ one should pose

$$\begin{aligned}
&u_z^{(2)}(r,0)-u_z^{(1)}(r,0)=\Delta u_z(r), && 0\le r<\infty,\\
&u_r^{(2)}(r,0)-u_r^{(1)}(r,0)=\Delta u_r(r), && 0\le r<\infty,
\end{aligned} \tag{6}$$

where $\Delta u_z(r)$, $\Delta u_r(r)$ are the unknown jumps of normal and shear displacements.

In dealing with the solution to this problem we use the method of the Hankel integral transforms. In this way we obtain the representation of displacements and stresses within the body via Hankel's transforms of jumps $H_0\equiv H_0[\Delta u_z(r),\ \xi]$ and $H_1\equiv H_1[\Delta u_r(r),\ \xi]$, defined by

$$H_0[\Delta u_z(r),\xi]=\int_0^\infty r\,\Delta u_z(r)\,J_0(r\xi)\,dr,$$
$$H_1[\Delta u_r(r),\xi]=\int_0^\infty r\,\Delta u_r(r)\,J_1(r\xi)\,dr, \tag{7}$$

where J_k stands for the Bessel function of the first kind of order k.

The remaining boundary conditions of the posed initial problem $(3)_{1\text{-}2}$ and $(3)_{5\text{-}6}$ then yields the system of dual integral equations written in a shortened form as

$$\begin{aligned}
&\int_0^\infty \xi^2\left(\tilde K_{1z}H_0+\tilde K_{1r}H_1\right)J_0(\xi r)d\xi=\frac{-P_{gas}-p}{C},0\le r\le a,\\
&\int_0^\infty \xi^2\left(\tilde K_{2z}H_0+\tilde K_{2r}H_1\right)J_1(\xi r)d\xi=0, \qquad 0\le r\le a,\\
&\int_0^\infty \xi\,H_0J_0(\xi r)\,d\xi=0, \qquad a\le r<\infty,\\
&\int_0^\infty \xi\,H_1J_1(\xi r)\,d\xi=0, \qquad a\le r<\infty,
\end{aligned} \tag{8}$$

with some regular kernels $\tilde K_{1z}$, $\tilde K_{1r}$, $\tilde K_{2z}$, $\tilde K_{2r}$.

Following Ufland (1977) we introduce the following representations which identically satisfy $(8)_{3\text{-}4}$:

$$\begin{aligned}
&H_0\left[\Delta u_z,\xi\right]=\xi^{-1}\int_0^a\varphi_z(t)\sin(\xi t)dt,\\
&H_1\left[\Delta u_r,\xi\right]=\xi^{-1}\int_0^a\varphi_r(t)\left(\frac{\sin(\xi t)}{\xi}-\cos(\xi t)\right)dt.
\end{aligned} \tag{9}$$

By substituting (9) into $(8)_{1\text{-}2}$ we arrive at the two integral equations for auxiliary functions φ_z, φ_r which can be reduced to the following form

$$\begin{aligned}
&\frac{\pi}{2}\frac{D}{\sqrt{A_1A_2}}\varphi_z(r)+\int_0^a\varphi_z(t)K_{1z}dt+\int_0^a\varphi_r(t)K_{1r}dt=\frac{-P_{gas}-p}{C},\\
&\frac{\pi}{2}\frac{(B+C)D}{A_1}\left(\varphi_r(r)+\int_0^r\frac{\varphi_r(t)dt}{t}\right)-\int_0^a\varphi_z(t)K_{2z}dt-\int_0^a\varphi_r(t)K_{2r}dt=0.
\end{aligned} \tag{10}$$

In the above, K_{1z}, K_{1r}, K_{2z}, K_{2r} are known complicated regular kernels (their formulae are too lengthy so we do not present them here). Furthermore, we confine ourselves to the case of different shear modulae of the subsequent layers $\mu_1\ne\mu_2$ in which

$$D=\frac{k_2\left(B+A_1k_2^2\right)-k_1\left(B+A_1k_1^2\right)}{2\left(k_1^2-k_2^2\right)} \tag{11}$$

provided k_1 and k_2 ($k_1<k_2$) are the real positive of the biquadratic in k

$$A_1Ck^4+\left(B^2+2BC-A_1A_2\right)k^2+A_2C=0\,. \tag{12}$$

In addition to (10), the condition for the unknown pressure of the gas P_{gas} is obtained from (5), using the formula $V=2\pi\int_0^a r\Delta u_z(r)dr$ and $(9)_1$, with the result

$$2\pi P_{gas}\int_0^a t\varphi_z(t)\,dt=g_0\,. \tag{13}$$

Thus, the problem at hand reduces to the set of integral equations (10) and (13) for the unknown functions φ_z, φ_r and scalar parameter P_{gas}.

4. ANALYSIS OF RESULTS

4.1. Numerical procedure

Due to the complex structure of equations (10) it is unlikely to obtain their solutions in the analytical form. For this reason, we apply a certain numerical procedure outlined briefly below.

We find the unknown functions $\varphi_z(r)$, $\varphi_r(r)$ in the space of continuous functions on the segment [0, a]. As the set of polynomials is the full set of functions in this space, these functions can be approximated with any, a priori given accuracy by polynomials

$$\begin{aligned}
&\varphi_{zN}(r)=c_{z1}r+c_{z2}r^3+\ldots+c_{zN}r^{2N-1},\\
&\varphi_{rM}(r)=c_{r1}r^2+c_{r2}r^4+\ldots+c_{rM}r^{2M},
\end{aligned} \tag{14}$$

where c_{zn} ($n=\overline{1,N}$) and c_{rm} ($m=\overline{1,M}$) stand for the coefficients.

Substituting expressions (14) for $\varphi_z(r)$, $\varphi_r(r)$ into integral equations (10) and satisfying them in the set of the collocation points (for equation $(10)_1$ at the points $\underline{r}_n=na/N$ ($n=\overline{1,N}$) and for equation $(10)_2$ at the points $\underline{r}_m=ma/M$ ($m=\overline{1,M}$), we arrive at a set of non-linear algebraic equations for the unknown coefficients c_{zn}, c_{rm} and parameter P_{gas}, being the discrete analogue of the equations (10) and (13). Its solution is found by Newton's method. The desired accuracy is achieved by increasing the power of approximating polynomials in (14).

4.2. Stress intensity factors

The physically meaningful parameters are the stress intensity factors (SIF) of mode I and II, defined conventionally by

$$\begin{aligned}
&K_{\text{I}}=\lim_{r\to a^+}\sqrt{2\pi(r-a)}\;\sigma_{zz}(r,0),\\
&K_{\text{II}}=\lim_{r\to a^+}\sqrt{2\pi(r-a)}\;\sigma_{rz}(r,0).
\end{aligned} \tag{15}$$

They may be determined in terms of the solutions of (10) as

$$K_{\text{I}}=-\frac{CD}{A_1}\sqrt{\frac{\pi}{a}}\,\varphi_z(a),\quad K_{\text{II}}=-\frac{CD}{A_1}\sqrt{\frac{\pi}{a}}\,\varphi_r(a). \tag{16}$$

4.3. Results

The described numerical procedure was performed on the simplifying assumption $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$ and for the following dimensionless parameters:

$$\overline{P}_{gas} = P_{gas}/\mu_1, \quad \overline{p} = p/\mu_1, \quad \overline{\lambda}_2 = \lambda_2/\mu_1 = 2,$$
$$\overline{h} = h/a, \quad \overline{g}_0 = g_0/\mu_1 a^3 = 1,$$
$$\overline{K}_I = K_I\sqrt{a}/\mu_1, \quad \overline{K}_{II} = K_{II}\sqrt{a}/\mu_1.$$

Fig. 2 shows the dependence of gas pressure on the external load. It is seen that with the increasing of the external load the internal pressure of the gas decreases. This relationship is characterized by the high gradient in the range of negative values of load, i.e. for the compressive applied pressure. While the external pressure becomes tensile, the gradient decreases and the value $\overline{P}_{gas}$ tends to zero.

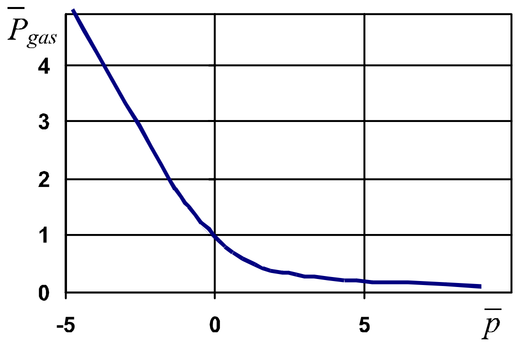

Fig. 2. Pressure of the gas versus the applied load

Variations of the SIFs of mode I and mode II due to the external load are demonstrated in Fig. 3 and Fig. 4, respectively. Similarly to the behavior of the curve $P_{gas} = P_{gas}(p)$ (see Fig. 2) the dependences $K_I = K_I(p)$ and $K_{II} = K_{II}(p)$ are nonlinear. Observe that if $p = 0$ then K_I and K_{II} are not equal to zero. Besides, these parameters remain positive for the compressive load. Figures 3 and 4 also depict the influence of the boundary on the SIFs. It can be seen that the closer to the boundary the crack is located, the more these parameters are.

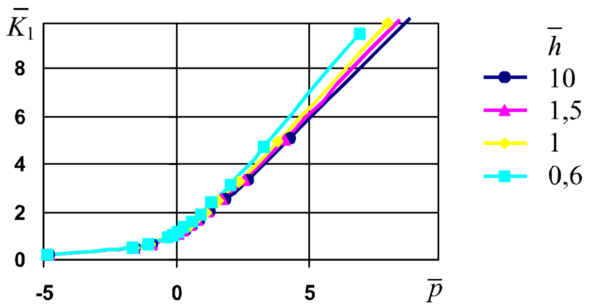

Fig. 3. SIF of mode I versus the applied load

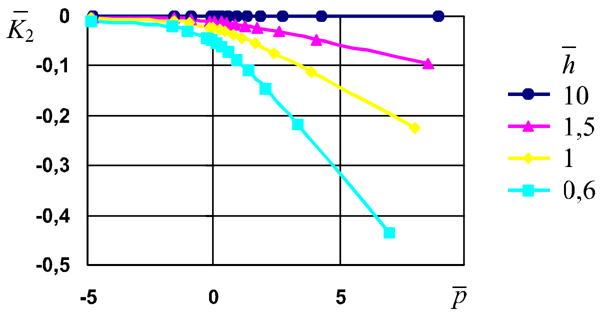

Fig. 4. SIF of mode II versus the applied load

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ZAGADNIENIE PÓŁNIESKONCZONEGO UWARSTWIONEGO OŚRODKA SPRĘŻYSTEGO ZAWIERAJĄCEGO SZCZELINĘ KOŁOWĄ WYPEŁNIONĄ GAZEM

Streszczenie: Niniejsza praca poświęcona jest zagadnieniu periodycznej dwuwarstwowej półprzestrzeni sprężystej osłabionej międzywarstwową szczeliną kołową wypełnioną gazem. Zastosowano przybliżone podejście oparte na liniowej teorii sprężystości z parametrami mikrolokalnymi w osiowo-symetrycznym przypadku. Używając transformacji Hankela, otrzymano układ dualnych równań całkowych, który sprowadzono do numerycznego rozwiązania równań całkowych. Zależności dotyczące wewnętrznego ciśnienia gazu i współczynników intensywności naprężeń zilustrowano graficznie.