MATHEMATICAL MODEL FOR ESTIMATING THE PERIOD OF CREEP-FATIGUE CRACK GROWTH IN CONSTRUCTION MATERIALS AT HIGH TEMPERATURE

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Abstract: In this work we propose mathematical model of important scientific and technical problem – estimating of remaining lifetime of constructions elements subjected to high temperature fatigue. The differential equation, with initial and final conditions, for assessing the remaining lifetime of three-dimensional solid was obtained. This mathematical model is formulated, based on energetic approach. Proposed approach gave us the possibility to combine fatigue and creep loadings in the single equation. Known in scientific materials experimental data confirmed the correctness of this model.

1. INTRODUCTION

To assess the remaining life of constructions items subjected to high temperature fatigue loading, creep growth of the crack must be taken into account in the bounds of one loading cycle, because many materials are subjected to the action of time-variable loading with large cycles. There are very few works devoted to the question of creep-fatigue crack growth in scientific papers, despite of fact that many constructions work under this kind of loading conditions. By now in this field of fracture mechanic there are known works which are based only on empirical researches (Tayra and Otani, 1986; Garofalo, 1970; Gladwin et al., 1988; Koterazawa,1994). In this work there is made an attempt to build mathematical model for describing such a process using energetic approach, in particular the equation of balance of energy changing rates. Similar application of energetic approach was carried out in papers by Andreykiv and Kit (2006) and Andreykiv and Sas (2006) where the models for assessing the lifetime of constructions subjected to high temperature fatigue and high temperature creep were described.

2. CYCLIC MATHEMATICAL MODEL CONSTRUCTION

Let us consider three-dimension solid with crack of area S_o (Fig. 1), subjected to action of high temperature T_o and time–variable cyclic loading p with hold period T.

It is assumed that the solid is heated uniformly to high temperature T_o. The crack is macroscopic and external tension loadings with parameter p applied in such a way that stress-deformation state is symmetric. The purpose of the problem is to find the time $t = t_o$ (the number of loading cycles $N = N_o$) when the crack will grow to the critical size S_o and the solid will fracture.

According to Andreykiv and Kit (2006) at the crack growth the equation of energetic balance is true.

$$Q + A = W + \Gamma + K \ . \tag{1}$$

Here A is work of external forces which is constant in our case. W - deformation energy of the solid which we can represent as following

$$W = W_{\mathrm{e}} + W_{\mathrm{p}}^{(1)}(S) + W_{\mathrm{p}}^{(2)}(t) - W_{\mathrm{p}}^{(3)}(t) - 2W_{\mathrm{p}}^{(4)}(t)\,, \tag{2}$$

W_e – elastic constituent of W; $W_p^{(1)}(S)$ – part of plastic energy that depends on crack area S; $W_p^{(2)}(t)$ – work of plastic deformations from external efforts at constant crack area during the stretching of fracture zone near the cracks contour, that depends on time t; $W_p^{(3)}(t)$ – work of plastic deformations during the unloading, which depends on t and is released when the area of the crack is constant; $W_p^{(4)}(t)$ – work of plastic deformations during the static loading; Γ – fracture energy that depends only on crack area S ; Q =const – the value of heating energy, which is born by external factors; K – kinetic energy which in our case is a small, thus we will neglect it.

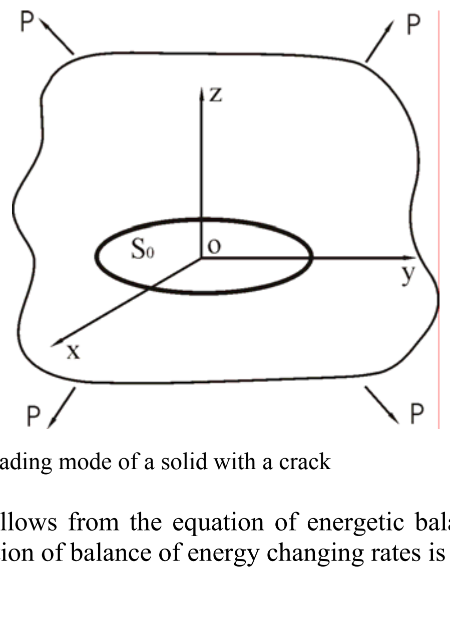

Fig. 1. Loading mode of a solid with a crack

It follows from the equation of energetic balance that the equation of balance of energy changing rates is

$$\frac{\partial A}{\partial t}=\frac{\partial W}{\partial t}+\frac{\partial \Gamma}{\partial t}. \quad (3)$$

Putting (2) in (3) we have

$$\frac{\partial}{\partial S}\left[\Gamma-\left(A-W_{\mathrm{e}}-W_{\mathrm{p}}^{(1)}\right)\right]\frac{\mathrm{d}S}{\mathrm{d}t}-\frac{\partial(W_{\mathrm{p}}^{(3)}+2W_{\mathrm{p}}^{(4)})}{\partial t}=0. \quad (4)$$

Since, we will find $V=\partial S/\partial t$ rate of changing of crack area during its growth

$$\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{\partial(W_{\mathrm{p}}^{(3)}+2W_{\mathrm{p}}^{(4)})}{\partial t}\Bigg/\frac{\partial}{\partial S}\left[\Gamma-\left(A-W_{\mathrm{e}}-W_{\mathrm{p}}^{(1)}\right)\right]. \quad (5)$$

Based on paper by Andreykiv and Sas (2006) expression in brackets can be represented as

$$\frac{\partial}{\partial l}\left[\Gamma-\left(A-W_{e}-W_{\mathrm{p}}^{(1)}\right)\right]=\gamma_{\mathrm{fC}}-\gamma_{\mathrm{t}}. \quad (6)$$

Here γ_t is specific work of plastic deformations during crack growth. Putting (6) in (5) we have

$$\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{\partial(W_{\mathrm{p}}^{(3)}+2W_{\mathrm{p}}^{(4)})}{\partial t}\Bigg/\left[\gamma_{\mathrm{fC}}-\gamma_{\mathrm{t}}\right]=\frac{\mathrm{d}S^{(\mathrm{f})}}{\mathrm{d}t}+\frac{\mathrm{d}S^{(\mathrm{c})}}{\mathrm{d}t}, \quad (7)$$

Multiplying equation (7) by hold period T and assuming that $dt = T\mathrm{d}N$, we will obtain the equation for determination the rate of crack growth for one loading cycle

$$\frac{\mathrm{d}S}{\mathrm{d}N}=\frac{\mathrm{d}S^{(\mathrm{f})}}{\mathrm{d}N}+\frac{\mathrm{d}S^{(\mathrm{c})}}{\mathrm{d}N}. \quad (8)$$

Separately creep and fatigue contributions can be determined as following

$$\frac{\mathrm{d}S^{(\mathrm{f})}}{\mathrm{d}N}=\frac{\partial W_{\mathrm{p}}^{(3)}}{\partial N}\Bigg/\left[\gamma_{\mathrm{fC}}-\gamma_{\mathrm{t}}\right], \quad (9)$$

$$\frac{\mathrm{d}S^{(\mathrm{c})}}{\mathrm{d}N}=2\frac{\partial W_{\mathrm{p}}^{(4)}}{\partial N}\Bigg/\left[\gamma_{\mathrm{fC}}-\gamma_{\mathrm{t}}\right]. \quad (10)$$

For completeness of mathematical model, the following initial and final conditions must be added to equation (8)

$$N=0,\ S(0)=S_0,\ N=N_*,\ S(N_*)=S_*, \quad (11)$$

where critical crack area S_* is deduced from energetic criterion

$$\gamma_{\mathrm{t}}(S_*)=\gamma_{\mathrm{fC}}. \quad (12)$$

Here γ_{fC} – specific fracture energy during crack growth; γ_t – specific work of plastic deformations in prefracture zone near the crack tip, which is determined as $\gamma_t = \sigma_{of}\delta_{max}$; N_* – period of precritical macrocrack growth; σ_{of} – averaged normal stress in fracture zone near the crack tip; δ_{max} – normal opening in the crack tip. Thus, kinetic equations (8) and conditions (11), (12) compose mathematical model for exploring the precritical crack growth in three-dimensional solids.

It can be assumed that during each loading cycle of continuous duration there is high temperature creep of material in prefracture zone, main period of time of which is withstand creep. Based on this, we can determinate approximately the opening of prefracture zone $\delta_{tmax}(x,\xi,t)$as

$$\delta_{\mathrm{t\,max}}(x,\xi,t)=\delta_{\max}(x,\xi)+\dot{\delta}_{\mathrm{t\,max}}(x,\xi,0)\cdot t \quad (13)$$

Here $\delta_{tmax}(x,\xi)$ is maximum opening of prefracture zone at the beginning of loading cycle; $\dot{\delta}_{\mathrm{t\,max}}(x,\xi,0)$ – rate of opening in prefracture zone during the creep deformation in a cycle. Then, based on results Andreykiv and Kit (2006), Andreykiv and Sas (2006) and Andreykiv and Lishinskaya (1999) the constituents of equation (7) can be described as following

$$\gamma_{\mathrm{t}}=\sigma_{\mathrm{of}}\delta_{\max}(0,t),\ \ \gamma_{\mathrm{f}}=\sigma_{\mathrm{of}}\delta_{\mathrm{fC}}, \quad (14)$$

$$W_{\mathrm{p}}^{(4)}(t)=\int_L\left\{\int_0^{l_{\mathrm{pt}}}\sigma_{\mathrm{of}}\left[\delta_{\max}(x,\xi)+\dot{\delta}_{\mathrm{t\,max}}(x,\xi)\cdot t\mathrm{d}x\right]-\int_0^{l_{\mathrm{p}}}\sigma_{\mathrm{of}}\delta_{\max}(x,\xi)\mathrm{d}x\right\}\mathrm{d}\xi, \quad (15)$$

where: lp – length of initial plastic zone near the crack tip; l_{pt} is length of plastic zone near the crack tip for the time of incubation period before the leap of the crack; $\sigma_{of} = \sigma_{0.2} + 0.5A\varepsilon_t^n$; $\sigma_{0.2}$ – yield strength of material; A, n – parameters of tensile stress-strain diagram. As in works Tayra and Otani (1986) and Garofalo (1970), the value $\delta_{tmax}(x,\xi)$ in prefracture zone approximately can be presented as

$$\delta_{\max}(x,\xi)\approx\delta_{\max}(0,\xi)\left(1-\frac{x}{l_{\mathrm{p}}}\right)^2, \quad (16)$$

and length of plastic zone lp in such form (Andreykiv and Sas, 2006):

$$l_p=\delta_{\max}(0,\xi)E\sigma_{\mathrm{of}}^{-1}. \quad (17)$$

Here E – modulus of elasticity; K_{Imax} – stress intensity factor. Putting (16) in (17) and conducting necessary calculation we will find

$$W_{\mathrm{p}}^{(4)}=\frac{E}{3}\int_L\left\{\left[\delta_{\max}(0,\xi)+\dot{\delta}_{\mathrm{t\,max}}(0,\xi)t\right]^2-\delta_{\max}^2(0,\xi)\right\}\mathrm{d}\xi. \quad (18)$$

And therefore equation (10) on the basis of correlations (18) and paper by Andreykiv and Sas (2006) for determination the creep contribution of crack growth rate becomes

$$\frac{\mathrm{d}S^{(\mathrm{c})}}{\mathrm{d}N}=1.33TE\int_L\dot{\delta}_{\mathrm{t\,max}}(0)\mathrm{d}\xi\left[\sigma_{\mathrm{of}}-\delta_{\mathrm{fC}}^{-1}L^{-1}\int_L\sigma_{\mathrm{of}}\delta_{\max}(0,\xi)\mathrm{d}\xi\right]^{-1}. \quad (19)$$

Based on Andreykiv and Sas (2006) we will present the rate of opening and opening of the crack as

$\dot{\delta}_{\max} = A_1 \left[\delta_{\max}(0)\delta_{\mathrm{fC}}^{-1} \right]^m$, $\delta_{\max}(0,\xi) = K_{\mathrm{I}\max}^2(\xi)\sigma_{\mathrm{of}}^{-1}E^{-1}$.

Put them in (19) we will have

$$\frac{\mathrm{d}S^{(\mathrm{c})}}{\mathrm{d}N} = 1.33TEA_1 \int_L \left[\delta_{\max}(0)\delta_{\mathrm{fC}}^{-1} \right]^m \mathrm{d}\xi \cdot \left[\sigma_{\mathrm{of}} - \delta_{\mathrm{fC}}^{-1} L^{-1} \int_L \sigma_{\mathrm{of}} \delta_{\max}(0,\xi)\,\mathrm{d}\xi \right]^{-1} \tag{20}$$

Let us consider the contribution $\mathrm{d}S^{(f)}/\mathrm{d}N$ represented by equation (9). According to results of work Shata and Terletska (1999) the value $\partial W_{\mathrm{p}}^{(3)} / \partial N$ we can determine as

$$\frac{\partial W_{\mathrm{p}}^{(3)}}{\partial N} = \frac{\varepsilon_{\mathrm{fc}}\alpha}{\delta_{\mathrm{fc}}} \int_L \int_0^{l_{\mathrm{p}}} \sigma_{\mathrm{of}} [\delta_{\mathrm{t}\max}(x,\xi,t) - \delta_{\mathrm{t}\min}(x,\xi,t)]\mathrm{d}x\mathrm{d}\xi \tag{21}$$

where $\varepsilon_{\mathrm{fC}}$ - critical value of materials deformation during the cyclic loading; α - coefficient that correlate static opening with cyclic opening of the crack (Szata and Terletska, 1999).

We consider the case when in every cycle the solid is subjected to static loading with hold time T, that is why the opening of the crack is large then in the case of pure fatigue. Regarding that the second stage is prevailing (creep) in loading cycle, we can write the difference of openings [$\delta_{\max}(x) - \delta_{\min}(x)$], using the results of works Andreykiv and Lishinska (1999) and Szata and Terletska (1999), in form

$$\delta_{\max}(x) - \delta_{\min}(x) = \frac{1}{2}\left(\delta_{\max}^{(\mathrm{f})}(x) + \dot{\delta}_{\mathrm{t}\max}(x)t \right)\left(1 - R^2\right)^2 \tag{22}$$

where R – coefficient of cycle asymmetry. We can assume that $R = 0$, since

$$\frac{\partial W_{\mathrm{p}}^{(3)}}{\partial N} = \frac{\varepsilon_{\mathrm{fc}}\alpha}{2\delta_{\mathrm{fc}}} \sigma_{\mathrm{of}} \int_L \int_0^{l_{\mathrm{p}}} [\delta_{\max}^{(\mathrm{f})}(x,\xi) + \dot{\delta}_{\mathrm{t}\max}(x,\xi)t]\mathrm{d}x\mathrm{d}\xi . \tag{23}$$

In prefracture zone the value $\delta_{\max}^{(\mathrm{f})}(x) + \dot{\delta}_{\mathrm{t}\max}(x)\cdot t$ we will present as

$$\delta_{\max}^{(\mathrm{f})}(x) + \dot{\delta}_{\mathrm{t}\max}(x)t = \left[\delta_{\max}^{(\mathrm{f})}(0,\xi) + \dot{\delta}_{\mathrm{t}\max}(0,\xi)t \right] \left(1 - \frac{x}{l_{\mathrm{p}}} \right)^2 . \tag{24}$$

Let us regard that the openings in the crack tip are constant along the contour of the crack. Therefore

$$\frac{\partial W_{\mathrm{p}}^{(3)}}{\partial N} = \frac{\varepsilon_{\mathrm{fc}}\alpha}{2\delta_{\mathrm{fc}}} L\sigma_{\mathrm{of}} \int_0^{l_{\mathrm{p}}} \left[\delta_{\max}^{(\mathrm{f})}(0) + \dot{\delta}_{\max}(0)t \right]\left(1 - \frac{x}{l_{\mathrm{p}}} \right)^2 \mathrm{d}x = \frac{\varepsilon_{\mathrm{fc}}\alpha}{6\delta_{\mathrm{fc}}} \sigma_{\mathrm{of}} (\delta_{\max}^{(\mathrm{f})}(0) + \dot{\delta}_{\max}(0)t) l_{\mathrm{p}} L \tag{25}$$

The following equations for calculation $\partial W_{\mathrm{p}}^{(3)} / \partial N$ can be used (Andreykiv and Lishinska, 1999; Shata and Terletska, 1999):

$$l_{\mathrm{p}} = \delta_{\max}(0)E\sigma_{\mathrm{of}}^{-1}, \quad \dot{\delta}_{\mathrm{t}\max} = A_1 \left[\delta_{\max}(0)\delta_{\mathrm{fC}}^{-1} \right]^m,$$

$$\delta_{\mathrm{fC}} = K_{\mathrm{fC}}^2 \sigma_{\mathrm{of}}^{-1} E^{-1} . \tag{26}$$

Putting (26) in (24) the $\partial W_{\mathrm{p}}^{(3)} / \partial N$ is following

$$\frac{\partial W_{\mathrm{p}}^{(3)}}{\partial N} = \frac{\varepsilon_{\mathrm{fc}}\sigma_{\mathrm{of}}\alpha}{6}\left[\delta_{\max}^{(\mathrm{f})}(0) + \dot{\delta}_{\max}(0)t \right]^2 E^2 K_{\mathrm{fC}}^{-2} L \tag{27}$$

Availing the correlations (13), (26) and

$$\delta_{\max} / \delta_{\mathrm{fC}} = K_{\mathrm{I}\max}^2 \Big/ K_{\mathrm{fC}}^2 , \tag{28}$$

equation (9) becomes

$$\frac{\mathrm{d}S^{(\mathrm{f})}}{\mathrm{d}N} = \left[K_{\mathrm{I}\max}^2 \sigma_{\mathrm{of}}^{-1} E^{-1} + A_1 T \left(K_{\mathrm{I}\max}^2 K_{\mathrm{fC}}^{-2} \right)^m \right]^2 \alpha E^2 L 4.5^{-1} (K_{\mathrm{fC}}^2 - K_{\mathrm{I}\max}^2)^{-1} \tag{29}$$

Uniting additions $\mathrm{d}S^{(\mathrm{f})} / \mathrm{d}N$ and $\mathrm{d}S^{(\mathrm{c})} / \mathrm{d}N$, according to formulas (20) and (28), the final form of kinetic equation (8) for estimating the creep-fatigue rate of crack growth can be finally written as

$$\frac{\mathrm{d}S}{\mathrm{d}N} = \left[K_{\mathrm{I}\max}^2 \sigma_{\mathrm{of}}^{-1} E^{-1} + A_1 T \left(K_{\mathrm{I}\max}^2 K_{\mathrm{fC}}^{-2} \right)^m \right]^2 \alpha E^2 L 4.5^{-1} (K_{\mathrm{fC}}^2 - K_{\mathrm{I}\max}^2)^{-1} + 1.3LEA_1\sigma_{\mathrm{of}} T \left(K_{\mathrm{I}\max}^2 K_{\mathrm{fC}}^{-2} \right)^m \left(1 - K_{\mathrm{I}\max}^2 K_{\mathrm{fC}}^{-2} \right)^{-1} \tag{30}$$

With initial and final conditions

$$N = 0, \ \ S(0) = S_0, \ \ N = N_*, \ \ S(N_*) = S_*, \tag{31}$$

$$K_{\mathrm{I}\max}(S_*) = K_{\mathrm{fC}} . \tag{32}$$

For plane spreading of the crack of length l in plate the correlations (30)-(32) will become

$$\frac{\mathrm{d}l}{\mathrm{d}N} = \left[K_{\mathrm{I}\max}^2 \sigma_{\mathrm{of}}^{-1} E^{-1} + A_1 T \left(K_{\mathrm{I}\max}^2 K_{\mathrm{fC}}^{-2} \right)^m \right]^2 \alpha E^2 4.5^{-1} (K_{\mathrm{fC}}^2 - K_{\mathrm{I}\max}^2)^{-1} + 1.3EA_1\sigma_{\mathrm{of}} T \left(K_{\mathrm{I}\max}^2 K_{\mathrm{fC}}^{-2} \right)^m \left(1 - K_{\mathrm{I}\max}^2 K_{\mathrm{fC}}^{-2} \right)^{-1}, \tag{33}$$

$$N = 0, \ \ l(0) = l_0, \ \ N = N_*, \ \ l(N_*) = l_*, \tag{34}$$

$$K_{\mathrm{I}\max}(l_*) = K_{\mathrm{fC}} . \tag{35}$$

Correlations (30)–(35) compose the mathematical model for determination the period N_* of precritical growth of creep-fatigue crack in solid.

3. APPROBATION OF THE MODEL

To confirm the efficiency and correctness of correlations (33)–(35) we will test the model comparing with experimental data (Garofalo, 1970) for stainless steel 321. For this kind of steel the mechanical characteristics are the following $E = 1.9 \cdot 10^5$ MPa; $\sigma_t = 450$ MPa, $K_{fc} = 90$ MPa m$^{0.5}$ for temperature 650 °C.

Based on experimental data for pure fatigue (Fig. 2) we will find coefficient $\alpha = 0.01$. Then, using the data from work Garofalo (1970) we will have the constants A_1 and m: $A_1 = 6\ 10^{-5}$, $m = 1.43$. Using the results of work Garofalo (1970) we can show that $dl^{(c)}/dN$ is significantly less then $dl^{(f)}/dN$, so we can neglect the constituent $dl^{(c)}/dN$, and the final kinetic equation (33) will become:

$$dl/dN = 10^4 \left[\frac{K_{\mathrm{I}\max}^2}{\sigma_{\mathrm{of}} E} + \right. \left. + 0.00072 \left(\frac{K_{\mathrm{I}\max}^2}{8100} \right)^{1.43} \right]^2 \bigg/ \left[1 - \frac{K_{\mathrm{I}\max}^2}{8100} \right]. \quad (36)$$

Equation (36) was compared with results of experimental data (Garofalo, 1970):

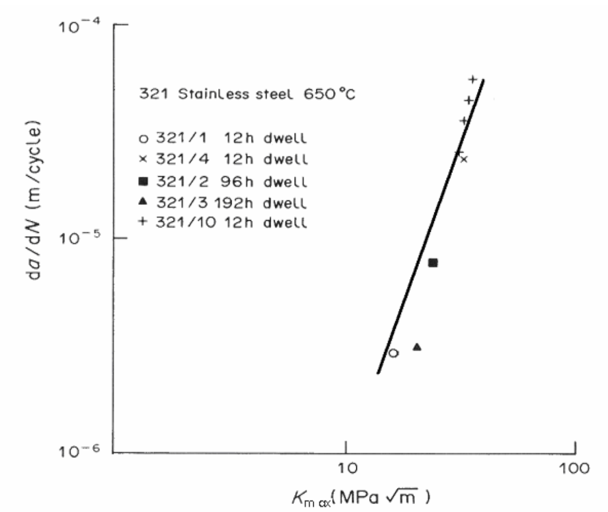

Fig. 2. Graphical comparison between theoretical results V~$K_{\mathrm{I}\max}$ (solid line) and experimental data for stainless steel 321 (Garofalo, 1970)

This comparing confirms the correctness of proposed mathematical model (33)-(35).

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MATEMATYCZNY MODEL OCENY CZASU WZROSTU PĘKNIĘCIA W WYNIKU PEŁZANIA – ZMĘCZENIA DLA MATERIAŁÓW KONSTRUKCYJNYCH PRZY WYSOKIEJ TEMPERATURZE

Streszczenie: W tej pracy zaproponowano matematyczny model obliczeniowy dla istotnego zagadnienia naukowo – inżynierskiego. tj. oceny czasu życia elementów konstrukcyjnych poddanych wysokotemperaturowemu zmęczeniu. Uzyskano różniczkowe równanie dla pewnych początkowych i końcowych warunków do oceny trwałości elementów trójwymiarowych. Powyższy model matematyczny sformułowano na podstawie podejścia energetycznego. Zaproponowane podejście umożliwia połączenie obciążeń zmęczeniowych i pełzania w jednym równaniu. Znane z literatury wyniki doświadczeń potwierdziły słuszność proponowanego modelu.