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On the Reynolds transport theorem for three phase systems with storage in interfaces

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Abstract In the paper, the Reynolds transport theorem (RTT) for three phase systems is developed, in terms associated with a moving control volume. The basic tools applied to the derivation are the generalized transport theorem by Truesdell and Toupin , and generalized surface transport theorem by Aris as well as Slattery. The final results referenced to a generic extensive quantity demonstrate the theorem in the integral instantaneous form. As a further illustration of applicability of the theorem relation developed some specific forms are deduced from such as for multiphase systems in terms of fixed control volume, surface systems and homogeneous spatial systems.

Keywords: Reynolds transport theorem; Three phases; Interface storage

1 Introduction

The model equations for transport phenomena are the conservation laws of extensive quantities (since abbreviated EQ) stored at the volume density η (intensive property, scalar, vector or tensor) within the material system Σ defined by constant mass m_Σ filling in variable volume Ω surrounded by boundary surface Γ moving at the local flow velocity $\mathbf{V}$. These laws

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expressed on the rate basis can be written in a unified integral form as:

$$\underbrace{\frac{d}{dt}\int\limits_{\Omega}\eta dV}_{\substack{\text{rate of accumulation}\\ \text{in moving system }\Sigma}} = \underbrace{\int\limits_{\Omega}\wp dV}_{\text{production rate}} + \underbrace{\int\limits_{\Gamma}\mathbf{J}\cdot\mathbf{n}dA}_{\text{transport rate}} , \tag{1}$$

where $\wp$ (scalar, vector or tensor), and $\mathbf{J}$ (vector or tensor) are a production rate density within Ω and transport flux (molecular and radiation transfer) across Γ, respectively, $\mathbf{n}$ is the outward unit normal vector to Γ, "$\cdot$" is the dot product and dt, dV and dA are differential increments of time, volume and surface area. Throughout the paper we assume the integral symbol $\int$ represents either the volume integral (triple – when accompanied with differential dV), surface integral (double – differential dA) or line integral (single – differential dl), respectively.

The Reynolds Transport Theorem (RTT) is a kinematic relation applied to express the rate of accumulation in the system Σ given as the left side term of Eq. (1) with the use of material coordinates (the Lagrangian description) in terms referenced to a selected domain of spatially prescribed configuration (the Eulerian description) of volume V (fixed or movable). As H. Lamb writes [8, p. 2], the germs of the first formulations of the theorem can be designated to Euler [3] in 1757. Since that time for many past years including Reynolds discovery in 1903 [14, p. 13, eq. (15)] until nowadays the problem has been subjected to intense research interest. In turn, forms of the RTT for homogeneous systems can be found now in numerous textbooks intended for students by, e.g. Fox and McDonald [4], as well as in monographs specialized for researchers interested in the field as those by, e.g. Aris [1], Rutkowski [10], or by Kundu and Cohen [7] as a more recent example.

However, number of RTT forms applicable to heterogeneous systems is significantly less despite that they can play the key role in numerous practical applications due to the physical effects which they can produce. The first to be mentioned here is by Truesdell and Toupin [12, p. 347] in 1960. They derived the RTT for a heterogeneous material system of volume Ω compound of two spatial homogeneous subsystems (phase 1 and phase 2 of densities η_1 and η_2, respectively) neighbouring across an interface (dividing surface) S of negligible storage abilities as follows [12, p. 468]:

$$\frac{d}{dt}\int\limits_{\Omega}\eta dV = \int\limits_{V}\frac{\partial\eta}{\partial t}dV + \int\limits_{R}\eta\mathbf{V}\cdot\mathbf{n}dA + \int\limits_{S}(\eta_1-\eta_2)\mathbf{U}\cdot\boldsymbol{\zeta}dA , \tag{2}$$

where V is an established fixed volume of reference defined spatially (Eulerian description) composed of the both phasic volumes and bounded by an external surface R of outward normal $\mathbf{n}$, S is the interface area and $\boldsymbol{\zeta}$ is the normal to S pointed from phase 1 to phase 2. Here $\mathbf{U}$ and $\mathbf{V}$ are the spatial velocities of the interface and phases, respectively.

Nevertheless, there are numerous examples when contribution of the interface storage into system storage can be essential. Consequently, for the cases in point Slattery [11] proposed the RTT of the following form

$$\frac{d}{dt}\left(\int\limits_{\Omega}\eta dV+\int\limits_{\Gamma}\eta_s dA\right)=\int\limits_{V}\left(\frac{D\eta}{Dt}+\eta\,\mathrm{div}\mathbf{V}\right)dV+ \\ +\int\limits_{S}\left(\frac{D_s\eta_s}{Dt}+\eta_s\,\mathrm{div}_s\mathbf{V}_s[\![\eta(\mathbf{V}-\mathbf{U})\cdot\boldsymbol{\zeta}]\!]\right)dA\ , \qquad (3)$$

where the fixed volume of reference V is composed of all phasic volumes involved and the overall interface area S comprises of all the interfaces dwelling within volume V. Above $\mathbf{U}$ and $\mathbf{V}_s$ are the spatial velocities of the interface and the surface system moving in, respectively. By $D\eta/Dt$ and $D_s\eta_s/Dt$ the material derivatives of spatial and surface densities η and η_s are denoted, respectively. div_s is the surface divergence differential operator and $\boldsymbol{\zeta}$ means the outward unit normal vector to the interface pointing into phase moving at $\mathbf{V}$. Therefore, the bracketed term $[\![\eta(\mathbf{V}-\mathbf{U})\cdot\boldsymbol{\zeta}]\!]$ of Eq. (3) reads as $[\eta^+(\mathbf{V}^+-\mathbf{U})\cdot\boldsymbol{\zeta}^+ +\eta^-(\mathbf{V}^- -\mathbf{U})\cdot\boldsymbol{\zeta}^-]$ and refers to jump condition for phasic spatial density η across the interface set in between two phases which properties are denoted by superscripts '+' and '–'.

This paper is purposed to derive the RTT relations for three phase systems of essential interfacial storage in terms related to referential volume CV (control volume) surrounded by boundary CS (control surface)) being in arbitrary motion with respect to fixed (inertial) reference frame. The final outcomes present the RTT in a general integral instantaneous form from which forms of the RTT suitable for specific systems are also derived.

Because of the ratio of interface area to system volume rises when a characteristic linear scale decreases then interest in the effect of interfacial storage is motivated by applications to problems when small systems are considered. Therefore, such the significance manifests primarily in systems densely structured by presence of small constituents such as waves, layers, bubbles droplets, etc., extending immensely the interface area and in turn interface contribution into the system storage. Worthy to note also are some applied fields where interface behaviour can be appreciable. Let's mention here transport phenomena in nano- and micro-channels and phase change processes affected by surface tension met in energy storage systems. Also interfacial dynamics can be important for widespread nuclear and oil technologies, car engine combustion of liquid fuels, food and pharmaceutical industry and also environmental applications such as wastes emissions of aerosols and soot, forest fires and detonation phenomena.

2 RTT for three phase systems

In Fig. 1 a three phase material system Σ is displayed occupying spatial domain Ω of boundary Γ split into phasic portions Ω_i $(i = 1, 2, 3)$. The system is composed of three spatial subsystems $\cup_i^3 \Sigma_{vi}$ separated by interfaces S and of the K surface subsystems $\cup_k^K \Sigma_{sk}$ dwelling in S.

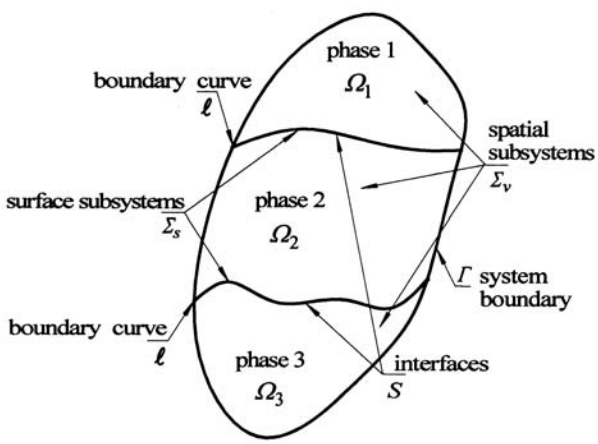

Figure 1. System Σ composed of spatial and surface subsystems.

System Σ passes through a movable CV bounded by CS, see in Fig. 2. In view of Σ is composed of spatial and surface subsystems the CV comprises both spatial phasic domains of total volume V and interfacial domains of aggregated area S, hence $CV = V \cup S$.

Volume V involves all the phasic volumes embedded in CV, hence $V = \cup_i^3 V_i$. The aggregated interfacial area S involves the interfaces placed within CV so that $S = \cup_k^K S_k$. The entire CS consists in $CS = R \cup C$ where R is the aggregated external boundary of phasic domains determined as $R = \cup_i^3 R_i$ with understanding that each individual R_i is designated for the entire external boundary of the i-th phasic volume. $C = \cup_k^K C_k$ stands for the aggregated boundary curve of all individual boundary curves C_k formed as intersection of CS and interface S_k. Following Slattery [11] the interface is considered here as a surface possessing also accumulative abilities with respect to EQ-ies such as mass, momentum, energy, and entropy.

Accumulation $\delta\Phi_\Sigma$ of an extensive quantity EQ in system Σ is determined by the difference in system storages Φ_Σ at $t+\delta t$ and t, hence

$$\underbrace{\delta\Phi_\Sigma}_{\substack{\text{accumulation in} \\ \Sigma \text{ during } \delta t}} = \underbrace{\Phi_\Sigma(t+\delta t)}_{\substack{\text{storage in } \Sigma \text{ at} \\ t+\delta t}} - \underbrace{\Phi_\Sigma(t)}_{\substack{\text{storage} \\ \text{in } \Sigma \text{ at } t}} . \tag{4}$$

In Fig. 2(a) the coincidence of Ω and CV is shown at an instant t. In such particular circumstances boundary Γ of system Σ traced by lowercase underlined letters $\underline{abcdefa}$ is superimposed upon boundary CS of CV indicated by ghijklg, hence $\underline{abcdefa} = ghijklg$, see in Fig.2(a). In turn, amounts of EQ stored at t within Σ and CV are the same, what gives

$$\underbrace{\Phi_\Sigma(t)}_{\substack{\text{storage in } \Sigma \\ \text{at } t}} = \underbrace{\Phi_{CV}(t)}_{\substack{\text{storage in} \\ CV \text{ at } t}} . \tag{5}$$

Let subsequent instant of time $t+\delta t$ be considered as displayed in Fig. 2(b) where system Σ is shown to be displaced partially out of CV. Hence, boundaries of Σ traced along $\underline{abcdefa}$ and CV marked as $ghijklg$ are shifted each other — see in Fig. 2(b).

In turn, system Σ leaves to CV some amount of EQ stored in region I ($\underline{afedjklga}$) and carries some amount of EQ stored in region II ($\underline{afediha}$) out of CV. Therefore, based on Fig. 2(b) one gets storage $\Phi_\Sigma(t+\delta t)$ expressed in terms referenced to CV as:

$$\underbrace{\Phi_\Sigma(t+\delta t)}_{\substack{\text{storage in } \Sigma \\ \text{at } t+\delta t}} = \underbrace{\Phi_{CV}(t+\delta t)}_{\substack{\text{storage in} \\ CV \text{ at } t+\delta t}} + \underbrace{\delta\Phi_{II}(\delta t)}_{\substack{\text{amount of } EQ \\ \text{carried by } \Sigma \text{ out} \\ \text{of } CV \text{ during } \delta t \\ \text{(region II)}}} - \underbrace{\delta\Phi_I(\delta t)}_{\substack{\text{amount of } EQ \\ \text{brought in } CV \\ \text{by } \Sigma \text{ during } \delta t \\ \text{(region I)}}} , \tag{6}$$

where $\delta\Phi_I(\delta t)$ is amount of EQ brought in region I of CV by moving Σ. Consequently, term $\delta\Phi_I(\delta t)$ stands for inflow of EQ into CV across CS, i.e. represents transport of EQ across CS in favour of storage in CV. Term $\delta\Phi_{II}(\delta t)$ of Eq. (6) is amount of EQ carried by moving Σ in course of time δt into region II out of CV — see in Fig. 2(b). Hence, term $\delta\Phi_{II}(\delta t)$ of Eq. (6) defines the transport of EQ at expense of storage in CV, i.e. efflux

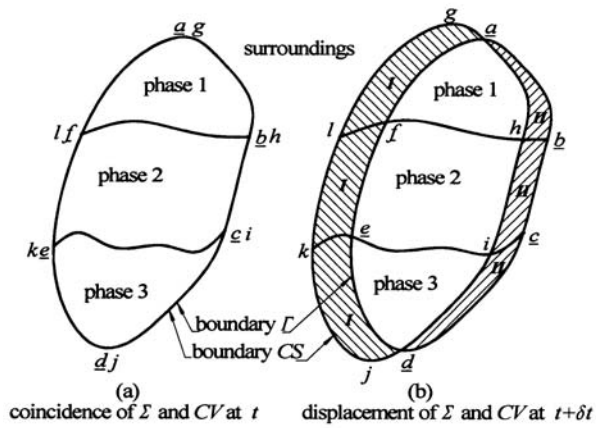

Figure 2. System Σ passing through a moving CV: (a) coinciding condition at time t, (b) displacement of Σ with respect to CV at time $t + \delta t$ (see description in the text).

of EQ out of CV. Note that the sum of transport terms shown in Eq. (6) expresses the net interchange between CV and its surroundings across the complete boundary $CS = R \cup C$. By substitution Eqs. (5) and (6) into Eq. (4) one obtains accumulation in system Σ expressed in terms of CV as follows:

$$\underbrace{\delta\Phi_\Sigma}_{\substack{\text{accumulation in}\\ \Sigma \text{ during } \delta t}} = \underbrace{\phi_{CV}(t+\delta t) - \Phi_{CV}(t)}_{\text{accumulation in moving CV}} + \underbrace{\delta\Phi_{II}(\delta t) - \delta\Phi_I(\delta t)}_{\text{transport across } CS} \; . \tag{7}$$

Storages $\Phi_{CV}(t)$ and $\Phi_{CV}(t + \delta t)$ of Eq. (7) include those in the spatial domains of volume V (denoted by Φ_V) and those in the interfacial domains of area S (described by Φ_S). Likewise, amounts of EQ transported by moving Σ refer to contributions made by macroscopic movements of both spatial (by $\delta_v\Phi$) and surface (by $\delta_s\Phi$) subsystems. With this understanding, corresponding terms are introduced into Eq. (7) and subsequently all the terms on both sides are divided by δt. Then by letting $\delta t \to 0$ one gets Eq. (7) expressed on the rate basis as:

$$\underbrace{\lim_{\delta t\to 0}\frac{\delta\Phi_\Sigma}{\delta t}}_{\substack{\text{accumulation}\\ \text{rate in system } \Sigma}} = \lim_{\delta t\to 0}\Big[\underbrace{\frac{\Phi_V(t+\delta t) - \Phi_V(t)}{\delta t}}_{\substack{\text{accumulation rate in spatial}\\ \text{phasic domains of volume } V\\ (1)}} + \underbrace{\frac{\delta_v\Phi_{II}(\delta t) - \delta_v\Phi_I(\delta t)}{\delta t}}_{\substack{\text{transport rate by movement of}\\ \text{spatial subsystems across } CS\\ (2)}} +$$

$$+\underbrace{\frac{\Phi_S(t+\delta t)-\Phi_S(t)}{\delta t}}_{\substack{\text{accumulation rate in}\\ \text{interfacial domains of area } S\\ (3)}}+\underbrace{\frac{\delta_s\Phi_{II}(\delta t)-\delta_s\Phi_I(\delta t)}{\delta t}}_{\substack{\text{transport rate by movement of}\\ \text{surface subsystems across } CS\\ (4)}}\Bigg] . \tag{8}$$

The left side of Eq. (8) converges at the accumulation rate of EQ within Σ to be given by

$$\lim_{\delta t\to 0}\frac{\delta\Phi_\Sigma}{\delta t}=\frac{d\Phi_\Sigma}{dt} . \tag{9}$$

Below, particular terms placed on the right side of Eq. (8) are converted into desired rate expressions referenced to CV that by $\delta t\to 0$ coincides Σ. In turn, the goal is approached, i.e. rate of accumulation in system Σ (the material description) becomes to be expressed in terms referenced to CV (the spatial or Eulerian description) what is the essence of each RTT.

2.1 Term (1) — accumulation rate in spatial phasic domains

Taking the limit when $\delta t\to 0$ the rate form of the first term of Eq. (8) becomes

$$\lim_{\delta t\to 0}\frac{\Phi_V(t+\delta t)-\Phi_V(t)}{\delta t}=\frac{d\Phi_V(t)}{dt} . \tag{10}$$

Because $V=\cup_i^3 V_i$, then the overall storage $\Phi_V(t)$ in spatial phasic domains is given by

$$\Phi_V(t)=\sum_i^3\Phi_{V_i}(t) . \tag{11}$$

Consequently, in view of Eq. (11) expression (10) writes up as

$$\frac{d\Phi_V(t)}{dt}=\sum_i^3\frac{d\Phi_{V_i}(t)}{dt} , \tag{12}$$

where

$$\Phi_{V_i}(t)=\int\limits_{V_i(t)}\eta_i(\underline{z},t)dV,\quad j=1,2,3 \tag{13}$$

is the storage within the i-th phasic domain of moving volume $V_i(t)$ at an instant t and by $\underline{z}$ the spatial coordinates are denoted. In Fig. 3 boundaries of volume $V_i(t)$ are illustrated. It is seen in Fig. 3 that the entire boundary $\Re_i$ of V_i is a closed surface $\Re_i=R_i+S_i$ where R_i is the external part of $\Re_i$

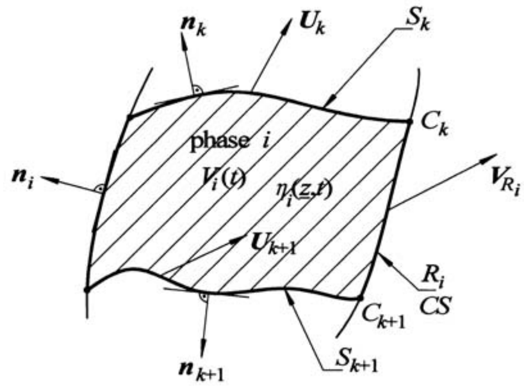

Figure 3. The phasic domain $V_i(t)$ bounded by external boundary R_i and moving interfaces S_k and S_{k+1}.

and $S_i = \sum_k^{K_i} S_k$ is the interfacial part of $\Re_i$ assembled of K_i interfaces S_k associated phase i.

Now the generalized transport theorem (Truesdell and Toupin [12]) is applied to determine each derivative of $\Phi_{V_i}(t)$ shown as the right side terms of relation (12) to obtain

$$\begin{aligned} \frac{d\Phi_{V_i}(t)}{dt} &= \frac{d}{dt}\int\limits_{V_i(t)} \eta_i(\underline{z},t)dV = \\ &= \int\limits_{V_i} \frac{\partial \eta_i}{\partial t} dV + \int\limits_{R_i} \eta_i \mathbf{V}_{R_i}\cdot \mathbf{n}_i dA + \sum_k^{K_i} \int\limits_{S_k} \eta_{i\rhd S_k} \mathbf{U}_k \cdot \mathbf{n}_k dA \ , \end{aligned} \tag{14}$$

where boundaries R_i and S_k (see in Fig. 3) are moving at velocities $\mathbf{V}_{R_i}$ and $\mathbf{U}_k$, respectively. $\mathbf{n}_i$ and $\mathbf{n}_k$ are the unit normal vectors to boundaries R_i and S_k, $(k = 1, 2)$, respectively, drawn outward with respect to $V_i(t)$. $\eta_{i\rhd S_k}$ is the spatial density of EQ stored in the i-th phase taken at infinitesimally close position to interface S_k. Equation (14) can be can be modified by the use of the Gauss's theorem (Kaplan [6]). Hence, the double integrals of Eq. (14) defined over a closed surface $\Re_i = R_i + S_i$ can be expressed by the triple integral defined over a spatial domain V_i of boundary $\Re_i = R_i + S_i$. Thus one obtains

$$\int\limits_{R_i} \eta_i \mathbf{V}_{R_i}\cdot \mathbf{n}_i dA + \sum_k^{K_i} \int\limits_{S_k} \eta_{i\rhd S_k} \mathbf{U}_k \cdot \mathbf{n}_k dA = \int\limits_{V_i} \mathrm{div}(\eta_i \mathbf{V}_{\Re_i}) dV \ . \tag{15}$$

By substitution Eq. (15) into Eq. (14) and subsequently Eq. (14) into Eq. (12) one gets

$$\frac{d\Phi_V(t)}{dt} = \int\limits_V \frac{\partial \eta}{\partial t} dV + \int\limits_V \mathrm{div}(\eta \mathbf{V}_{\Re}) dV \ , \tag{16}$$

where

$$\int\limits_V \frac{\partial \eta}{\partial t} dV = \sum_i^3 \int\limits_{V_i} \frac{\partial \eta_i}{\partial t} dV \tag{17}$$

and

$$\int\limits_V \mathrm{div}(\eta \mathbf{V}_{\Re}) dV = \sum_i^3 \int\limits_{V_i} \mathrm{div}(\eta_i \mathbf{V}_{\Re_i}) dV \ . \tag{18}$$

Consequently, Eq. (16) defines term (1) of Eq. (8).

2.2 Term (2) — transport rate by movement of spatial subsystems

This term corresponds to the contribution in the rate of accumulation in system Σ due to movements of the spatial subsystems relative to the CV. One can see in Fig. 2(b) that this effect refers to phases engaged in spatial regions I and II and hence six terms are involved (three per one spatial region). Considering time increment δt sufficiently small the 2nd term of Eq. (8) can be expressed in the rate form as:

$$\lim_{\delta t \to 0} \frac{\delta_v \Phi_{II}(\delta t) - \delta_v \Phi_I(\delta t)}{\delta t} =$$

$$= \lim_{\delta t \to 0} \frac{\sum_i^3 \delta t \frac{d\Phi_{II,i}}{dt} - \sum_i^3 \delta t \frac{d\Phi_{I,i}}{dt}}{\delta t} = \lim_{\delta t \to 0} \left(\sum_i^3 \frac{d\Phi_{II,i}}{dt} - \sum_i^3 \frac{d\Phi_{I,i}}{dt} \right) \ . \tag{19}$$

The derivative $d\Phi_{I,i}/dt$ of Eq. (19) referenced to the i-th spatial portion of region I of volume $\delta V_{I,i}$ bounded by $\Gamma_{I,i} \cup R_{I,i} \cup \delta S_{I,i}$, see in Fig. 4(a), can be evaluated by the generalized transport theorem (Truesdell and Toupin [12]) to obtain

$$\begin{aligned} \frac{d\Phi_{I,i}}{dt} \quad = \quad & \int\limits_{\delta V_{I,i}} \frac{\partial \eta_i}{\partial t} dV - \int\limits_{\Gamma_{I,i}} \eta_i \mathbf{V}_i \cdot \mathbf{n}_i dA + \\ & + \int\limits_{R_{I,i}} \eta_i \mathbf{V}_{R_{I,i}} \cdot \mathbf{n}_i dA + \sum_k^{K_i} \int\limits_{\delta S_{I,k}} \eta_{i \triangleright \delta S_{I,k}} \mathbf{U}_k \cdot \mathbf{n}_k dA \ , \end{aligned} \tag{20}$$

where $\delta S_{I,k}$ is the k-th portion of $\delta S_{I,i}$ and $\delta S_{I,i} = \sum_k^{K_i} \delta S_{I,k}$ — see in Fig. 4(a). Note in Eq. (20), the unit normal vector drawn outward of $\delta V_{I,i}$ at boundary $\Gamma_{I,i}$ is $-\mathbf{n}_i$ because it is expressed by unit normal $\mathbf{n}_i$ at boundary Γ_i drawn outward of spatial subsystem Ω_i.

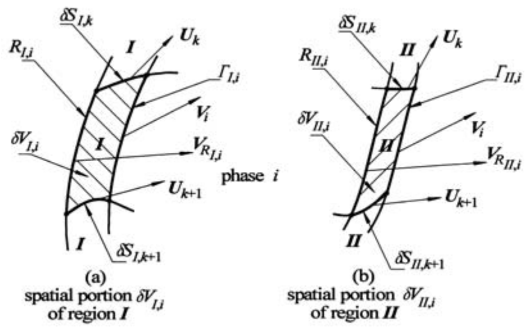

Figure 4. The spatial portions of region I of volume $\delta V_{I,i}$ and region II of volume $\delta V_{II,i}$ accompanied by associated boundaries.

Likewise, for the i-th (it refers to the i-th phase) spatial portion of region II of volume $\delta V_{II,i}$ bounded by $\Gamma_{II,i} \cup R_{II,i} \cup \delta S_{II,i}$, see in Fig.4 (b), one gets the accumulation rate $d\Phi_{II,i}/dt$ given as:

$$\frac{d\Phi_{II,i}}{dt} = \int\limits_{\delta V_{II,i}} \frac{\partial \eta_i}{\partial t} dV + \int\limits_{\Gamma_{II,i}} \eta_i \mathbf{V}_i \cdot \mathbf{n}_i dA + \\ - \int\limits_{R_{II,i}} \eta_i \mathbf{V}_{R_{II,i}} \cdot \mathbf{n}_i dA + \sum_k^{K_i} \int\limits_{\delta S_{II,k}} \eta_{i \triangleright \delta S_{II,k}} \mathbf{U}_k \cdot \mathbf{n}_k dA \ , \quad (21)$$

where $\delta S_{II,k}$ is the k-th portion of $\delta S_{II,i}$ and $\delta S_{II,i} = \sum_k^{K_i} \delta S_{II,k}$ — see in Fig. 4(b). Note in Eq. (21), the unit normal vector drawn outward of $\delta V_{II,i}$ at boundary $R_{II,i}$ is $-\mathbf{n}_i$ because it is expressed by unit normal $\mathbf{n}_i$ at boundary R_i drawn outward of phasic volume V_i. Substituting Eqs. (20) and (21) into Eq. (19) and subsequently determining the limit of each term by letting $\delta t \to 0$ one finds

- $$(\delta t \to 0) \to (\delta V_{I,i} \to 0) \to \lim_{\delta t \to 0} \int\limits_{\delta V_{I,i}} \frac{\partial \eta_i}{\partial t} dV = 0 \ , \quad (22)$$

- $$(\delta t \to 0) \to (\delta V_{II,i} \to 0) \to \lim_{\delta t \to 0} \int\limits_{\delta V_{II,i}} \frac{\partial \eta_i}{\partial t} dV = 0 \ , \quad (23)$$

- $(\delta t \to 0) \to (\Gamma_{I,i} \to R_{I,i})$ and $(\Gamma_{II,i} \to R_{II,i})$ therefore

$$\lim_{\delta t\to 0}\Big(\int\limits_{\Gamma_{II,i}} \eta_i \mathbf{V}_i \cdot \mathbf{n}_i dA - \int\limits_{R_{II,i}} \eta_i \mathbf{V}_{R_{II,i}} \cdot \mathbf{n}_i dA +$$
$$+ \int\limits_{\Gamma_{I,i}} \eta_i \mathbf{V}_i \cdot \mathbf{n}_i dA - \int\limits_{R_{I,i}} \eta_i \mathbf{V}_{R_{I,i}} \cdot \mathbf{n}_i dA \Big) =$$
$$= \int\limits_{R_{II,i}} \eta_i (\mathbf{V}_i - \mathbf{V}_{R_{II,i}}) \cdot \mathbf{n}_i dA + \int\limits_{R_{I,i}} \eta_i (\mathbf{V}_i - \mathbf{V}_{R_{I,i}}) \cdot \mathbf{n}_i dA =$$
$$= \int\limits_{R_i} \eta_i (\mathbf{V}_i - \mathbf{V}_{R_i}) \cdot \mathbf{n}_i dA \ , \tag{24}$$

where $R_i = R_{I,i} + R_{II,i}$ is the entire external boundary of the phasic volume V_i,

- $(\delta t \to 0) \to (\delta S_{I,k} \to 0)$, and $(\delta S_{II,k} \to 0)$ therefore

$$\lim_{\delta t\to 0}\Big(\int\limits_{\delta S_{II,k}} \eta_{i\triangleright S_{II,k}} \mathbf{U}_k \cdot \mathbf{n}_k dA - \int\limits_{\delta S_{I,k}} \eta_{i\triangleright S_{I,k}} \mathbf{U}_k \cdot \mathbf{n}_k dA \Big) = 0 \ . \tag{25}$$

Involving the resulted expressions (22)–(25) into Eq. (19) one obtains term (2) of Eq. (8) in the rate form given as:

$$\lim_{\delta t\to 0} \frac{\delta_v \Phi_{II}(\delta t) - \delta_v \Phi_I(\delta t)}{\delta t} = \int\limits_R \eta (\mathbf{V} - \mathbf{V}_R) \cdot \mathbf{n} dA \ , \tag{26}$$

where

$$\int\limits_R \eta (\mathbf{V} - \mathbf{V}_R) \cdot \mathbf{n} dA = \sum_i^3 \int\limits_{R_i} \eta_i (\mathbf{V}_i - \mathbf{V}_{R_i}) \cdot \mathbf{n}_i dA \ . \tag{27}$$

2.3 Term (3) — accumulation rate in interfacial domains

By letting $\delta t \to 0$ and taking the limit, the rate form of term (3) of Eq. (8) is

$$\lim_{\delta t\to 0} \frac{\Phi_S(t+\delta t) - \Phi_S(t)}{\delta t} = \frac{d\Phi_S(t)}{dt} \ . \tag{28}$$

The total area of interfaces embedded within CV is $S = \cup_k^K S_k$. Hence the overall storage $\Phi_S(t)$ determined at an instant t within the interfacial domain of aggregated area S is given by

$$\Phi_S(t) = \sum_k^K \Phi_{S_k}(t) \ . \tag{29}$$

Equation (28) in view of Eq. (29) can be written as:

$$\lim_{\delta t \to 0} \frac{\Phi_S(t+\delta t) - \Phi_S(t)}{\delta t} = \frac{d\Phi_S(t)}{dt} = \sum_k^K \frac{d\Phi_{S_k}}{dt} \ . \tag{30}$$

The following integral describes storage $\Phi_{S_k}(t)$ in the interface of area $S_k(t)$ at an instant t

$$\Phi_{S_k}(t) = \int\limits_{S_k(t)} \eta_s(y^1, y^2, t)\sqrt{a(y^1, y^2, t)}dy^1 dy^2 \ , \tag{31}$$

where a is the determinant of the surface metric tensor and y^1, y^2 are the surface coordinates. Note, all the properties used in Eq. (31) refer to the interface S_k. Hence, the corresponding subscript k is omitted in the integrand of Eq. (31). Taking into considerations dependency of integral (31) on η_s, a and S_k, the accumulation rate $d\Phi_{S_k}/dt$ shown in Eq. (30) can be expressed now by the surface transport theorem (Aris [1])

$$\frac{d\Phi_{S_k}}{dt} = \int\limits_{S_k} \Big(\frac{\partial \eta_{s,k}}{\partial t} + \eta_{s,k} \,\mathrm{div}_s \mathbf{U}_k \Big) dA + \int\limits_{C_k} \eta_{s,k}(\mathbf{V}_{C_k} - \mathbf{U}_k) \cdot \mathbf{b}_k dl \ , \tag{32}$$

where $\mathrm{div}_s \mathbf{U}_k$ is the surface divergence of the interface spatial velocity $\mathbf{U}_k$, $\mathbf{V}_{C_k}$ is the spatial velocity of boundary curve C_k formed by intersection of S_k and CS, $\mathbf{b}_k$ is the unit surface vector tangent to the surface and normal to the curve C_k directed outward of S_k — see in Fig. 5.

By substitution Eq. (32) into Eq. (30), then taking the limit and performing summation one gets term (3) of Eq. (8) in the rate form given by

$$\lim_{\delta t \to 0} \frac{\Phi_S(t+\delta t) - \Phi_S(t)}{\delta t} = \frac{d\Phi_S(t)}{dt} = \\ = \int\limits_S \Big(\frac{\partial \eta_s}{\partial t} + \eta_s \,\mathrm{div}_s \mathbf{U} \Big) dA + \int\limits_C \eta_s(\mathbf{V}_C - \mathbf{U}) \cdot \mathbf{b} dl \ , \tag{33}$$

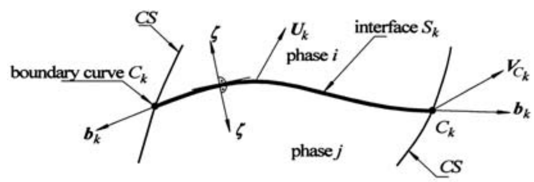

Figure 5. Interface S_k in between two phases — see description in the text.

where the right side reads as:

$$\int_S \Big(\frac{\partial \eta_s}{\partial t} + \eta_s \operatorname{div}_s \mathbf{U}\Big) dA + \int_C \eta_s (\mathbf{V}_C - \mathbf{U}) \cdot \mathbf{b} dl =$$
$$= \sum_k^K \int_{S_k} \Big(\frac{\partial \eta_{s,k}}{\partial t} + \eta_{s,k} \operatorname{div}_s \mathbf{U}_k\Big) dA + \sum_k^K \int_{C_k} \eta_{s,k} (\mathbf{V}_{C_k} - \mathbf{U}_k) \cdot \mathbf{b}_k dl \quad (34)$$

and the differential area is $dA = \sqrt{a} dy^1 dy^2$. Note that $(\mathbf{V}_{C_k} - \mathbf{U}_k)$ is a tangential vector field.

2.4 Term (4) — transport rate by movement of surface subsystems

This term corresponds to contributions in the rate of accumulation in the system made by movement of surface subsystems across boundary CS of the CV. We construct the rate form of the 4th term of Eq. (8) by letting $\delta t \to 0$ and taking the limit what can be written as:

$$\lim_{\delta t \to 0} \frac{\delta_s \Phi_{II}(\delta t) - \delta_s \Phi_I(\delta t)}{\delta t} =$$
$$= \lim_{\delta t \to 0} \frac{\sum_k^K \delta t \frac{d\Phi^{(s)}_{II,k}}{\delta t} - \sum_k^K \delta t \frac{d\Phi^{(s)}_{I,k}}{\delta t}}{\delta t} = \lim_{\delta t \to 0} \Big(\sum_k^K \frac{d\Phi^{(s)}_{II,k}}{\delta t} - \sum_k^K \frac{d\Phi^{(s)}_{I,k}}{\delta t} \Big). \quad (35)$$

Each term on the right side of relation (35) can be developed based on the generalized surface transport theorem (Slattery [11]). Consequently, accumulation rate $d\Phi^{(s)}_{I,k}/dt$ in the interface portion $\delta S_{I,k}$, see in Fig. 6(a), located in region I surrounded by boundary curves $C_{I,k}$ and $\ell_{I,k}$ is

$$\frac{d\Phi^{(s)}_{I,k}}{\delta t} = \int\limits_{\delta S_{I,k}} \Big(\frac{\partial \eta_{s,k}}{\partial t} - \mathrm{grad}_s \eta_{s,k} \cdot \mathbf{U}_k - 2H_k \eta_{s,k} \mathbf{U}_k \cdot \boldsymbol{\zeta}_k\Big) dA + \\ + \int\limits_{C_{I,k}} \eta_{s,k} \mathbf{V}_{C_k} \cdot \mathbf{b}_k dl - \int\limits_{\ell_{I,k}} \eta_{s,k} \mathbf{V}_{s_k} \cdot \mathbf{b}_k dl \ , \tag{36}$$

where $\mathbf{V}_{s_k}$ is the spatial velocity of the surface system flowing in $\delta S_{I,k}$ and H_k is the mean curvature of $\delta S_{I,k}$. $\ell_{I,k}$ is the boundary curve formed by intersection of system boundary Γ and $\delta S_{I,k}$. Likewise, accumulation rate $d\Phi^{(s)}_{II,k}/dt$ in interface portion $\delta S_{II,k}$ located in region II , see in Fig. 6(b), surrounded by boundary curves $C_{II,k}$ and $\ell_{II,k}$ is

$$\frac{d\Phi^{(s)}_{II,k}}{\delta t} = \int\limits_{\delta S_{II,k}} \Big(\frac{\partial \eta_{s,k}}{\partial t} - \mathrm{grad}_s \eta_{s,k} \cdot \mathbf{U}_k - 2H_k \eta_{s,k} \mathbf{U}_k \cdot \boldsymbol{\zeta}_k\Big) dA + \\ - \int\limits_{C_{II,k}} \eta_{s,k} \mathbf{V}_{C_k} \cdot \mathbf{b}_k dl + \int\limits_{\ell_{II,k}} \eta_{s,k} \mathbf{V}_{s_k} \cdot \mathbf{b}_k dl \ . \tag{37}$$

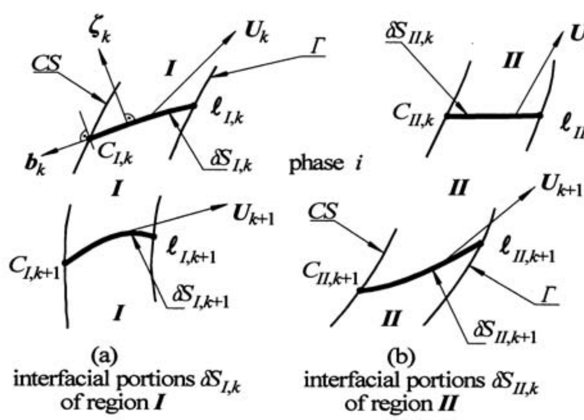

Figure 6. Interfacial portions $\delta S_{I,k}$ and $\delta S_{II,k}$, ($k = 1, 2$), of region I and region II, respectively, accompanied by associated boundary curves.

Accordingly Eq. (35) the limits of each term of Eqs. (36) and (37) when $\delta t \to 0$ must be determined. Therefore by letting $\delta t \to 0$ one finds the limits to be expressed as:

- $(\delta t \to 0) \to (\delta S_{I,k} \to 0) \to \lim\limits_{\delta t \to 0} \int\limits_{\delta S_{I,k}} \Big(\frac{\partial \eta_{s,k}}{\partial t} +$

$$- \operatorname{grad}_s \eta_{s,k} \cdot \mathbf{U}_k - 2H_k \eta_{s,k} \mathbf{U}_k \cdot \boldsymbol{\zeta}_k \Big) dA = 0 \; , \tag{38}$$

- $(\delta t \to 0) \to (\ell_{I,k} \to C_{I,k}), \quad \text{thus}$

$$\lim_{\delta t \to 0} \Big(\int\limits_{\ell_{I,k}} \eta_{s,k} \mathbf{V}_{s,k} \cdot \mathbf{b}_k dl - \int\limits_{C_{I,k}} \eta_{s,k} \mathbf{V}_{C_k} \cdot \mathbf{b}_k dl \Big) = \int\limits_{C_{I,k}} \eta_{s,k} (\mathbf{V}_{s,k} - \mathbf{V}_{C_k}) \cdot \mathbf{b}_k dl \; , \tag{39}$$

- $(\delta t \to 0) \to (\delta S_{II,k} \to 0) \to \lim\limits_{\delta t \to 0} \int\limits_{\delta S_{II,k}} \Big(\frac{\partial \eta_{s,k}}{\partial t} +$

$$- \operatorname{grad}_s \eta_{s,k} \cdot \mathbf{U}_k - 2H_k \eta_{s,k} \mathbf{U}_k \cdot \boldsymbol{\zeta}_k \Big) dA = 0 \; , \tag{40}$$

- $(\delta t \to 0) \to (\ell_{II,k} \to C_{II,k}), \quad \text{thus}$

$$\lim_{\delta t \to 0} \Big(\int\limits_{\ell_{II,k}} \eta_{s,k} \mathbf{V}_{s,k} \cdot \mathbf{b}_k dl - \int\limits_{C_{II,k}} \eta_{s,k} \mathbf{V}_{C_k} \cdot \mathbf{b}_k dl \Big) =$$

$$= \int\limits_{C_{II,k}} \eta_{s,k} (\mathbf{V}_{s,k} - \mathbf{V}_{C_k}) \cdot \mathbf{b}_k dl \; . \tag{41}$$

The results given by Eqs. (38)–(41) substituted into Eq. (35) provide term (4) of Eq. (8) expressed in the rate form as:

$$\lim_{\delta t \to 0} \frac{\delta_s \Phi_{II}(\delta t) - \delta_s \Phi_I(\delta t)}{\delta t} =$$

$$= \lim_{\delta t \to 0} \Big(\sum_k^K \frac{d\Phi_{II,k}^{(s)}}{\delta t} - \sum_k^K \frac{d\Phi_{I,k}^{(s)}}{\delta t} \Big) = \int\limits_C \eta_s (\mathbf{V}_s - \mathbf{V}_C) \cdot \mathbf{b} dl \; , \tag{42}$$

where

$$\int\limits_C \eta_s (\mathbf{V}_s - \mathbf{V}_C) \cdot \mathbf{b} dl = \sum_k^K \int\limits_{C_k} \eta_{s,k} (\mathbf{V}_{s,k} - \mathbf{V}_{C_k}) \cdot \mathbf{b}_k dl$$

and $C_k = C_{I,k} + C_{II,k}$ is the entire boundary curve of interface S_k, also $C = \cup_k^K C_k$.

2.5 General form of RTT for three phase systems

By substitution expression (9) onto left side of Eq. (8) and relations (16), (26), (33) and (42) into the right side of Eq. (8) we can generalize the Reynolds transport theorem for three phase systems to be expressed as follows:

$$\frac{d\Phi_\Sigma}{dt} = \underbrace{\int\limits_V \frac{\partial \eta}{\partial t}\cdot dV + \int\limits_V \mathrm{div}(\eta \mathbf{V}_{\Re})\cdot dV + \int\limits_R \eta(\mathbf{V}-\mathbf{V}_R)\cdot \mathbf{n} dA +}_{\text{phasic terms}} \tag{43}$$
$$\underbrace{+\int\limits_S \Big(\frac{\partial \eta_s}{\partial t} + \eta_s\,\mathrm{div}_s \mathbf{U}\Big) dA + \int\limits_C \eta_s(\mathbf{V}_C - \mathbf{U})\cdot \mathbf{b} dl + \int\limits_C \eta_s(\mathbf{V}_s - \mathbf{U}_C)\cdot \mathbf{b} dl}_{\text{interfacial terms}}\ .$$

The third term on the right side of Eq. (43) can be modified by the use of the Gauss's theorem for spatial domains (Kaplan [6]). Thus one gets

$$\int\limits_R \eta(\mathbf{V}-\mathbf{V}_R)\cdot \mathbf{n} dA =$$
$$= \int\limits_V \mathrm{div}\big[\eta(\mathbf{V}-\mathbf{V}_{\Re})\big] dV + \int\limits_S [\![\eta_{\triangleright S}(\mathbf{V}_{\triangleright S}-\mathbf{U})\cdot \boldsymbol{\zeta}]\!]\, dA\ , \tag{44}$$

where

$$\int\limits_S \eta_{\triangleright S}(\mathbf{V}_{\triangleright S}-\mathbf{U})\cdot \boldsymbol{\zeta} dA =$$
$$= \sum_k^K \int\limits_{S_k} \big[\eta^+_{\triangleright S_k}(\mathbf{V}^+_{\triangleright S_k}-\mathbf{U}_k)\cdot \boldsymbol{\zeta}^+_k + \eta^-_{\triangleright S_k}(\mathbf{V}^-_{\triangleright S_k}-\mathbf{U}_k)\cdot \boldsymbol{\zeta}^-_k\big] dA \tag{45}$$

and superscripts '+' and '–' refer to spatial properties of the phases on either side of interface, respectively. Subscript $\triangleright S_k$ means that value of a phasic property referenced is taken at infinitesimally close spatial position to S_k. Substitution of Eq. (44) into Eq. (43) yields the final form of the RTT attempted as follows:

$$\underbrace{\frac{d\Phi_\Sigma}{dt}}_{\substack{\text{accumulation}\\ \text{in } \Sigma}} = \underbrace{\int\limits_V \frac{\partial \eta}{\partial t} dV + \int\limits_V \operatorname{div}(\eta \mathbf{V}_\Re) dV}_{\text{accumulation in moving spatial phasic domains of volume } V} + \underbrace{\int\limits_V \operatorname{div}\big[\eta(\mathbf{V}-\mathbf{V}_\Re)\big] dA}_{\text{transport by movement of spatial subsystems across } \Re} +$$

$$+ \underbrace{\int\limits_S \Big(\frac{\partial \eta_s}{\partial t} + \eta_s \operatorname{div}_s \mathbf{U}\Big) dA + \int\limits_C \eta_s (\mathbf{V}_C - \mathbf{U}) \cdot \mathbf{b} dl}_{\text{accumulation in interfacial domains of area } S} +$$

$$+ \underbrace{\int\limits_C \eta_s (\mathbf{V}_s - \mathbf{U}_C) \cdot \mathbf{b} dl}_{\text{transport by movement of surface subsystems across bounding curves } C} + \underbrace{\int\limits_S [\![\eta_{\triangleright S} (\mathbf{V}_{\triangleright S} - \mathbf{U}) \cdot \boldsymbol{\zeta}]\!] dA}_{\text{transport between interface } S \text{ and bulk phases}} \quad (46)$$

Note in Eq. (46) the spatial divergence operator div and surface divergence operator div_s are different, see formulas for these operators given by Slattery [11]. The form of RTT given by Eq. (46) is the most general because it expresses the rate of accumulation in a three phase system in terms of moving and deformable CV of arbitrary prescribed configuration in which EQ can be accumulated both by the phases and interface. Worthy to mention is applicability of RTT relation (46) also to multiphase systems provided that particular terms can account in proper number of spatial and surface subsystems involved.

3 Particular forms of RTT for multiphase systems

A general form of the RTT given by Eq. (46) can be easy modified to express specific RTT formulations valid for many problems of common practical interest.

3.1 RTT for three phase systems in terms of fixed CV

Diverse modelling of multiphase flows can be developed using CV of fixed configuration. The RTT for such practical cases can be derived by application of Eq. (46) in which the case of fixed CV implies that:

- external boundaries R_i, $i = 1, 2, 3$, are fixed, hence $\mathbf{V}_{R_1} = \mathbf{V}_{R_2} = \mathbf{V}_{R_3} = 0$, and thus one gets

$$\int\limits_V \mathrm{div}(\eta\mathbf{V}_{\Re})dV + \int\limits_V \mathrm{div}\big[\eta(\mathbf{V}-\mathbf{V}_{\Re})\big]dV + \int\limits_S [\![\eta_{\triangleright S}(\mathbf{V}_{\triangleright S}-\mathbf{U})\cdot\boldsymbol{\zeta}]\!]\,dA = \\ = \int\limits_V \mathrm{div}(\eta\mathbf{V})dV + \int\limits_S [\![\eta_{\triangleright S}(\mathbf{V}_{\triangleright S}-\mathbf{U})\cdot\boldsymbol{\zeta}]\!]\,dA\,, \qquad (47)$$

- boundary curves C_k move at $\mathbf{V}_{C_k} = \mathbf{U}_k$, $k = 1, \ldots, K$, what results in the following line integral of Eq. (46) vanishes, hence

$$\int\limits_C \eta_s(\mathbf{V}_C - \mathbf{U})\cdot\mathbf{b}dl = 0 \qquad (48)$$

and subsequent line integral becomes

$$\int\limits_C \eta_s(\mathbf{V}_s - \mathbf{V}_C)\cdot\mathbf{b}dl = \int\limits_C \eta_s(\mathbf{V}_s - \mathbf{U})\cdot\mathbf{b}dl\,. \qquad (49)$$

Expression (49) can be transformed by the use of the Green's theorem (for surface domains surrounded by a curve) based on which a suitable surface integral defined on the surface vector field can be related to the line integral along a closed curve bounding the surface (Aris [1]) as follows:

$$\int\limits_C \eta_s(\mathbf{V}_s - \mathbf{U})\cdot\mathbf{b}dl = \int\limits_S \mathrm{div}_s(\eta_s\dot{\mathbf{y}})dA\,, \qquad (50)$$

where

$$\int\limits_S \mathrm{div}_s(\eta_s\dot{\mathbf{y}})dA = \sum_k^K \int\limits_{S_k} \mathrm{div}_s(\eta_{s,k}\dot{\mathbf{y}}_k)dA \qquad (51)$$

and

$$\dot{\mathbf{y}}_k = \mathbf{V}_{s,k} - \mathbf{U}_k \qquad (52)$$

is the relative velocity $\dot{\mathbf{y}}_k$ of the surface system $\Sigma_{s,k}$ wholly defined in the surface S_k. Now Eq. (46) can be modified with the use of Eqs. (47)–(51) to obtain the RTT for three phase systems in terms of the fixed CV as:

$$\begin{aligned}
\underbrace{\frac{d\Phi_\Sigma}{dt}}_{\text{accumulation in }\Sigma} &= \underbrace{\int\limits_V \frac{\partial \eta}{\partial t} dV}_{\text{accumulation in phasic volumes of fixed } CV} + \underbrace{\int\limits_V \operatorname{div}(\eta \mathbf{V}) dV}_{\text{transport across boundaries of phasic volumes of fixed } CV} + \\
&+ \underbrace{\int\limits_S \Big(\frac{\partial \eta_s}{\partial t} + \eta_s \operatorname{div}_s \mathbf{U}\Big) dA}_{\text{accumulation in moving interface } S \text{ within fixed } CV} + \underbrace{\int\limits_S \operatorname{div}_s \big[\eta_s (\mathbf{V}_s - \mathbf{U}\big] dA}_{\text{transport across bounding curve } C \text{ moving along fixed } CS} + \\
&+ \underbrace{\int\limits_S [\![\eta_{\triangleright S} (\mathbf{V}_{\triangleright S} - \mathbf{U}) \cdot \boldsymbol{\zeta}]\!] \, dA}_{\text{transport between interface } S \text{ and bulk phases}}
\end{aligned} \tag{53}$$

Taking into account Eq. (52) one can express Eq. (53) in a form

$$\begin{aligned}
\frac{d\Phi_\Sigma}{dt} &= \int\limits_V \frac{\partial \eta}{\partial t} dV + \int\limits_V \operatorname{div}(\eta \mathbf{V}) dV + \\
&+ \int\limits_S \Big(\frac{\partial \eta_s}{\partial t} + \operatorname{grad}_s \eta_s \cdot \dot{\mathbf{y}} + \eta_s \operatorname{div}_s \mathbf{V}_s\Big) dA + \\
&+ \int\limits_S [\![\eta_{\triangleright S} (\mathbf{V}_{\triangleright S} - \mathbf{U}) \cdot \boldsymbol{\zeta}]\!] \, dA \,,
\end{aligned} \tag{54}$$

where term $\operatorname{grad}_s \eta_s \cdot \dot{\mathbf{y}} + \eta_s \operatorname{div}_s \mathbf{V}_s$ can be also written as

$$\operatorname{grad}_s \eta_s \cdot \dot{\mathbf{y}} + \eta_s \operatorname{div}_s \mathbf{V}_s = - \operatorname{grad}_s \eta_s \cdot \mathbf{U} + \operatorname{div}_s (\eta_s \mathbf{V}_s) \,. \tag{55}$$

Now the surface divergence theorem (Slattery [11]) can be applied to term $\int\limits_S \operatorname{div}_s (\eta_s \mathbf{V}_s) dA$ what gives

$$\int\limits_S \operatorname{div}_s (\eta_s \mathbf{V}_s) dA = \int\limits_C \eta_s \mathbf{V}_s \cdot \mathbf{b} dl - \int\limits_S 2H \eta_s \mathbf{V}_s \cdot \boldsymbol{\zeta} dA \,. \tag{56}$$

Employing the results of Eqs. (55) and (56) into Eq. (54) yields the RTT in a form

$$\begin{aligned}\frac{d\Phi_\Sigma}{dt} &= \int\limits_V \frac{\partial \eta}{\partial t} dV + \int\limits_V \mathrm{div}(\eta \mathbf{V}) dV + \int\limits_C \eta_s \mathbf{V}_s \cdot \mathbf{b} dl + \\ &+ \int\limits_S \left\{ \frac{\partial \eta_s}{\partial t} + \mathrm{grad}_s \eta_s \cdot \mathbf{U} - 2H\eta_s \mathbf{V}_s \cdot \boldsymbol{\zeta} \right\} dA + \\ &+ \int\limits_S [\![\eta_{\triangleright S}(\mathbf{V}_{\triangleright S} - \mathbf{U}) \cdot \boldsymbol{\zeta}]\!] \, dA \ . \end{aligned} \tag{57}$$

Worthy to mention the RTT given by Eq. (57) is of the same form as that proposed by Slattery [11]. Note that Eq. (57) remains also valid for multiphase systems provided that sums determining particular terms are extended to proper number of phases and interfaces involved. If the interfaces involved do not possess any storage abilities then $\eta_s = 0$ what substituted into Eq. (57) gives

$$\frac{d\Phi_\Sigma}{dt} = \int\limits_V \frac{\partial \eta}{\partial t} dV + \int\limits_V \mathrm{div}(\eta \mathbf{V}) dV + \int\limits_S [\![\eta_{\triangleright S}(\mathbf{V}_{\triangleright S} - \mathbf{U}) \cdot \boldsymbol{\zeta}]\!] \, dA \ . \tag{58}$$

Expression (58) is the RTT given in terms of fixed CV for multiphase systems with interfaces of negligible storage abilities. In turn the RTT of expression (58) is widely applied to formulate differential forms of phasic and interfacial balances which are well known as the jump conditions (Drew and Passman [2], Ishii and Hibiki [5]) for bulk phasic properties at the phase interface.

3.2 RTT for surface systems

Let us consider an interface S located in between two spatial bulk systems (say '+' and '–') and surrounded by bounding curve C. The interface is an open surface system in motion with flow of surface subsystems (fluid particles) within and exchanges EQ with its surficial surroundings by flow across curve C as well as with its spatial surroundings due to flow across the superficial sides. Let $\mathbf{V}_s$ be the spatial velocity of the surface system flowing in S and $\mathbf{U}$ is velocity of interface S. Taking into account the only interfacial terms shown in Eq. (53) the RTT for the case in point becomes

$$\frac{d\Phi_{\Sigma_s}}{dt} = \underbrace{\int_S \frac{\partial \eta_s}{\partial t} dA}_{\text{(a)}} + \underbrace{\int_S \eta_s \,\mathrm{div}_s \mathbf{U} dA}_{\text{(b)}} + \underbrace{\int_S \mathrm{div}_s(\eta_s \dot{\mathbf{y}}) dA}_{\text{(c)}} + \underbrace{\int_S \left[\eta^+_{\triangleright S}(\mathbf{V}^+_{\triangleright S} - \mathbf{U}) \cdot \boldsymbol{\zeta}^+ + \eta^-_{\triangleright S}(\mathbf{V}^-_{\triangleright S} - \mathbf{U}) \cdot \boldsymbol{\zeta}^-\right] dA}_{\text{(d)}} \; . \qquad (59)$$

Equation (59) expresses rate of accumulation in the surface system with reference to interface moving at $\mathbf{U}$. The corresponding terms are: accumulation rate in the "frozen" interface (a), rate of accumulation due to interface stretching (b), transport across bounding curve C (c) and transport between the interface and bulk phases (d). Two specific issues can result from relation (59). If $\mathbf{U} \cdot \boldsymbol{\zeta}^+ = \mathbf{U} \cdot \boldsymbol{\zeta}^- = 0$ the interface is stationary (Slattery [11]) what subsequently implemented into Eq. (59) leads to

$$\frac{d\Phi_{\Sigma_s}}{dt} = \int_S \left[\frac{\partial \eta_s}{\partial t} + \eta_s \,\mathrm{div}_s \mathbf{U} + \mathrm{div}_s(\eta_s \dot{\mathbf{y}})\right] dA + \int_S (\eta^+_{\triangleright S} \mathbf{V}^+_{\triangleright S} \cdot \boldsymbol{\zeta}^+ + \eta^-_{\triangleright S} \mathbf{V}^-_{\triangleright S} \cdot \boldsymbol{\zeta}^-) dA \; . \qquad (60)$$

However, a case in which there is no exchange of EQ between phases and interface is perhaps more common. This situation results in $\mathbf{V}^+_{\triangleright S} \cdot \boldsymbol{\zeta}^+ = \mathbf{U} \cdot \boldsymbol{\zeta}^+$ and $\mathbf{V}^-_{\triangleright S} \cdot \boldsymbol{\zeta}^- = \mathbf{U} \cdot \boldsymbol{\zeta}^-$. Consequently, relation (59) reduces to

$$\frac{d\Phi_{\Sigma_s}}{dt} = \int_S \left[\frac{\partial \eta_s}{\partial t} + \eta_s \,\mathrm{div}_s \mathbf{U} + \mathrm{div}_s(\eta_s \dot{\mathbf{y}})\right] dA \; . \qquad (61)$$

Note, surface integral (corresponding integrand is $\mathrm{div}_s(\eta_s \dot{\mathbf{y}})$) describing transport term of Eq. (61) can be transformed to the line integral by application of Green's theorem of Eq. (50), thus one gets

$$\underbrace{\frac{d\Phi_{\Sigma_s}}{dt}}_{\substack{\text{accumulation in}\\ \text{surface system}}} = \underbrace{\int\limits_S \Big(\frac{\partial \eta_s}{\partial t} + \eta_s \,\mathrm{div}_s \mathbf{U}\Big)\sqrt{a}dy^1 dy^2}_{\text{acculumation in moving interface of area } S} + \underbrace{\int\limits_C \eta_s(\mathbf{V}_s - \mathbf{U})\cdot \mathbf{b} dl}_{\substack{\text{transport across bounding curve}\\ C \text{ surrounding moving interface of}\\ \text{area } S}} \tag{62}$$

Equations (61) and (62) express rate of accumulation in the surface system in terms of interface S moving at $\mathbf{U}$ at no phasic flow interactions. Note, Eq. (61) obtained in this paper as a particular case of the RTT given by Eq. (46) takes the same form as those given by Aris [1], Slattery [11], Ishii and Hibiki [5]. Since the relative velocity $\dot{\mathbf{y}}$ is defined by Eq. (52) then Eq. (61) can be also written as:

$$\frac{d\Phi_{\Sigma_s}}{dt} = \int\limits_S \Big(\frac{D_s \eta_s}{Dt} + \eta_s \,\mathrm{div}_s \mathbf{V}_s\Big)\sqrt{a}dy^1 dy^2 \ , \tag{63}$$

where the surface material derivative is given by (see Slattery [11])

$$\frac{D_s \eta_s}{Dt} = \frac{\partial \eta_s}{\partial t} = \mathrm{grad}_s \eta_s \cdot \dot{\mathbf{y}} \ . \tag{64}$$

If the surface system is fixed with interface S then $\mathbf{V}_s$=$\mathbf{U}$, hence $\dot{\mathbf{y}} = 0$. Consequently Eq. (62) becomes as:

$$\frac{d\Phi_{\Sigma_s}}{dt} = \int\limits_S \Big(\frac{\partial \eta_s}{\partial t} + \eta_s \,\mathrm{div}_s \mathbf{U}\Big)\sqrt{a}dy^1 dy^2 \ . \tag{65}$$

Equation (65) expresses the rate of change in the thermodynamic surface system fixed with respect to interface in terms of spatial movement of the interface. If $\mathbf{U} = 0$ the interface is fixed. Subsequently Eq. (62) with the help of Eq. (50) becomes

$$\frac{d\Phi_{\Sigma_s}}{dt} = \int\limits_S \Big[\frac{\partial \eta_s}{\partial t} + \mathrm{div}_s(\eta_s \mathbf{V}_s)\Big]\sqrt{a}dy^1 dy^2 \ , \tag{66}$$

what expresses to the rate of change in the thermodynamic surface system moving at $\mathbf{V}_s = \dot{\mathbf{y}}$ in a fixed interface. If, however, $\eta_s = const.$ then Eq. (66) results in

$$\frac{d\Phi_{\Sigma_s}}{dt} = \int_S \eta_s \, \mathrm{div}_s \mathbf{V}_s \sqrt{a} dy^1 dy^2 \; . \tag{67}$$

3.3 Cases of homogenous spatial systems

Now let the system Σ be a spatial pure homogeneous. Hence the RTT for this limit developed in terms of moving and deforming CV becomes to be also given by Eq. (46) if all the interfacial terms vanish. In turn Eq. (46) with substitution $CV = \sum_i V_i$ surrounded by $CS = \sum_i R_i$ becomes the RTT for homogeneous spatial systems expressed in terms of moving CV as

$$\frac{d\Phi_\Sigma}{dt} = \int_{CV} \frac{\partial \eta}{\partial t} dV + \int_{CV} \mathrm{div}(\eta \mathbf{V}_{CS}) dV + \int_{CV} \mathrm{div}[\eta(\mathbf{V} - \mathbf{V}_{CS})] dV \; . \tag{68}$$

The case of a fixed CV can be frequently of particular interest. Then $\mathbf{V}_{CS} = 0$, what results that the limits of integration become fixed also. In turn, by substitution $\mathbf{V}_{CS} = 0$ into Eq. (68) one gets a form of RTT as:

$$\frac{d\Phi_\Sigma}{dt} = \int_{CV} \frac{\partial \eta}{\partial t} dV + \int_{CV} \mathrm{div}(\eta \mathbf{V}) dV \tag{69}$$

expressing the rate of accumulation in moving homogeneous spatial systems in terms of fixed CV. Note, that Eqs. (68) and (69) are of the same forms as given for the case in well-known popular textbooks, e.g. Munson, Young and Okiishi [9], White [13].

4 Concluding remarks

The RTT is a basic tool in development of the local instantaneous model equations together with corresponding jump conditions based on which averaged models can be derived. The form of RTT given by Eq. (46) is the most general because it expresses the rate of accumulation in a three phase system in terms of moving and deformable CV of arbitrary prescribed configuration in which EQ can be accumulated both within the phases and interfaces. Based on the general form of RTT as derived in the paper a few forms of the RTT are also formulated such as for three phase systems in terms of fixed CV and for a few cases of surface systems. Worthy to mention is applicability of RTT relation developed also to multiphase systems provided that particular terms can account in contributions done by all spatial and surface subsystems involved.

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