DETERMINING OF THE UNCERTAINTY OF CALCULATIONS IN THE CASE OF MODELS NONLINEARLY DEPENDENT ON THEIR PARAMETERS

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Summary

The problem of determining the uncertainty of results of calculations carried out e.g. during expert's activities within the reconstruction of road accidents has been addressed in this paper.

The total differential formula that is often used to estimate the uncertainty of calculation results has the form of a Taylor series where the terms comprising higher-order partial derivatives have been omitted. Then, a question arises whether the omission of these terms, if they exist, may have a considerable impact on the accuracy of uncertainty calculation results. The effect of the omission of these terms has been analysed in this paper with taking as an example the estimation of uncertainty of the stopping distance of a braking motorcycle. It has been found that in the case of mathematical models where the final result nonlinearly depends on model parameters, the differences obtained may be significant.

Keywords: uncertainty, total differential, Taylor series.

1. Introduction

The issue of uncertainty of calculations in the analysis of road accidents is particularly important because even a small modification in the input parameters often adopted for calculations may sometimes result in a change in the allocation of responsibility to individual participants in a road accident.

The uncertainty of calculations is a quantity dependent on the inaccuracy of estimation of the parameter values necessary for the calculations.

Let us assume that we should estimate the uncertainty of calculation of a quantity Y that is a function of parameters $(x_1, x_2, \dots x_n)$, i.e. $Y = f(x_1, x_2, \dots x_n)$. To determine the uncertainty ΔY caused by the inaccuracy $\Delta x_1, \Delta x_2, \dots, \Delta x_n$ of estimation of the function arguments,

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we may use a Taylor series expansion. If the infinitely small increments in the independent variables are superseded by finite increment values then the following may be written:

$$f(x_1+\Delta x_1, x_2+\Delta x_2, \dots, x_n+\Delta x_n) =$$
$$f(x_1, x_2, \dots, x_n) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \dots + \frac{\partial^2 f}{\partial x_1^2}\cdot\frac{(\Delta x_1)^2}{2!} + \frac{\partial^2 f}{\partial x_2^2}\cdot\frac{(\Delta x_2)^2}{2!} \tag{1}$$
$$+ \dots + 2\cdot\frac{\partial^2 f}{\partial x_1 \partial x_2}\cdot\frac{\Delta x_1 \Delta x_2}{2!} + \dots$$

Based on (1), the difference $\Delta Y = f(x_1+\Delta x_1, x_2+\Delta x_2, \dots, x_n+\Delta x_n) - f(x_1, x_2, \dots, x_n)$ may be determined, which is a measure of uncertainty of the function $Y = f(x_1, x_2, \dots, x_n)$. If this function linearly depends on parameters $(x_1, x_2, \dots, x_n)$ then each of the higher-order partial derivatives is equal to zero. Then:

$$\Delta Y = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \dots + \frac{\partial f}{\partial x_n}\Delta x_n \tag{2}$$

This expression is very similar to that representing the total differential. The total differential defines the maximum possible value of the uncertainty ΔY; therefore, the absolute values of partial derivatives are normally taken for the calculations.

If the partial derivative with respect to x_i is denoted by $\frac{\partial f}{\partial x_i} = f_{xi}^{(1)}$ then, based on (2), we will obtain the following total differential formula, which is often used in the uncertainty analysis:

$$\Delta^{(1)}Y = |f_{x1}^{(1)}|\,\Delta x_1 + |f_{x2}^{(1)}|\,\Delta x_2 + \dots + |f_{xn}^{(1)}|\,\Delta x_n = \sum_{i=1}^{n}\left|f_{xi}^{(1)}\right|\Delta x_i \tag{3}$$

We have to remember, however, that this formula has been derived with making an assumption that the function $Y = f(x_1, x_2, \dots, x_n)$ linearly depends on parameters $(x_1, x_2, \dots, x_n)$. If otherwise, the higher-order derivatives would be non-zero and if they were taken into account as well then the calculated value of uncertainty ΔY would be higher.

Let us consider a case that the function $Y = f(x_1, x_2, \dots, x_n)$ nonlinearly depends on parameters $(x_1, x_2, \dots, x_n)$ and that there are non-zero second-order partial derivatives. In such a situation, additional terms will be present in formula (3). If we introduce the following symbols:

$\frac{\partial^2 f}{\partial x_i^2} = f_{xi}^{(2)}$ – the second derivative of function $Y = f(x_1, x_2, \dots, x_n)$ with respect to a variable x_i, with $i=1,\dots,n$

$\frac{\partial^2 f}{\partial x_i \partial x_j} = f_{xi,xj}^{(2)}$ – the second mixed derivative of function $Y = f(x_1, x_2, \dots, x_n)$ with respect to variables x_i, x_j, with $i,j=1,\dots,n$

then, having applied the Schwartz theorem about the equality of mixed partials, we will obtain the following formula for the uncertainty:

$$\Delta^{(2)}Y = \sum_{i=1}^{n}\left|f_{xi}^{(1)}\right|\Delta x_i + \sum_{i=1}^{n}\left|\frac{f_{xi}^{(2)}}{2}\right|(\Delta x_i)^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\left|f_{xi,xj}^{(2)}\right|\Delta x_i \Delta x_j \tag{4}$$

2. Motorcycle stopping distance

The application of the dependencies presented above will be illustrated by an example of determining the stopping distance S_Z of a motorcycle braking as hard as possible:

$$S_Z = V(t_r + t_0 + \frac{t_n}{2}) + \frac{V^2}{2\mu g f_n}[m] \tag{5}$$

where:
t_r – driver's reaction time [s],
V – initial motorcycle speed [m/s],
t_0 – braking system response time [s],
t_n – deceleration build-up time [s],
μ – coefficient of tyre adhesion to the road surface,
g – acceleration of gravity [m/s^2].

The practical braking efficiency of a motorcycle may be determined with the use of a ratio f_n representing the share of individual wheels in the maximum achievable total efficiency of motorcycle braking with the use of both brakes of brakes. The values of this ratio are as follows [4]:

- For braking of the rear wheel only, f_n = 40 to 45%;
- For braking of the front wheel only, f_n = 50 to 75%;
- For braking of both (front and rear) wheels, f_n = up to 95%.

For the calculations carried out, the following input values were assumed:

- Initial motorcycle speed V= 14 m/s,
- Time of driver's reaction to an unexpected situation: t_r =1.4 s,
- Response time of the hydraulic braking system: t_0 =0.25 s,
- Braking force build-up time at hard braking t_n =0.3 s,
- Coefficient of tyre adhesion to dry asphalt road surface μ=0.85,
- Acceleration of gravity (constant) g=9.81 m/s^2,
- Braking efficiency ratio f_n=0.7.

For the input values as above, the nominal motorcycle stopping distance calculated from formula (5) was S_Z = 42m.

Let us determine the uncertainty of the above calculation from total differential formula (3). The motorcycle stopping distance is a function of the following parameters:

$$S_Z = f(t_r, t_0, t_n, \mu, V, g, f_n)$$

Based on formula (3), the following may be written:

$$\Delta^{(1)} S_Z = \left|f^{(1)}_{t_r}\right| \Delta t_r + \left|f^{(1)}_{t_0}\right| \Delta t_0 + \left|f^{(1)}_{t_n}\right| \Delta t_n + \left|f^{(1)}_{\mu}\right| \Delta \mu + \left|f^{(1)}_{V}\right| \Delta V + \left|f^{(1)}_{f_n}\right| \Delta f_n \tag{6}$$

The following inaccuracies of the estimation of specific parameters were assumed:

ΔV=2m/s; Δt_r=0,2s; Δt_0=0,1s; Δt_n=0,1s; $\Delta\mu$=0,2; Δg=0; Δf_n=0,2

The uncertainty value obtained from the calculation was $\Delta^{(1)} S_Z$ = 22 m.

Formula (6) (total differential) is suitable for determining the uncertainty in the case of linear dependence of the motorcycle stopping distance on all the parameters. In the case under consideration, however, the stopping distance is a nonlinear function of motorcycle speed V. Therefore, the uncertainty value should be determined with taking into account the terms comprising higher-order partial derivatives as well. Based on (4), we obtain:

$$\begin{aligned}
\Delta^{(2)} S_Z = {} & \left|f^{(1)}_{t_r}\right| \Delta t_r + \left|f^{(1)}_{t_0}\right| \Delta t_0 + \left|f^{(1)}_{t_n}\right| \Delta t_n + \left|f^{(1)}_{\mu}\right| \Delta \mu + \left|f^{(1)}_{V}\right| \Delta V + \left|f^{(1)}_{f_n}\right| \Delta f_n \\
& + \frac{1}{2}\left|f^{(2)}_{t_r}\right| (\Delta t_r)^2 + \frac{1}{2}\left|f^{(2)}_{t_0}\right| (\Delta t_0)^2 + \frac{1}{2}\left|f^{(2)}_{t_n}\right| (\Delta t_n)^2 + \frac{1}{2}\left|f^{(2)}_{\mu}\right| (\Delta \mu)^2 \\
& + \frac{1}{2}\left|f^{(2)}_{V}\right| (\Delta V)^2 + \frac{1}{2}\left|f^{(2)}_{f_n}\right| (\Delta f_n)^2 \\
& + \left|f^{(2)}_{t_r t_0}\right| \Delta t_r \Delta t_0 + \left|f^{(2)}_{t_r t_n}\right| \Delta t_r \Delta t_n + \left|f^{(2)}_{t_r \mu}\right| \Delta t_r \Delta \mu + \left|f^{(2)}_{t_r V}\right| \Delta t_r \Delta V + \left|f^{(2)}_{t_r f_n}\right| \Delta t_r \Delta f_n \\
& + \left|f^{(2)}_{t_0 t_n}\right| \Delta t_0 \Delta t_n + \left|f^{(2)}_{t_0 \mu}\right| \Delta t_0 \Delta \mu + \left|f^{(2)}_{t_0 V}\right| \Delta t_0 \Delta V + \left|f^{(2)}_{t_0 f_n}\right| \Delta t_0 \Delta f_n \\
& + \left|f^{(2)}_{t_n \mu}\right| \Delta t_n \Delta \mu + \left|f^{(2)}_{t_n V}\right| \Delta t_n \Delta V + \left|f^{(2)}_{t_n f_n}\right| \Delta t_n \Delta f_n \\
& + \left|f^{(2)}_{\mu V}\right| \Delta \mu \Delta V + \left|f^{(2)}_{\mu f_n}\right| \Delta \mu \Delta f_n \\
& + \left|f^{(2)}_{V f_n}\right| \Delta V \Delta f_n
\end{aligned} \tag{7}$$

The uncertainty value obtained from this calculation is $\Delta^{(2)} S_Z$ = 29 m.

3. Conclusion

The terms omitted in the total differential formula (3) were found to work out at 7 m, which exceeds 30% of the uncertainty value calculated from formula (3). This example shows that if the calculated value is a nonlinear function of some parameters then the higher-order derivatives should also be taken into account because their influence may be significant.

In typical calculations of the dynamics of motion of an automotive vehicle, it rarely happens that parameters occur for which derivatives of an order higher than two have non-zero values; therefore, an uncertainty determination procedure where terms with first- and second-order derivatives are used, i.e. the one as presented herein, will be sufficient in most cases.

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