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THEORETICAL CONCEPTS OF THE FOURIER BOUNDARY ELEMENT METHOD

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Abstract. *The traditional Boundary Element Method (BEM) [4] is a collection of numerical techniques for solving some partial differential equations. The classical BEM produces fully populated coefficients matrix. With Galerkin Boundary Element Method (GBEM) is possible to produce the symmetric coefficients matrix [5]. The Fourier BEM is a more general numerical approach and allows to avoid problems with singular integrals [1,3]. The article presents the main aspects of Fourier BEM equations and the comparison of GBEM and Fourier BEM formulation.*

Keywords: Boundary Element Method, Galerkin approach, Fourier approach, numerical integration

Teoretyczne podstawy metody elementów brzegowych Fouriera

Streszczenie. *Tradycyjna metoda elementów brzegowych(MEB) [4] prowadzi w efekcie do rozwiązania układu równań liniowych z pełną macierzą współczynników. Stosując podejście Galerkina ostateczny układ równań liniowych jest reprezentowany macierzą symetryczną [5]. W podejściu Fouriera, współczynniki układu równań wyznaczane są w przestrzeni Fouriera co pozwala uniknąć problemów z całkowaniem całek nieosobliwych [1,3]. W artykule zaprezentowano podstawowe założenia MEB Fouriera oraz porównanie z MEB Galerkina.*

Słowa kluczowe: metoda elementów brzegowych Galerkina i Fouriera, całkowanie numeryczne

Introduction

Basic integral equation (BIE) for the Boundary Element Method (BEM) is constructed by the convolution with the fundamental solution [1,3]. Figure 1 presents domain $\Omega \subset R^n$ with Dirichlet and Neuman boundary.

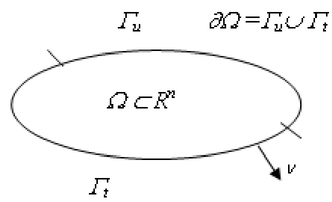

Fig.1. The domain Ω
Rys.1.Dziedzina Ω

The basic principles of traditional BEM are presented for the paradigmatic example of the *n*-dimensional stationary heat conduction described by:

$$\Delta u(x) = -f(x), x \in \Omega \subset R^n, \Delta = \sum_{k=1}^{n} \partial^2 / \partial x_k^2$$
$$u(x) = u_\Gamma(x), \quad x \in \Gamma_u \subset \Omega, \tag{1}$$
$$t(x) = t_\Gamma(x), \quad x \in \Gamma_t \subset \Omega.$$

where:

- Δ - Laplace operator,
- u – the unknown quantity,
- f - the known volume sources in Ω.

The flux on the boundary is:

$$t = A_t u = -\partial \nu\, u = -\nu \cdot \nabla u,$$

where:

- ∇, ν - the gradient and the outer unit normal,
- $A_t = -\nu \cdot \nabla$ - the boundary operator,
- $\partial / \partial x_k$ - the partial derivatives denotes ∂_k,
- $\mathbf{x}$ – *n*-dimensional vector,
- $d\mathbf{x}$ – the short form for dx_1dx_2 (or $dx_1dx_2dx_3$).

To obtain a well posed problem, half of the boundary data (either u on Γ_u or t on Γ_t) should be defined by boundary conditions, i.e. $\Gamma_u \cup \Gamma_t = \partial\Omega$.

1. The Galerkin Boundary Element Method

Most of the numerical methods are based on a weak form of the differential equation [4]. Basic integral equation for the BEM is constructed by the convolution with the fundamental solution *U(x)*.The fundamental solutions inherit their singular character from the Dirac distribution. Unfortunately analytic formulas for the fundamental solution can only be found for simple differential operators. Nevertheless, as long as the coefficients of the differential operator are constant, the existence of the fundamental solution can always be assured [1].

The known and unknown boundary quantities u, t are approximated by a sum of polynomial trial functions ϕ_u^i, ϕ_t^i with the coefficients u^i, t^i :

$$u(x) \approx \sum_i^{N_u} u^i \phi_u^i(x), \qquad t(x) \approx \sum_i^{N_t} t^i \phi_u^i(x). \tag{2}$$

For convergence reasons, the trial functions for the u should be at least linear, for the t it is sufficient to take constant trial functions.

Galerkin BIE lead to the algebraic system of BIEs [1, 5]:

$$\sum_i K_u^{ji} u^i = F_u^j + \sum_i H_u^{ji} t^i - \sum_i G_u^{ji} u^i, \tag{3}$$
$$\sum_i K_t^{ji} t^i = F_t^t + \sum_i H_t^{ji} t^i - \sum_i G_t^{ji} u^i, \tag{4}$$

where the vectors and matrices are defined as follows:

$$F_u^j := \int_{\Gamma_x} \phi_t^j(x) \int_\Omega f(y) U(x-y) d\Omega_y d\Gamma_x,$$
$$H_u^{ji} := \int_{\Gamma_x} \phi_t^j(x) \int_{\Gamma_y} \phi_t^i(y) U(x-y) d\Gamma_y d\Gamma_x,$$
$$G_u^{ji} := \int_{\Gamma_x} \phi_t^j(x) \int_{\Gamma_y} \phi_u^i(y) A_t^i U(x-y) d\Gamma_y d\Gamma_x,$$
$$K_u^{ji} := \int_{\Gamma_x} \phi_t^j(x) \kappa(x) \phi_u^i(x) d\Gamma_x$$
$$F_t^j := \int_{\Gamma_x} \phi_u^j(x) \int_\Omega f(y) A_t^j U(x-y) d\Omega_y d\Gamma_x,$$
$$H_t^{ji} := \int_{\Gamma_x} \phi_u^j(x) \int_{\Gamma_y} \phi_t^i(y) A_t^j U(x-y) d\Gamma_y d\Gamma_x,$$
$$G_t^{ji} := \int_{\Gamma_x} \phi_u^j(x) \int_{\Gamma_y} \phi_u^i(y) A_t^j A_t^i U(x-y) d\Gamma_y d\Gamma_x,$$
$$K_t^{ji} := \int_{\Gamma_x} \phi_u^j(x) (\kappa \phi_t^i + \phi_u^i A_t^j \kappa) d\Gamma_x,$$

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and κ is defined in [1].

The Galerkin BIE lead to the final matrix system:

$\sum_i A^{ji} X^i = Y^j$, where the matrix $\boldsymbol{A}$ is fully populated and symmetric [1, 5].

The traditional Galerkin BIE are reformulated to Fourier BIE by means of the convolution and Parserval theorem [1, 6].

2. The distribution theory

Distributions are objects which generalize functions. They extend the concept of derivative to all locally integral functions and are used to formulate generalized solutions of partial differential equations. They are important in physics and engineering where many non-continuous problems naturally lead to differential equations whose solutions or initial conditions are distributions, such as the Dirac delta distribution [6].

The basic idea is to identify functions with abstract linear functionals on a space of well-behaved ***test functions***. Operators on distributions can be understood by moving them to the test function.

For example, let:

$u: R \rightarrow R,$

be a locally integrable, and let :

$\phi: R \rightarrow R$,

be a smooth (infinitely differentiable) function with compact support (i.e., identically zero outside of some bounded set). The function ϕ is the ***test function*** and:

$$u(\phi) = \langle u, \phi \rangle = \int_R u\phi dx < \infty \; . \tag{5}$$

This is a real number which depends on ϕ. The function u is then a continuous linear functional on the space which consists of all the test functions ϕ. The set of generalized functions u include all linear and continuous functionals. They are defined by some test functions ϕ. Properties of the test functions define the set of ***generalized functions***.

The test functions:

$\phi(x) \in D(\Omega) = C_0^\infty(\Omega)$,

are bounded, posses a compact support and are infinitely continuously differentiable. They and all their derivatives vanish at the boundary.

Distribution:

$u \in D'(\Omega)$

is defined by the scalar product with the test function ϕ :

$$u(\phi) = \langle u, \phi \rangle = \int_{R^n} u(x)\overline{\phi(x)} dx < \infty \; . \tag{6}$$

The differentiation of generalized functions is defined as:

$$\langle \partial_k u, \phi \rangle = -\langle u, \partial_k \phi \rangle, \phi \in C_0^\infty \; . \tag{7}$$

Because of the definition of the test function ϕ, distributions are infinitely differentiable. Jumps and singularities can be differentiated [1].

The differentiation of the product of two distribution u_1, u_2 is defined as:

$$\partial_k (u_1 u_2) = u_2 \partial_k u_1 + u_1 \partial_k u_2 \; . \tag{8}$$

The convolution of distributions:

$$u_1 * u_2 = \int_{R^n} u_1(x-y) u_2(y) dy, x, y \in R^n \; , \tag{9}$$

is defined if only u_1 or u_2 has compact support.

For special distributions u_1 and u_2, u_3 is [1]:

commutation: $u_1 * u_2 = u_2 * u_1$,

association: $(u_1 * u_2) * u_3 = u_1 * (u_2 * u_3)$.

The differentiation of a convolution product is:

$$\partial^\alpha (u_1 * u_2) = (\partial^\alpha u_1) * u_2 = u_1 * (\partial^\alpha u_2) \; . \tag{10}$$

By using a larger space of test functions, it is possible to define the ***tempered distributions***, useful for the Fourier transform in generality. All tempered distributions have a Fourier transform, but not all distributions have one [1].

For tempered distributions $u \in S'$, the Fourier transform $\hat{u}$ is defined by the transform of the test function ϕ:

$$\hat{u}(\varphi) = u(\hat{\phi}), \phi \in S \; . \tag{11}$$

The invariance of the scalar product concerning the Fourier transform is called Parseval's identity:

$$\langle u_1, u_2 \rangle = \frac{1}{(2\pi)^n} \langle \hat{u}_1, \hat{u}_2 \rangle \; , \tag{12}$$

$$\int_{R^n} u_1(x)\overline{u_2}(x) dy = \frac{1}{(2\pi)^n} \int_{R^n} \hat{u}_1(\hat{x})\overline{\hat{u}_2}(\hat{x}) d\hat{x} \; .$$

It is possible to define the Fourier transform of tempered distributions. These include all the integrable functions, as well as well-behaved functions of polynomial growth and distributions of compact support, and have the added advantage that the Fourier transform of any tempered distribution is again a tempered distribution.

If the support of distribution u is contained in the convex and compact domain $\Omega \subseteq R^n$ with a cutoff distribution χ (eq. 27), then the transform of u has no local singularities. Consequently every distribution with compact support has a Fourier transform which is an entire analytic function in C^n [1].

The special distributions

The n-dimensional Dirac distribution:

$$\delta(x) = \prod_{k=1}^n \delta(x_k) \tag{13}$$

is defined by:

$\int_{R^n} \delta(x) dx = 1, x \in R^n$;

$\delta(x) = 0$ for all $|x| \neq 0$.

$\delta(x)$ is concentrated on a single point $x=0$ with infinite density:

$\delta(0) \rightarrow \infty$

but finite measure:

$\int \delta \, dx = 1$.

The Dirac distribution is the identity object concerning convolution:

$$u = u * \delta = \int_{R^n} u(y)\delta(x-y) dy, x, y \in R^n \; , \tag{14}$$

and its Fourier transform is:

$$\delta(x) \stackrel{F}{\longleftrightarrow} 1, x \in R^n \; . \tag{15}$$

The Fourier transform of the translated Dirac-distribution is defined for complex $y \in C^n$:

$$\delta(x-y) \stackrel{F}{\longleftrightarrow} e^{-i<y,\hat{x}>}, x, y \in C^n \tag{16}$$

All differentiations can be expressed by a convolution with $\delta^\alpha = \partial^\alpha \delta$:

$$u * (\partial^\alpha \delta) = \int_{R^n} u(y) \partial^\alpha \delta(x-y) dy = \partial^\alpha u(x), x, y \in R^n \; . \tag{17}$$

The product with another distribution u can be simplified by the relation:

$$u(x) \partial_x^\alpha \delta(x-y) = \sum_{l=0}^{k} \binom{k}{k-l} (-1)^l \partial^l u(y) \partial_x^{k-l} \delta(x-y) \; . \tag{18}$$

If the boundary $\partial\Omega$ is described by $\psi(x)=0$ with the function $\psi\in C^{\infty}$ then the integration of a distribution u along this boundary can be described by the scalar product:

$$\langle\delta(\psi),u\rangle=\int_{\psi=0}u(x)dx\,. \tag{19}$$

The gradient $\nabla\psi$ of the hypersurface $\psi=0$ is $\nabla\psi\neq 0$ for $\psi=0$ and the outer unit normal of this hypersurface is:

$$\nu=-\frac{\nabla\psi}{|\nabla\psi|}\,, \tag{20}$$

and:

$$\nabla\delta(\psi)=\delta'(\psi)\nabla\psi\,. \tag{21}$$

Distributions can be represented by series useful for practical computations. For the Dirac distribution, there are many ways to construct a sequence of functions or regular distributions $\phi_k(x), k=1,2,...$, which converges to Dirac distribution.

One dimensional examples for these sequences are [1]:

a) $\phi_k(x)=\frac{1}{\varepsilon_k}\begin{cases}e^{\varepsilon^2/(x^2-\varepsilon^2)}, & |x|<\varepsilon\\ 0 & ,|x|>\varepsilon\end{cases}$

b) $\phi_k(x)=\frac{k}{\pi}\frac{1}{1+k^2x^2}$

c) $\phi_k(x)=\frac{k}{\sqrt{\pi}}e^{-k^2x^2}$

d) $\phi_k(x)=\frac{1}{k\pi}\frac{\sin^2 kx}{x^2}$.

These sequences are useful for the multiplication of distributions $H(x)\delta(x)=\delta(x)/2$ and the evaluation of the free terms in Boundary Integral Equation (BIE).

The Heaviside distribution is obtained by the integration of the Dirac distribution:

$$H(x)=\int_{-\infty}^{x}\delta(y)dy=\begin{cases}1 & x>0\\ 0 & x<0\end{cases}. \tag{22}$$

In the literature, there are several definitions for the value at $x=0$. For the linear distribution it is determined by:

$$\langle H(x),\delta(x)\rangle=\kappa=\frac{1}{2}, x\in R^1\,. \tag{23}$$

For the multidimensional Heaviside distribution, the cutoff distribution for a domain $\Omega\in R^n$ is defined:

$$\chi(x):=\begin{cases}1 & x\in\Omega\\ \kappa(x) & x\in\partial\Omega\\ 0 & x\notin\overline{\Omega}=\Omega\cup\partial\Omega\end{cases}, \tag{24}$$

which can be expressed by:

$$\chi(x)=H(\psi(x)), \tag{25}$$

with a function $\psi\in C^{\infty}(R^n)$.

The integration of a distribution u over the domain Ω can be described by:

$$\langle H(\psi),u\rangle=\int_{\psi\geq 0}u(x)dx\,. \tag{26}$$

The value $\kappa(x)$ on the boundary $\partial\Omega$ is uniquely defined [1]:

$\kappa(x)=\frac{1}{2}$ for smooth part of a boundary,

$\kappa(x)=\frac{1}{4}$ for a rectangular corner,

$\kappa(x)=\frac{\theta(x)}{2\pi}$ for a arbitrary angles θ.

The gradient of the cutoff distribution leads to a definition of the normal vector of the boundary even for non-smooth $\partial\Omega$.

The main advantage of the theory of distribution is that it re-establishes differentiation as the simple procedure and all quantities are differentiable even if they exhibit singularities and jumps [1].

3. Fourier BEM

To obtain the Fourier transform of the BIE, all quantities have to be extended from Ω to R_n. It can be done by defining **a cutoff distribution χ.** All quantities are multiplied by χ and finally transformed into Fourier space. Mathematically this extension and transformation is justified only in the frame of the theory of distributions [1, 6].

The unknown quantity u extends only over Ω and may jumps across the boundary $\partial\Omega$. In the theory of distribution this is described by a multiplication of $u\in C^{\infty}(R^n)$ with ***the cutoff distribution*** χ:

$$u(x)\to\chi(x)u(x),\chi(x):=\begin{cases}1 & x\in\Omega\\ \kappa(x) & x\in\partial\Omega\\ 0 & x\notin\overline{\Omega}=\Omega\cup\partial\Omega\end{cases}. \tag{27}$$

χ can be expresses for smooth boundaries by a generalized multi-dimensional ***Heaviside*** distribution:

$$\chi(x)=H(\psi(x))\,, \tag{28}$$

where:

$$H(x)=\int_{-\infty}^{x}\delta(y)dy=\begin{cases}1 & x>0\\ 0 & x<0\end{cases},$$

and $\psi\in C^{\infty}$ describes as the hypersurface of the boundary $\partial\Omega$.

The main advantage of the distributional BIEs is that the integrals extend formally over the entire R_n and therefore the Fourier transformation can be applied to these integral equation.

Distributional trial and test functions

For the definition of the trial functions it is needed to define a cutoff distribution [1]:

- for a rectangle element:
  $\chi^0(x):=H(x_1)H(1-x_1)\delta(x_2), x\in R^2$,
  $\chi^0(x):=H(x_1)H(1-x_1)H(x_2)H(1-x_2)\delta(x_3), x\in R^3$,
- for a triangular element:
  $\chi^0(x):=H(x_1)H(x_2)H(1-x_1-x_2)\delta(x_3), \quad x\in R^3$.

The trial functions are obtained by multiplying $\chi^0(x)$ and $p^0(x)\in C^{\infty}(R^n)$:

$\phi^0(x):=\chi^0(x)p^0(x)$.

The trial functions $\phi^i_{u,t}(x)$ on arbitrary straight elements are obtained by translation and/or dilation operators:

$$\begin{aligned}T^i&:\phi^0\to\phi^i_{u,t}=\phi^0(x-b^i),\\ D^i&:\phi^0\to\phi^i_{u,t}=\phi^0(a^ix),\end{aligned} \tag{29}$$

with the translation vector $\mathbf{b}_i$ and the dilation matrix $\mathbf{a}_i$.

Finally the unknown and the known quantities on the boundaries are approximated by:

$$\delta(x)u(x) \approx \sum_i^{N_u} u^i \phi_u^i(x),\ \delta(\psi)\nabla\psi(x)\cdot\nabla u(x) \approx \sum_i^{N_t} t^i \phi_t^i(x)$$

Distributional Galerkin BIE

The distributional Galerkin BIE for the temperature is [1]:

$$\left\langle \phi_t^j, u_\chi \right\rangle = \left\langle \phi_t^j, f_\chi * U \right\rangle + \sum_i^{N_t} t^i \left\langle \phi_t^j, \phi_t^i * U \right\rangle - \sum_i^{N_u} u^i \left\langle \phi_t^j, \phi_u^i * A_t^i U \right\rangle \quad (30)$$

and for the flux:

$$-\left\langle \phi_u^j, A_t^j u_\chi \right\rangle = -\left\langle \phi_u^j, f_\chi * A_t^j U \right\rangle - \sum_i^{N_t} t^i \left\langle \phi_u^j, \phi_t^i * A_t^j U \right\rangle + + \sum_i^{N_u} u^i \left\langle \phi_u^j, \phi_u^i * A_t^j A_t^i U \right\rangle \quad (31)$$

To simplify the notation the following notes were defined:

scalar product: $\langle a,b \rangle = \int_{R^n} a(x)b(x)dx$,

convolution: $a * b = \int_{R^n} a(y)b(x-y)dy$,

$u_\chi = u(x)\chi(x)$,

$f_\chi = f(x)\chi(x)$.

Fourier BEM

The *n*-dimension Fourier transform:

$$F(u) = \hat{u}, u \in L_1(R^n), i = \sqrt{-1}$$

is defined as:

$$\hat{u}(\hat{x}) = \int_{R^n} u(x) e^{-i<x,\hat{x}>} dx, <x,\hat{x}> = \sum_{k=1}^{n} x_k \hat{x}_k \ . \quad (32)$$

The basic of Fourier BEM are two known theorems of the Fourier transformation.

The theorem of Parseval states the invariance of energy or work with respect to the dimensional Fourier transformation:

$$\int_{R^n} \phi(x)u(x)dx = \frac{1}{(2\pi)^n} \int_{R^n} \hat{\phi}(-\hat{x})\hat{u}(\hat{x})d\hat{x}, \quad x,\hat{x} \in R^n \quad (33)$$

The convolution theorem links the convolution in the original space to a simple multiplication in the transformed space:

$$\int_{R^n} \phi(y)u(x-y)dy \quad \overset{F}{\longleftrightarrow} \quad \hat{\phi}(\hat{x})\hat{u}(\hat{x}) \quad (34)$$

In the notation introduced earlier these two theorems may be described as:

$$\left\langle \phi(x), u(x) \right\rangle = \frac{1}{(2\pi)^n} \left\langle \hat{\phi}(-\hat{x}), \hat{u}(\hat{x}) \right\rangle , \quad (35)$$

$$\phi(x) * u(x) \quad \overset{F}{\longleftrightarrow} \quad \hat{\phi}(\hat{x})\hat{u}(\hat{x}) \ . \quad (36)$$

The inner integrals in BIE are convolutions so:

$$f_\chi * U \quad \overset{F}{\longleftrightarrow} \quad \hat{f}_\chi(\hat{x})\ \hat{U}(\hat{x}) \quad (37)$$

$$\phi_t^i * U \quad \overset{F}{\longleftrightarrow} \quad \hat{\phi}_t^i(\hat{x})\ \hat{U}(\hat{x}) \quad (38)$$

$$\phi_t^i * A_t^i U \quad \overset{F}{\longleftrightarrow} \quad \hat{\phi}_t^i(\hat{x})\ \hat{A}_t^i \hat{U}(\hat{x}) \quad (39)$$

The Fourier transformation of the differential equation converts the differential operator $P(\partial)$ to an algebraic expression $\hat{P}(\hat{x})$:

$$\Delta u(x) = -f(x) \ \overset{F}{\longleftrightarrow} \ -|\hat{x}|^2 \hat{u}(\hat{x}) = -\hat{f}(\hat{x}) \ , \quad (40)$$

where:

$$\hat{P} = -|\hat{x}|^2 = -\sum_k^n \hat{x}_k^2 \ .$$

The Fourier fundamental solution $\hat{U}$ is the response to a single unit force:

$$f(x) = \delta(x) \overset{F}{\longleftrightarrow} \hat{f}(\hat{x}) = 1 \ . \quad (41)$$

It has to be solved:

$$\Delta U(x) = -\delta(x) \ \overset{F}{\longleftrightarrow} \ -|\hat{x}|^2 \hat{U}(\hat{x}) = -1 \ . \quad (42)$$

which is achieved in the transformed space by the inversion of $\hat{P}$:

$$\hat{U}(\hat{x}) = \frac{1}{|\hat{x}|^2} \ . \quad (43)$$

This procedure can be applied to all linear differential operators with constants coefficients. The Fourier fundamental solution is always known and often has simple structure [1].

$\hat{U}$ is singular in the point $|\hat{x}| = 0$ and hence not uniquely determined in the original space.

The zeroes of the denominator correspond to additional homogenous solutions of the differential equation:

$$\Delta u(x) = 0 \ \overset{F}{\longleftrightarrow} \ -|\hat{x}|^2 \hat{u}(\hat{x}) = 0 \ . \quad (44)$$

For straight elements, the normal vector ν^i is locally independent of x. Hence, the transform of the fundamental flux and of the hyper singular term is:

$$A_t^i U = -\nu^i \cdot \nabla U \overset{F}{\longleftrightarrow} \hat{A}_t^i \hat{U} = -\hat{\nu}^i \cdot i\hat{x}\hat{U} \quad (45)$$

$$A_t^j A_t^i U = \nu^j \cdot \nabla(\nu^i \cdot \nabla U)$$

$$\overset{F}{\longleftrightarrow} \hat{A}_t^j \hat{A}_t^i \hat{U} = \hat{\nu}^j \cdot i\hat{x}(\hat{\nu}^i \cdot i\hat{x})\hat{U}$$

BEM is based on Green's functions i.e. on special fundamental solutions. The Fourier BEM method analysed by [1] is especially of interest for cases where the fundamental solution is not known.

The transform of the cutoff distribution χ_0 is:

- for reference element in R^2:

$$\chi^0(x) = H(x_1)H(1-x_1)\delta(x_2)$$
$$\overset{F}{\longleftrightarrow} \hat{\chi}^0(\hat{x}) = \frac{i}{\hat{x}_1}(e^{-i\hat{x}_1} - 1) \quad (46)$$

- for reference element in R^3:

$$\chi^0(x) = H(x_1)H(1-x_1)H(x_2)H(1-x_2)\delta(x_3)$$
$$\overset{F}{\longleftrightarrow} \hat{\chi}^0(\hat{x}) = \frac{i}{\hat{x}_1}(e^{-i\hat{x}_1} - 1)\frac{i}{\hat{x}_2}(e^{-i\hat{x}_2} - 1) \quad (47)$$

Elements of the arbitrary polynomial degree are constructed via multiplication by $p_0(x)$ in the original space or an analytical convolution in the transformed space:

$$\phi^0(x) = \chi^0(x)p^0(x)$$
$$\overset{F}{\longleftrightarrow} \hat{\phi}^0(\hat{x}) = \frac{1}{(2\pi)^n} \hat{\chi}^0(\hat{x}) * \hat{p}^0(\hat{x})$$

Table 1. shows the trial functions for some two-dimensional line elements [1].

Table 1. Trial functions in R^2
Tabela 1. funkcje testowe w R^2

Constant	$\phi^0 = \chi^0$	$\overset{F}{\longleftrightarrow}$	$\hat{\phi}^0 = \hat{\chi}^0$
Linear	$\phi^1 = (1-x_1)\chi^0$	$\overset{F}{\longleftrightarrow}$	$\hat{\phi}^1 = (1-i\partial_1)\hat{\chi}^0$
	$\phi^2 = x_1\chi^0$	$\overset{F}{\longleftrightarrow}$	$\hat{\phi}^2 = i\partial_1\hat{\chi}^0$
Parabolic	$\phi^1 = (2x_1^2 - 3x_1 + 1)\chi^0$	$\overset{F}{\longleftrightarrow}$	$\hat{\phi}^1 = (-2\partial_1^2 - 3i\partial_1 + 1)\hat{\chi}^0$
	$\phi^2 = (4x_1^2 - 4x_1)\chi^0$	$\overset{F}{\longleftrightarrow}$	$\hat{\phi}^2 = (-4\partial_1^2 - 4i\partial_1)\hat{\chi}^0$
	$\phi^3 = (2x_1^2 - x_1)\chi^0$	$\overset{F}{\longleftrightarrow}$	$\hat{\phi}^3 = (-2\partial_1^2 - i\partial_1)\hat{\chi}^0$

For straight elements and for arbitrary polynomial trial functions $p_0(x)$, the transformed expressions are analytically known in R^2 and R^3 [1].

The discretized Fourier BIE lead to an algebraic system identical to that obtained in the original space, where the matrices are computed in the transformed space (eq. 3,4) but now, the matrices are computed in the transformed space:

$$F_u^j = \frac{1}{(2\pi)^n} \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{f}(\hat{x})\hat{U}(\hat{x}) \right\rangle , \quad (48)$$

$$H_u^{ji} = \frac{1}{(2\pi)^n} \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{\phi}_t^i(\hat{x})\hat{U}(\hat{x}) \right\rangle ,$$

$$G_u^{ji} = \frac{1}{(2\pi)^n}\left\langle \hat{\phi}_t^j(-\hat{x}), \hat{\phi}_u^i(\hat{x})\hat{A}_t^i\hat{U}(\hat{x})\right\rangle,$$

$$K_u^{ji} := \frac{1}{(2\pi)^n}\left\langle \hat{\phi}_t^j(-\hat{x}), \hat{p}_u^i(\hat{x})\right\rangle,$$

$$F_t^j = \frac{1}{(2\pi)^n}\left\langle \hat{\phi}_u^j(-\hat{x}), \hat{f}(\hat{x})\hat{A}_t^j\hat{U}(\hat{x})\right\rangle,$$

$$H_t^{ji} = \frac{1}{(2\pi)^n}\left\langle \hat{\phi}_u^j(\hat{x}), \hat{\phi}_t^i(\hat{x})\hat{A}_t^j\hat{U}(\hat{x})\right\rangle,$$

$$G_t^{ji} = \frac{1}{(2\pi)^n}\left\langle \hat{\phi}_u^j(\hat{x}), \hat{\phi}_u^i(\hat{x})\hat{A}_t^j\hat{A}_t^i\hat{U}(\hat{x})\right\rangle,$$

$$K_t^{ji} := \frac{1}{(2\pi)^n}\left\langle \hat{\phi}_u^j(-\hat{x}), \hat{p}_t^i\kappa^i + \hat{A}_t^j\hat{p}_u^i\kappa^i\right\rangle.$$

4. Numerical example

The comparison of GBEM and Fourier BEM formulation is presented for the boundary integral equations limited to constant elements and 2D space. As the test example, the Dirichlet problem of the Poisson equation is considered [2].

The Dirichlet problem for Poisson equation:

$$\Delta u(x) = -f(x), x \in \Omega,$$
$$u(x) = u_\Gamma = 0, x \in \Gamma,$$

is solved in a quadratic two-dimensional domain $\Omega = [0,1]\times[0,1]$ at the boundaries $u=0$. The interior is subjected to stationary heat source f. The boundary $\partial\Omega$ is divided into eight elements (fig.2.).

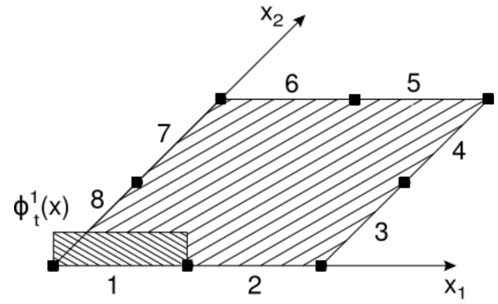

Fig.2. Quadratic domain Ω with eight boundary elements and constant trial function
Fig.2. Kwadratowe domeny Ω z ośmiu elementów brzegowych oraz stała funkcja próbna

The fundamental solution and its transform for the Laplacian Δ are [1]:

$$U = \frac{1}{2\pi}\ln\sqrt{x_1^2 + x_2^2} \overset{F}{\longleftrightarrow} \hat{U} = -\frac{1}{\hat{x}_1^2 + \hat{x}_2^2}$$

Taking into account that $u=0$ at the boundaries, the general system of BIE can be reduced to:

$$0 = \left\langle \phi_t^j, f_\chi * U\right\rangle + \sum_i^{N_t} t^i\left\langle \phi_t^j, \phi_t^i * U\right\rangle$$

$$\overset{F}{\longleftrightarrow} 0 = \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{f}_\chi\hat{U}\right\rangle + \sum_i^{N_t} t^i\left\langle \hat{\phi}_t^j(-\hat{x}), \hat{\phi}_t^i\hat{U}\right\rangle.$$

A uniform heat source f is assumed over Ω:

$$f_\chi(x) = f_0 H(x_1)H(1-x_1)H(x_2)H(1-x_2)$$

$$\overset{F}{\longleftrightarrow} \hat{f}_\chi(\hat{x}) = -f_0\frac{(e^{-i\hat{x}_1}-1)(e^{-i\hat{x}_2}-1)}{\hat{x}_1\hat{x}_2}.$$

For the R^2 elements the cuttoff distribution definition is(eq. 46):

$$\chi^0 = H(x_1)H(1-x_1)\delta(x_2) \quad , \quad x \in R^2,$$

and it's Fourier transform is described by:

$$\chi^0(x) = H(x_1)H(1-x_1)\delta(x_2) \overset{F}{\longleftrightarrow} \hat{\chi}^0(\hat{x}) = \frac{i}{\hat{x}_1}(e^{-i\hat{x}_1}-1)$$

In our example, for constant elements (table 1), the trial function and it's Fourier transform is:

$$\phi^0 = \chi^0 \overset{F}{\longleftrightarrow} \hat{\phi}^0 = \hat{\chi}^0.$$

The trial functions should be defined for eight constant elements (with different coordinates). For every element, the coefficients for Heaviside and Dirac distribution should be modified to receive the value of the product: $H(x_1)H(1-x_1)\delta(x_2)$ to be equal to one inside the element and zero outside.
From the definition, the Dirac distribution is equal to one only for $x=0$, and Heaviside distribution is equal to one for $x>0$.
Additionally, the Fourier transform for dilation and translation operators is described as (eq. 29):

$$\phi^0(x-b) \overset{F}{\longleftrightarrow} \hat{\phi}^0(\hat{x})e^{-ib\hat{x}}, \quad \phi^0(ax) \overset{F}{\longleftrightarrow} \frac{1}{a}\hat{\phi}^0(a\hat{x}).$$

Taking all together, we have the constant trial and test function for the flux t [1]:

$$\phi_t^1 = H(x_1)H(1-2x_1)\delta(x_2) \overset{F}{\longleftrightarrow}$$
$$\hat{\phi}_t^1 = \frac{i}{\hat{x}_1/2}\left(e^{-i\hat{x}_1/2}-1\right)\frac{1}{2} = i\frac{(e^{-i\hat{x}_1/2}-1)}{\hat{x}_1},$$

$$\phi_t^2 = H(2x_1-1)H(1-x_1)\delta(x_2) \overset{F}{\longleftrightarrow}$$
$$\hat{\phi}_t^2 = \frac{1}{2}\frac{i}{\hat{x}_1/2}e^{-i\hat{x}_1/2}\left(e^{-i\hat{x}_1/2}-1\right) = i\frac{(e^{-i\hat{x}_1}-e^{-i\hat{x}_1/2})}{\hat{x}_1}.$$

$$\phi_t^3 = H(x_2)H(1-2x_2)\delta(x_1-1) \overset{F}{\longleftrightarrow}$$
$$\hat{\phi}_t^3 = \frac{i}{\hat{x}_2/2}\left(e^{-i\hat{x}_2/2}-1\right)\frac{1}{2}e^{-i\hat{x}_1} = i\frac{(e^{-i\hat{x}_2/2}-1)e^{-i\hat{x}_1}}{\hat{x}_2}$$

$$\phi_t^4 = H(2x_2-1)H(1-x_2)\delta(x_1-1) \overset{F}{\longleftrightarrow}$$
$$\hat{\phi}_t^4 = \frac{1}{2}\frac{i}{\hat{x}_2/2}e^{-i\hat{x}_2/2}\left(e^{-i\hat{x}_2/2}-1\right)e^{-i\hat{x}_1}$$
$$= i\frac{(e^{-i\hat{x}_2}-e^{-i\hat{x}_2/2})e^{-i\hat{x}_1}}{\hat{x}_2}$$

$$\phi_t^5 = H(x_1)H(1-2x_1)\delta(x_2-1) \overset{F}{\longleftrightarrow}$$
$$\hat{\phi}_t^5 = \frac{i}{\hat{x}_1/2}\left(e^{-i\hat{x}_1/2}-1\right)\frac{1}{2}e^{-i\hat{x}_2} = i\frac{(e^{-i\hat{x}_1/2}-1)e^{-i\hat{x}_2}}{\hat{x}_1}$$

$$\phi_t^6 = H(2x_1-1)H(1-x_1)\delta(x_2-1) \overset{F}{\longleftrightarrow}$$
$$\hat{\phi}_t^6 = \frac{1}{2}\frac{i}{\hat{x}_1/2}e^{-i\hat{x}_1/2}\left(e^{-i\hat{x}_1/2}-1\right)e^{-i\hat{x}_2}$$
$$= i\frac{(e^{-i\hat{x}_1}-e^{-i\hat{x}_1/2})e^{-i\hat{x}_2}}{\hat{x}_1}$$

$$\phi_t^7 = H(x_2)H(1-2x_2)\delta(x_1) \overset{F}{\longleftrightarrow}$$
$$\hat{\phi}_t^7 = \frac{i}{\hat{x}_2/2}\left(e^{-i\hat{x}_2/2}-1\right)\frac{1}{2} = i\frac{(e^{-i\hat{x}_2/2}-1)}{\hat{x}_2}$$

$$\phi_t^8 = H(2x_2-1)H(1-x_2)\delta(x_1) \overset{F}{\longleftrightarrow}$$
$$\hat{\phi}_t^8 = \frac{1}{2}\frac{i}{\hat{x}_2/2}e^{-i\hat{x}_2/2}\left(e^{-i\hat{x}_2/2}-1\right) = i\frac{(e^{-i\hat{x}_2}-e^{-i\hat{x}_2/2})}{\hat{x}_2}$$

In the original space the system of equations to solve is [2]:

$$\sum_i H_u^{ji}t^i = -F_u^j,$$

where:

$$H_u^{ji} := \int_{\Gamma_x}\phi_t^j(x)\int_{\Gamma_y}\phi_t^i(y)U(x-y)d\Gamma_y d\Gamma_x,$$

$$F_u^j := \int_{\Gamma_x}\phi_t^j(x)\int_{\Omega} f(y)U(x-y)d\Omega_y d\Gamma_x.$$

The symbolic Matlab calculation of the matrix $\boldsymbol{H}$ coefficients are exactly described in [2] and for example:

$$H^{11} = \int_0^{1/2}\int_0^{1/2} U(x_1 - x_2)dx_2 dx_1 =$$

$$= \frac{1}{2\pi}\int_0^{1/2}\int_0^{1/2} \ln\sqrt{(x_1 - x_2)^2}\, dx_2 dx_1$$

The numerical calculation of singular integrals was widely discussed in [3], where the regularisation method was introduced. The analogical equations system is to solve in Fourier space but now coefficients are calculated according eq. 48:

$$H_u^{ji} = \frac{1}{(2\pi)^n}\left\langle \hat{\phi}_t^j(-\hat{x}), \hat{\phi}_t^i(\hat{x})\hat{U}(\hat{x})\right\rangle$$

$$F_u^j = \frac{1}{(2\pi)^n}\left\langle \hat{\phi}_t^j(-\hat{x}), \hat{f}(\hat{x})\hat{U}(\hat{x})\right\rangle$$

In our example we have:

$$\sum_i^8 t^i \left\langle \hat{\phi}_t^j(-\hat{x}), \hat{\phi}_t^i \hat{U}\right\rangle = -\left\langle \hat{\phi}_t^j(-\hat{x}), \hat{f}_\chi \hat{U}\right\rangle, j = 1,...,8$$

and:

$$H^{11} = \frac{1}{(2\pi)^2}\left\langle \hat{\phi}_t^1(-\hat{x}), \hat{\phi}_t^1(\hat{x})\hat{U}(\hat{x})\right\rangle$$

$$= \frac{1}{(2\pi)^2}\int_{R^2} \frac{\left[i(e^{i\hat{x}_1/2} - 1)\right]\cdot\left[i(e^{-i\hat{x}_1/2} - 1)\right]}{-\hat{x}_1\hat{x}_1\left(-\hat{x}_1^2 - \hat{x}_2^2\right)} dx_1 dx_2$$

To calculate H_{11} value in Fourier space the following identities are used:

$$\left[(e^{ix/2} - 1)\right]\cdot\left[(e^{-ix/2} - 1)\right] = \cdot\left(e^{ix/2}e^{-ix/2} - e^{ix/2} - e^{-ix/2} + 1\right) =$$

$$\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)\cdot\left(\cos\frac{x}{2} - i\sin\frac{x}{2}\right) - \left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right) -$$

$$\left(\cos\frac{x}{2} - i\sin\frac{x}{2}\right) + 1 = \left(\cos^2\frac{x}{2} - i^2\sin^2\frac{x}{2}\right) - \left(2\cos\frac{x}{2}\right)$$

$$+1 = 1 - \left(2\cos\frac{x}{2}\right) + 1 = 2 - \left(2\cos\frac{x}{2}\right) = -2(\cos\frac{x}{2} - 1),$$

$$\int_{R^1} \frac{-1}{(2\pi)^2}\frac{1}{\left(\hat{x}_1^2 + \hat{x}_2^2\right)} = \frac{-1}{(2\pi)}\frac{\mathrm{sgn}(\hat{x}_1)}{2\hat{x}_1}$$ (from [1]).

Finally:

$$H^{11} = \frac{1}{(2\pi)^2}\int_{R^2} \frac{\left[i(e^{i\hat{x}_1/2} - 1)\right]\cdot\left[i(e^{-i\hat{x}_1/2} - 1)\right]}{-\hat{x}_1\hat{x}_1\left(-\hat{x}_1^2 - \hat{x}_2^2\right)} d\hat{x}_1 d\hat{x}_2 =$$

$$\int_{R^1} \frac{\left[(e^{i\hat{x}_1/2} - 1)\right]\cdot\left[(e^{-i\hat{x}_1/2} - 1)\right]}{\hat{x}_1^{\;2}} \int_{R^1} \frac{-1}{(2\pi)^2}\frac{1}{\left(\hat{x}_1^2 + \hat{x}_2^2\right)} d\hat{x}_2 d\hat{x}_1 =$$

$$\frac{-1}{(2\pi)}\int_R -2(\cos\frac{\hat{x}_1}{2} - 1)\frac{\mathrm{sgn}(\hat{x}_1)}{2\hat{x}_1^3} d\hat{x}_1 =$$

$$\frac{1}{(2\pi)}\int_R \frac{\mathrm{sgn}(\hat{x}_1)}{\hat{x}_1^3}(\cos\frac{\hat{x}_1}{2} - 1)d\hat{x}_1 =$$

$$\frac{1}{(2\pi)}\left(\int_{-\infty}^0 \frac{-1}{\hat{x}_1^3}(\cos\frac{\hat{x}_1}{2} - 1)d\hat{x}_1 + \int_0^\infty \frac{1}{\hat{x}_1^3}(\cos\frac{\hat{x}_1}{2} - 1)d\hat{x}_1\right) =$$

$$\frac{1}{(2\pi)}\left(2\int_0^\infty \frac{1}{\hat{x}_1^3}(\cos\frac{\hat{x}_1}{2} - 1)d\hat{x}_1\right)$$

The numerical value of the last integral calculated in Matlab is equal: -0.0904.

The relative error between H_{11} calculated symbolically in original space and numerically in Fourier space is 3.65%.

5. Conclusion

To obtain the Fourier transform of the Galerkin BIE, all quantities have to be extended from original Ω space to Rn. Mathematically this extension and transformation is justified only in the frame of the theory of distributions.

Due to the equivalence of the terms in the original and the transformed space which is state by Parseval theorem, all the vectors and matrices have the same values as would be obtained by a traditional BEM approach. The further algorithm of the BEM can be taken without any modification.

The Fourier BEM method is more difficult than the standard BEM method but is specially of interest for cases where the fundamental solution is not known. This aspect of the Fourier BEM method requires further study.

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