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Generalizing trade–off directions in multiobjective optimization*

by

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Abstract: We consider a general multiobjective optimization problem with five basic optimality principles: efficiency, weak and proper Pareto optimality, strong efficiency and lexicographic optimality. We generalize the concept of trade-off directions defining them as some optimal surface of appropriate cones. In convex optimization, the contingent cone can be used for all optimality principles except lexicographic optimality, where the cone of feasible directions is useful. In nonconvex case the contingent cone and the cone of locally feasible directions with lexicographic optimality are helpful. We derive necessary and sufficient geometrical optimality conditions in terms of corresponding trade-off directions for both convex and nonconvex cases.

Keywords: generalized trade-off directions, multiobjective optimization, geometrical characterization, convex and nonconvex optimization, optimality principles

1. Introduction

The overall goal in multiobjective optimization is to find a compromise between several conflicting objectives which is best-fit to the needs of a decision maker. This compromise is usually referred to as an optimality principle. Various mathematical definitions of the optimality principle can be derived in several different ways depending on the needs of the solution approaches used. Moreover, sometimes the use of one definition may be more advantageous than some other due to computational complexity reasons.

The usage of trade-offs as a tool containing essential information about compromise has been suggested in a series of papers (see, e.g., Sakawa and Yano, 1990), where certain scalarizing functions were used to define the concept. Another approach, proposed in Kaliszewski and Michalowski (1995, 1997) consists in generating solutions

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satisfying some pre-specified bounds on trade-off information by means of a scalarizing function. In Henig and Buchanan (1997) for convex (including nondifferentiable) problems, the concept of trade-offs has been generalized into a cone of trade-off directions, which was defined as a Pareto optimal surface of a contingent (tangent) cone located at the point considered.

The usage of contingent and normal cones as well as the cone of feasible directions is a natural choice in the case of convex optimization (see, e.g., Rockafellar, 1970, 1981). In nonconvex optimization, the main difficulty arises due to the fact that the contingent cone as well as the cone of feasible directions may lose convexity. Two additional types of cones have been shown to be helpful - tangent cone and cone of local feasible directions (see, e.g., Clarke, 1983). The guaranteed property of convexity of these cones assures that they can be used to overcome some difficulties which appear in nonconvex optimization. However, in nonconvex case, tangent cones do not necessarily represent the shape of the set considered even locally and the relation to trade-off directions is lost. Therefore, to define trade-off directions in nonconvex case, we must use nonconvex contingent cones as it was suggested originally in Lee and Nakayama (1997) for smooth problems and later generalized for not necessarily differentiable problems in Miettinen and Mäkelä (2002).

The aim of this paper is to describe necessary and sufficient optimality conditions in terms of trade-off directions for both convex and nonconvex cases. The paper is organized as follows. In Section 2, we formulate a general multiobjective problem and introduce five basic optimality principles, which are the most common in multiobjective optimization. We give traditional definitions and geometrical ones via appropriate cones. For every optimality principle considered, we define generalized trade-off directions for convex and nonconvex cases in Section 3. Giving up convexity naturally means that we need local instead of global analysis. Section 4 presents the main results showing interrelation between optimal solutions and corresponding generalized trade-off directions. The results are presented for convex and nonconvex cases and summarized in two schemes. Section 5 is devoted to some illustrative examples in biobjective case. Final remarks appear in Section 6.

2. Basic optimality principles

We consider general multiobjective optimization problems of the following form:

$$\min_{\mathbf{x}\in S} \{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})\},$$

where $f_i : \mathbf{R}^n \to \mathbf{R}$ are *objective functions* for all $i \in I_k := \{1,\ldots,k\}$. The *decision vector* $\mathbf{x}$ belongs to the nonempty *feasible set* $S \subset \mathbf{R}^n$. The image of the feasible set is denoted by $Z \subset \mathbf{R}^k$, i.e. $Z := f(S)$. Elements of Z are termed *objective vectors* and they are denoted by $\mathbf{z} = f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x}))^T$. Additionally, for non-convex case we assume

(i) $f_i : \mathbf{R}^n \to \mathbf{R}$ are continuous for all $i \in I_k$;

(ii) $f(B(\mathbf{x};\varepsilon))$ open for all $\mathbf{x} \in S$ and $\varepsilon > 0$, where $B(\mathbf{x};\varepsilon)$ is *an open ball* with radius ε and center $\mathbf{x}$.

The Minkowski sum of two sets A and E is defined by $A+E = \{a+e \mid a \in A,\ e \in E\}$. The interior, closure, convex hull and complement of a set A are denoted by $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{conv} A$ and A^C, respectively.

A set A is a *cone* if $\lambda \mathbf{x} \in A$ whenever $\mathbf{x} \in A$ and $\lambda > 0$. We denote the positive orthant of $\mathbf{R}^k$ by $\mathbf{R}^k_+ = \{\mathbf{d} \in \mathbf{R}^k \mid d_i \geq 0 \text{ for every } i \in I_k\}$. The positive orthant is also known as *standard ordering cone*. The negative orthant $\mathbf{R}^k_-$ is defined respectively. Note, that $\mathbf{R}^k_-$ and $\mathbf{R}^k_+$ are closed convex cones.

In what follows, the notation $\mathbf{z} < \mathbf{y}$ for $\mathbf{z}, \mathbf{y} \in \mathbf{R}^k$ means that $z_i < y_i$ for every $i \in I_k$ and, correspondingly, $\mathbf{z} \leq \mathbf{y}$ stands for $z_i \leq y_i$ for every $i \in I_k$.

Simultaneous optimization of several objectives for multiobjective optimization problem is not a straightforward task. Contrary to the the single objective case, the concept of optimality is not unique in multiobjective cases.

Below we give traditional definitions of five well-known and most fundamental principles of optimality (see, e.g., Ehrgott, 2005; Henig, 1982; Miettinen, 1999).

Weak Pareto Optimality. An objective vector $\mathbf{z}^* \in Z$ is *weakly Pareto optimal* if there does not exist another objective vector $\mathbf{z} \in Z$ such that $z_i < z_i^*$ for all $i \in I_k$.

Pareto optimality or efficiency. An objective vector $\mathbf{z}^* \in Z$ is *Pareto optimal* or *efficient* if there does not exist another objective vector $\mathbf{z} \in Z$ such that $z_i \leq z_i^*$ for all $i \in I_k$ and $z_j < z_j^*$ for at least one index $j \in I_k$.

Proper Pareto Optimality. An objective vector $\mathbf{z}^* \in Z$ is *properly Pareto optimal* if there exists no objective vector $\mathbf{z} \in Z$ such that $\mathbf{z} \in Z^* + C$ for some convex cone C, $\mathbf{R}^k_- \setminus \{0\} \subset \operatorname{int} C$, attached to $\mathbf{z}^*$. Any Pareto optimal objective vector which is not properly Pareto optimal is called *improperly Pareto optimal*.

Strong Efficiency. An objective vector $\mathbf{z}^* \in Z$ is *strongly Pareto optimal* if for all $i \in I_k$ there exists no objective vector $\mathbf{z} \in Z$ such that $z_i < z_i^*$ or, in other words, $\mathbf{z}^* \in Z$ optimizes all $z_i,\ i \in I_k$.

Lexicographic Optimality. An objective vector $\mathbf{z}^* \in Z$ is *lexicographically optimal* if for all other objective vector $\mathbf{z} \in Z$ one of the following two conditions holds:
1) $\mathbf{z} = \mathbf{z}^*$
2) $\exists\, i \in I_k :\ (z_i^* < z_i) \wedge (\forall j \in I_{i-1} :\ z_j^* = z_j)$, where $I_0 = \emptyset$.

Next we redefine the five sets of efficient solutions by using appropriate ordering cones. It is trivial to verify that the definitions of optimality and efficiency formulated above are equivalent to those following below.

DEFINITION 1 *The weakly Pareto optimal set is*

$$WP(Z) := \{\mathbf{z} \in Z \mid (\mathbf{z} + \operatorname{int} \mathbf{R}^k_-) \cap Z = \emptyset\};$$

the Pareto optimal set is

$$PO(Z) := \{\mathbf{z} \in Z \mid (\mathbf{z} + \mathbf{R}^k_- \setminus \{0\}) \cap Z = \emptyset\};$$

the properly Pareto optimal set is defined as

$$PP(Z) := \{\mathbf{z} \in Z \mid (\mathbf{z} + C \setminus \{0\}) \cap Z = \emptyset\}$$

for some convex cone C (chosen beforehand for all $\mathbf{z} \in Z$*) such that* $\mathbf{R}^k_- \setminus \{0\} \subset \operatorname{int} C$*; the strongly efficient set is*

$$SE(Z) := \{\mathbf{z} \in Z \mid (\mathbf{z} + (\mathbf{R}^k_+)^C) \cap Z = \emptyset\};$$

and the lexicographically optimal set is

$$LO(Z) = \{\mathbf{z} \in Z \mid (\mathbf{z} + (C^k_{\text{lex}})^C) \cap Z = \emptyset\},$$

where $(C^k_{\text{lex}})^C$ *is a complement cone to the lexicographic cone which is defined as follows*

$$C^k_{\text{lex}} := \{0\} \cup \{\mathbf{d} \in \mathbf{R}^k \mid \exists i \in I_k \text{ such that } d_i > 0 \text{ and } d_j = 0 \; \forall j < i\}.$$

Note that

$$SE(Z) \subset PP(Z) \subset PO(Z) \subset WP(Z),$$

and

$$LO(Z) \subset PP(Z) \subset PO(Z) \subset WP(Z).$$

The corresponding local analogues of the five optimality sets $LWP(Z)$, $LPO(Z)$, $LPP(Z)$, $LSE(Z)$, $LLO(Z)$ can be defined in a similar way if we assume that the corresponding optimality conditions hold within some open ball $f(B(\mathbf{x};\delta) \cap S)$, $\delta > 0$. To guarantee the existence of an open neighborhood, we use in nonconvex case two additional assumptions (**i**) and (**ii**) on function $f(\mathbf{x})$. In a convex case, local and global concepts are equal. Note that $LSE(Z) \subset LPP(Z) \subset LPO(Z) \subset LWP(Z)$ and $LSE(Z) \subset LLO(Z) \subset LPO(Z) \subset LWP(Z)$.

3. Generalized trade-off directions

The concept of trade-offs in multiobjective optimization is a key point to define compromise between conflicting objectives. It can be used to describe solutions which linearly approximate the feasible region and which are mutually non-dominated with respect to the given optimality principle. The trade-off directions can be used in many algorithms requiring specifying directions which may lead fast to the solution that is most preferred by the decision maker (see e.g. Branke et al., 2008 and Miettinen, 1999).

Since the contingent cones linearly approximate the shape of the feasible set, equally well in both convex (global approximation) and nonconvex (local approximation) cases, they can be used to define the generalized trade-off directions. A (weakly) Pareto surface of the contingent cone serves for that purpose.

Next we define several geometrical basic cones (see, e.g., Rockafellar, 1970).

DEFINITION 2 *The contingent cone of a set* $Z \subset \mathbf{R}^k$ *at* $\mathbf{z} \in Z$ *is defined as*

$$K_{\mathbf{z}}(Z) := \{\mathbf{d} \in \mathbf{R}^k \mid \text{ there exist } t_j \searrow 0 \text{ and } \mathbf{d}_j \to \mathbf{d} \text{ such that } \mathbf{z} + t_j \cdot \mathbf{d}_j \in Z\}.$$

DEFINITION 3 *The cone of feasible directions of a set* $Z \subset \mathbf{R}^k$ *at* $\mathbf{z} \in Z$ *is denoted by*

$$D_{\mathbf{z}}(Z) := \{\mathbf{d} \in \mathbf{R}^k \mid \text{ there exists } t > 0 \text{ such that } \mathbf{z} + t \cdot \mathbf{d} \in Z\}.$$

We introduce the following definition which provides regularity condition for Z at $\mathbf{z} \in Z$.

DEFINITION 4 *(see, e.g., Aubin, Frankowska, 2008) The set Z is called regular at $\mathbf{z} \in Z$ if $D_{\mathbf{z}}(Z) = K_{\mathbf{z}}(Z)$.*

Notice that regularity condition is equivalent to the Karush-Kuhn-Tucker regularity condition (or the so-called KKT constraint qualification) $D_{\mathbf{z}}(Z) = \text{cl}\, A_{\mathbf{z}}(Z)$, where $A_{\mathbf{z}}(Z)$ is a cone of attainable directions, which is also known as inner contingent cone, i.e. $\text{cl}\, A_{\mathbf{z}}(Z) = K_{\mathbf{z}}(Z)$ (see, e.g., Bazaraa, Sherali, Shetty, 2006).

In nonconvex case, the cone of feasible directions $D_{\mathbf{z}}(Z)$ does not describe the shape of Z locally. Thus, we introduce a cone of locally feasible directions, which reflects the shape of Z locally (see, e.g., Mäkelä, Neittaanmäki, 1992).

DEFINITION 5 *The cone of locally feasible directions of a set $Z \subset \mathbf{R}^k$ at $\mathbf{z} \in Z$ is denoted by*

$$F_{\mathbf{z}}(Z) = \{\mathbf{d} \in \mathbf{R}^k \mid \text{ there exists } t > 0 \text{ such that } \mathbf{z} + \tau \cdot \mathbf{d} \in Z \text{ for all } \tau \in (0,t]\}.$$

The following definition provides local regularity condition for Z at $\mathbf{z} \in Z$.

DEFINITION 6 *The set Z is called locally regular at $\mathbf{z} \in Z$ if $F_{\mathbf{z}}(Z) = K_{\mathbf{z}}(Z)$.*

For nonconvex cases, Clarke (1983) has defined a convex tangent cone in the following way:

DEFINITION 7 *The tangent cone of a set $Z \subset \mathbf{R}^k$ at $\mathbf{z} \in Z$ is given by the formula*

$$\begin{gathered} T_{\mathbf{z}}(Z) = \{\mathbf{d} \in \mathbf{R}^k \mid \\ \text{for all } t_j \searrow 0 \text{ and } \mathbf{z}_j \to \mathbf{z} \text{ with } \mathbf{z}_j \in Z, \\ \text{there exists } \mathbf{d}_j \to \mathbf{d} \text{ with } \mathbf{z}_j + t_j \cdot \mathbf{d}_j \in Z\}. \end{gathered}$$

The following basic relations can be derived from the definitions of the concepts used in Mäkelä, Neittaanmäki (1992), and Rockafellar (1981).

LEMMA 1 *For the cones $K_{\mathbf{z}}(Z)$, $D_{\mathbf{z}}(Z)$, $T_{\mathbf{z}}(Z)$ and $F_{\mathbf{z}}(Z)$ we have the following*

a) *$K_{\mathbf{z}}(Z)$ and $T_{\mathbf{z}}(Z)$ are closed and $T_{\mathbf{z}}(Z)$ is convex.*
b) $0 \in K_{\mathbf{z}}(Z) \cap D_{\mathbf{z}}(Z) \cap T_{\mathbf{z}}(Z) \cap F_{\mathbf{z}}(Z)$.
c) $Z \subset \mathbf{z} + D_{\mathbf{z}}(Z)$.
d) $\text{cl}\, F_{\mathbf{z}}(Z) \subset K_{\mathbf{z}}(Z) \subset \text{cl}\, D_{\mathbf{z}}(Z)$.
e) $T_{\mathbf{z}}(Z) \subset K_{\mathbf{z}}(Z)$.
f) *If Z is convex, then $\text{cl}\, F_{\mathbf{z}}(Z) = T_{\mathbf{z}}(Z) = K_{\mathbf{z}}(Z) = \text{cl}\, D_{\mathbf{z}}(Z)$. Moreover $F_{\mathbf{z}}(Z) = D_{\mathbf{z}}(Z)$.*

Note that, under convexity assumption, for any $\mathbf{z} \in Z$ we have $\text{cl}\, F_{\mathbf{z}}(Z) = K_{\mathbf{z}}(Z)$ (see, e.g., Rockafellar, 1981), i.e. local regularity defines a bit stronger requirement on a local structure of a set than the convexity assumption. At the same time local regularity does not necessarily imply $\text{cl}\, D_{\mathbf{z}}(Z) = K_{\mathbf{z}}(Z)$, the condition which is guaranteed under

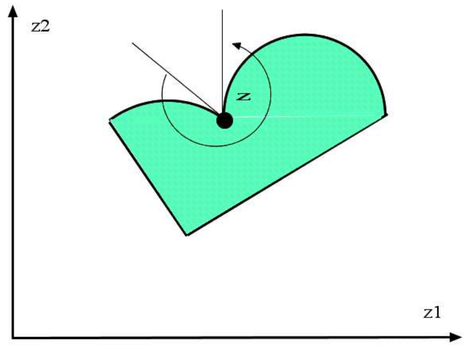

Figure 1. Nonconvex contingent cone $K_{\mathbf{z}}(Z)$

$\mathrm{cl}\, D_{\mathbf{z}}(Z) = K_{\mathbf{z}}(Z)$	$\Leftarrow$	Regularity
$\Uparrow$		
Convexity	$\Rightarrow$	Tangent regularity
$\Downarrow$		
$\mathrm{cl}\, F_{\mathbf{z}}(Z) = K_{\mathbf{z}}(Z)$	$\Leftarrow$	Local regularity

Figure 2. Interconnection between various types of regularity

convexity assumption. For more advance properties of tangent and contingent cones as well as some other related concepts of cones, the reader may consult, e.g., Aubin, Frankowska (2008).

Even though contingent cones are generally nonconvex, their convexity is guaranteed under special circumstances (see, e.g., Aubin, Frankowska, 2008 and Rockafellar, 1970).

DEFINITION 8 *The set Z is called tangentially regular at $\mathbf{z} \in Z$ if $T_{\mathbf{z}}(Z) = K_{\mathbf{z}}(Z)$.*

Trivially, we can see that e.g. convex sets are always tangentially regular.

Note that in order to formulate some of optimality conditions we use four different assumptions about structural properties of Z - convexity, tangent regularity, regularity and local regularity. In general, all these are different and do not directly imply each other. The interconnections between the four regularity assumptions are presented in Fig. 2. Also note that assuming all three types of regularity may not guarantee the property of convexity as Fig. 3 shows. Indeed, in this example all four cones are the same (contingent cone, tangent cone as well as cones of feasible and locally feasible directions), and they are equal to the half-space which is located above (including the tangent line itself) the tangent line at $\mathbf{z} \in Z$.

The sets of generalized trade-off directions can be defined as follows

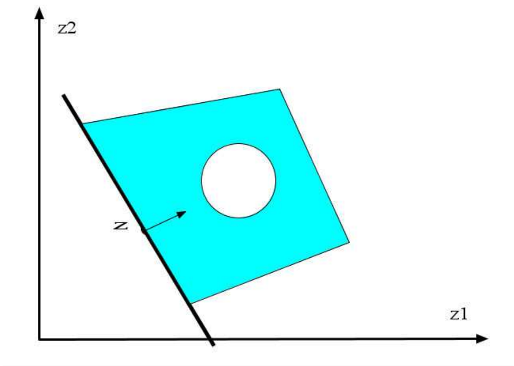

Figure 3. Z is regular, tangentially and locally regular at $\mathbf{z} \in Z$, but not convex

DEFINITION 9 *The sets of generalized trade-off directions are defined as follows:*
- *in case of weak Pareto optimality:* $G_{WP}(Z) := WP(K_{\mathbf{z}}(Z))$;
- *in case of Pareto optimality (efficiency):* $G_{PO}(Z) := PO(K_{\mathbf{z}}(Z))$;
- *in case of proper Pareto optimality:* $G_{PP}(Z) := PO(K_{\mathbf{z}}(Z))$;
- *in case of strong efficiency:* $G_{SE}(Z) := SE(K_{\mathbf{z}}(Z))$;
- *in case of lexicographic optimality:* $G_{LO}(Z) := LO(F_{\mathbf{z}}(Z))$.

Note that $G_{PO}(Z) = G_{PP}(Z)$ by definition, since Pareto optimality can be seen as an extreme case of proper Pareto optimality with $C = \mathbf{R}^k_-$. It is also easy to see that in convex case $LO(F_{\mathbf{z}}(Z)) = LO(D_{\mathbf{z}}(Z))$ and $SE(K_{\mathbf{z}}(Z)) = SE(D_{\mathbf{z}}(Z))$. This follows directly from the definitions and Lemma 1.

Notice that, since two solutions are considered to be mutually lexicographically non-dominated if they have the same objective vectors, we have to use the cone of feasible directions in the definition of the set of generalized trade-off directions in case with lexicographic optimality. Indeed, the set of generalized trade-off directions in case with local lexicographic optimality is either empty or only one point **0** (zero vector, origin of $F_{\mathbf{z}}(Z)$), so it becomes indifferent if $D_{\mathbf{z}}(Z)$ is closed or open, what is not true in cases with other types of local optimality.

4. Main results

4.1. Convex case

Here we formulate and prove the basic results concerning relations between optimality and the corresponding set of generalized trade-off directions in convex case.

THEOREM 1 *Let Z be convex. If* $\mathbf{z} \in WP(Z)$, *then* $G_{WP}(Z) \neq \emptyset$.

This result directly follows from the result of forthcoming Theorem 6 and the fact that $WP(Z) \subset LWP(Z)$.

THEOREM 2 *Let Z be convex. If $\mathbf{z} \in PO(Z)$, then $G_{PO}(Z) \neq \emptyset$ under assumption that Z is regular. Moreover, $G_{PO}(Z) \neq \emptyset$ implies $\mathbf{z} \in PO(Z)$.*

Proof. Assume $\mathbf{z} \in PO(Z)$. Suppose that $G_{PO}(Z) = \emptyset$. Then $(\mathbf{d}+\mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap K_{\mathbf{z}}(Z) \neq \emptyset$ for all $\mathbf{d} \in K_{\mathbf{z}}(Z)$. Taking $\mathbf{d} = \mathbf{0}$ ($\mathbf{0} \in K_{\mathbf{z}}(Z)$), we get $(\mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap K_{\mathbf{z}}(Z) \neq \emptyset$, and due to regularity assumption $(\mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap D_{\mathbf{z}}(Z) \neq \emptyset$. The last contradicts the initial assumption that $\mathbf{z} \in PO(Z)$ (see Theorem 2, Miettinen, Mäkelä, 2001).

Now assume $\mathbf{y} \in G_{PO}(Z)$, $\mathbf{y} \neq \mathbf{0}$, then $(\mathbf{y}+\mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap K_{\mathbf{z}}(Z) = \emptyset$ and $(\mathbf{y}+\mathbf{R}^k_- \setminus \{\mathbf{0}\})$ $\cap \operatorname{cl} D_{\mathbf{z}}(Z) = \emptyset$ under assumption that Z is convex. Then $(\mathbf{y}+\mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap D_{\mathbf{z}}(Z) = \emptyset$, and hence (due to linearity and convexity of $D_{\mathbf{z}}(Z)$ in convex case), we have $(\mathbf{z}+\mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap D_{\mathbf{z}}(Z) = \emptyset$,. Thus, we have $(\mathbf{z}+\mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap Z = \emptyset$, i.e. $\mathbf{z} \in PO(Z)$. This ends the proof.

THEOREM 3 *Let Z be convex. The solution $\mathbf{z} \in PP(Z)$ if and only if $G_{PP}(Z) \neq \emptyset$.*

This result directly follows from the result of forthcoming Theorem 9 and the fact that convex set is always tangentially regular and the fact that $PP(Z) \subset LPP(Z)$.

THEOREM 4 *Let Z be convex. The solution $\mathbf{z} \in SE(Z)$ if and only if $G_{SE}(Z) \neq \emptyset$, or equivalently $G_{SE}(Z) = \{\mathbf{0}\}$.*

Proof. First we show that $\mathbf{z} \in SE(Z)$ if and only if $\mathbf{0} \in G_{SE}(Z)$. Indeed (see Corollary 3.1, Mäkelä, Nikulin, 2009),

$$\mathbf{z} \in SE(Z) \Leftrightarrow K_{\mathbf{z}}(Z) \cap \mathbf{R}^k_+ = K_{\mathbf{z}}(Z) \Leftrightarrow$$

$$(\mathbf{0} + (\mathbf{R}^k_+)^C) \cap K_{\mathbf{z}}(Z) = \emptyset \Leftrightarrow \mathbf{0} \in G_{SE}(Z).$$

Now it remains to show that if $\mathbf{y} \in K_{\mathbf{z}}(Z)$, $\mathbf{y} \neq \mathbf{0}$, then $\mathbf{y} \notin G_{SE}(Z)$. Indeed, if $\mathbf{y} \in K_{\mathbf{z}}(Z)$, $\mathbf{y} \neq \mathbf{0}$, then $\mathbf{y} \in \mathbf{R}^k_+$. Thus, $\mathbf{0} \in (\mathbf{y} + (\mathbf{R}^k_+)^C) \cap K_{\mathbf{z}}(Z)$, and then $\mathbf{y} \notin G_{SE}(Z)$. This ends the proof.

THEOREM 5 *Let Z be convex. The solution $\mathbf{z} \in LO(Z)$ if and only if $G_{LO}(Z) \neq \emptyset$, or equivalently $G_{LO}(Z) = \{\mathbf{0}\}$.*

Proof. First we show that $\mathbf{z} \in LO(Z)$ if and only if $\mathbf{0} \in G_{LO}(Z)$. Indeed (see Corollary 4.1, Mäkelä, Nikulin, 2009),

$$\mathbf{z} \in LO(Z) \Leftrightarrow D_{\mathbf{z}}(Z) \cap C^k_{lex} = D_{\mathbf{z}}(Z) \Leftrightarrow$$

$$(\mathbf{0} + (C^k_{lex})^C) \cap D_{\mathbf{z}}(Z) = \emptyset \Leftrightarrow \mathbf{0} \in G_{LO}(Z).$$

Now it remains to show that if $\mathbf{d} \in D_{\mathbf{z}}(Z)$, $\mathbf{d} \neq \mathbf{0}$, then $\mathbf{d} \notin G_{LO}(Z)$. Indeed, if $\mathbf{d} \in D_{\mathbf{z}}(Z)$, $\mathbf{d} \neq \mathbf{0}$, then $\mathbf{d} \in C^k_{lex}$ and $-\mathbf{d} \in (C^k_{lex})^C$, i.e. $\mathbf{d} + (-\mathbf{d}) = \mathbf{0} \in D_{\mathbf{z}}(Z)$, and then $(\mathbf{d} + (C^k_{lex})^C) \cap D_{\mathbf{z}}(Z) \neq \emptyset$. Thus, $\mathbf{d} \notin G_{LO}(Z)$. This ends the proof.

4.2. Nonconvex case

Here we formulate and prove the basic results concerning relations between optimality and the corresponding set of generalized trade-off directions in nonconvex case. Notice also that assumption (ii), made at the beginning of the paper and not used at all in convex case, in nonconvex settings is going to play a more significant role. The actual meaning of this assumption is solely technical: it serves to prevent some degenerate cases such as e.g. mapping an open ball to a point.

THEOREM 6 *(see Miettinen, Mäkelä, 2003)*
If $\mathbf{z} \in LWP(Z)$*, then* $G_{WP}(Z) \neq \emptyset$.

To prove the next theorem we need one known result:

THEOREM 7 *(see Miettinen, Mäkelä, 2001)*
If $\mathbf{z} \in LPO(Z)$*, then*

$$(\mathbf{z} + \mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap F_{\mathbf{z}}(Z) = \emptyset.$$

THEOREM 8 *If* $\mathbf{z} \in LPO(Z)$*, then* $G_{PO}(Z) \neq \emptyset$ *under the assumption that* Z *is locally regular. Moreover,* $G_{PO}(Z) \neq \emptyset$ *implies* $\mathbf{z} \in LPO(Z)$ *under the assumptions that* Z *is both locally and tangentially regular.*

Proof. Assume $\mathbf{z} \in LPO(Z)$. Suppose that $G_{PO}(Z) = \emptyset$. Then $(\mathbf{d} + \mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap K_{\mathbf{z}}(Z) \neq \emptyset$ for all $\mathbf{d} \in K_{\mathbf{z}}(Z)$. Taking $\mathbf{d} = \mathbf{0}$ ($\mathbf{0} \in K_{\mathbf{z}}(Z)$), we get $(\mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap K_{\mathbf{z}}(Z) \neq \emptyset$, and due to local regularity assumption $(\mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap F_{\mathbf{z}}(Z) \neq \emptyset$. The last contradicts (due to Theorem 7) the initial assumption that $\mathbf{z} \in LPO(Z)$.

Now assume $\mathbf{y} \in G_{PO}(Z),\ \mathbf{y} \neq \mathbf{0}$; then

$$(\mathbf{y} + \mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap K_{\mathbf{z}}(Z) = \emptyset, \text{ and}$$
$$(\mathbf{y} + \mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap F_{\mathbf{z}}(Z) = \emptyset$$

under the assumption that Z is locally regular. If Z is tangentially regular, then (due to linearity and convexity of $F_{\mathbf{z}}(Z)$ under tangent regularity), we have $(\mathbf{z} + \mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap F_{\mathbf{z}}(Z) = \emptyset$. Thus, we have $\mathbf{z} \in LPO(Z)$. This ends the proof.

THEOREM 9 *(see Miettinen, Mäkelä, 2002) If* $\mathbf{z} \in LPP(Z)$*, then* $G_{PP}(Z) \neq \emptyset$*. Moreover,* $G_{PP}(Z) \neq \emptyset$ *implies* $\mathbf{z} \in LPP(Z)$ *under the assumption that* Z *is tangentially regular.*

To prove the next theorem we are going to use one known result:

THEOREM 10 *(see Mäkelä, Nikulin, 2009) If* $\mathbf{z} \in LSE$*, then*

$$K_{\mathbf{z}}(Z) \cap \mathbf{R}^k_+ = K_{\mathbf{z}}(Z).$$

THEOREM 11 *If* $\mathbf{z} \in LSE(Z)$*, then* $G_{SE}(Z) \neq \emptyset$*, or equivalently* $G_{SE}(Z) = \{\mathbf{0}\}$.

Proof. Let $\mathbf{z} \in LSE(Z)$. Then by Theorem 10

$$K_{\mathbf{z}}(Z) \cap \mathbf{R}^k_+ = K_{\mathbf{z}}(Z).$$

Then it follows that

$$(\mathbf{0} + \mathbf{R}^k_- \setminus \{\mathbf{0}\}) \cap K_{\mathbf{z}}(Z) = \emptyset \Rightarrow \mathbf{0} \in G_{SE}(Z).$$

Now it remains to show that if $\mathbf{y} \in K_{\mathbf{z}}(Z),\ \mathbf{y} \neq \mathbf{0}$, then $\mathbf{y} \notin G_{SE}(Z)$. Indeed, if $\mathbf{y} \in K_{\mathbf{z}}(Z),\ \mathbf{y} \neq \mathbf{0}$, then $\mathbf{y} \in \mathbf{R}^k_+$. Thus, $\mathbf{0} \in (\mathbf{y} + (\mathbf{R}^k_+)^C) \cap K_{\mathbf{z}}(Z)$, and then $\mathbf{y} \notin G_{SE}(Z)$. This ends the proof.

THEOREM 12 *If* $\mathbf{z} \in LLO(Z)$, *then* $G_{LO}(Z) \neq \emptyset$, *or equivalently* $G_{LO}(Z) = \{\mathbf{0}\}$.

Proof. Let $\mathbf{z} \in LLO(Z)$. Then (see Theorem 5, Mäkelä, Nikulin, 2009)

$$F_{\mathbf{z}}(Z) \cap C^k_{lex} = F_{\mathbf{z}}(Z).$$

Now it remains to show that if $\mathbf{d} \in F_{\mathbf{z}}(Z),\ \mathbf{d} \neq \mathbf{0}$, then $\mathbf{d} \notin G_{LO}(Z)$. Indeed, if $\mathbf{d} \in F_{\mathbf{z}}(Z),\ \mathbf{d} \neq \mathbf{0}$, then $\mathbf{d} \in C^k_{lex}$ and $-\mathbf{d} \in (C^k_{lex})^C$, i.e. $\mathbf{d} + (-\mathbf{d}) = \mathbf{0} \in F_{\mathbf{z}}(Z)$, and then $(\mathbf{d} + (C^k_{lex})^C) \cap F_{\mathbf{z}}(Z) \neq \emptyset$. Thus, $\mathbf{d} \notin G_{LO}(Z)$. This ends the proof.

5. Examples

We will illustrate geometrical meaning of the basic results formulated above via the following examples in biobjective case.

To construct the example, we will use the following norms in an arbitrary q–dimensional vector space $\mathbf{R}^q$:

- L_1 or *linear* norm

$$||\mathbf{y}||_1 := \sum_{i \in I_q} |y_i|,\ \mathbf{y} \in \mathbf{R}^q;$$

- L_2 or *Euclidean* norm

$$||\mathbf{y}||_2 := \sqrt{\sum_{i \in I_q} (y_i)^2},\ \mathbf{y} \in \mathbf{R}^q;$$

- L_∞ or *Chebyshev* norm

$$||\mathbf{y}||_\infty := \max_{i \in I_q} |y_i|,\ \mathbf{y} \in \mathbf{R}^q.$$

The first example describes the results in convex case.

Convex case example

Let $\mathbf{z} := f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, where $f_1(\mathbf{x}) = x_1$ and $f_2(\mathbf{x}) = x_2$. Assume that the sets of feasible solutions are given as

$$X_1 := \Big\{\mathbf{x} \mid\ ||\mathbf{x}||_1 \leq 1\Big\},$$

$$X_2 := \Big\{ \mathbf{x} \mid \ ||\mathbf{x}||_2 \leq 1 \Big\},$$
$$X_3 := \Big\{ \mathbf{x} \mid \ ||\mathbf{x}||_\infty \leq 1 \Big\}.$$

Then, respectively, we have

$$\begin{aligned} Z_1 &:= \Big\{ (f_1(\mathbf{x}), f_2(\mathbf{x})) : \ \mathbf{x} \in X_1 \Big\} = \Big\{ \mathbf{z} \mid \ ||\mathbf{z}||_1 \leq 1 \Big\}, \\ Z_2 &:= \Big\{ (f_1(\mathbf{x}), f_2(\mathbf{x})) : \ \mathbf{x} \in X_2 \Big\} = \Big\{ \mathbf{z} \mid \ ||\mathbf{z}||_2 \leq 1 \Big\}, \\ Z_3 &:= \Big\{ (f_1(\mathbf{x}), f_2(\mathbf{x})) : \ \mathbf{x} \in X_3 \Big\} = \Big\{ \mathbf{z} \mid \ ||\mathbf{z}||_\infty \leq 1 \Big\}. \end{aligned}$$

Fig. 4 represents Z_1 in objective space, and $\mathbf{z} = (-1, 0)$. Then we have

i)

$$F_{\mathbf{z}}(Z_1) = D_{\mathbf{z}}(Z_1) = K_{\mathbf{z}}(Z_1) = T_{\mathbf{z}}(Z_1) =$$
$$\{\mathbf{z} \mid z_2 \geq -z_1 - 1, \ z_2 \leq z_1 + 1, \ -1 \leq z_1\},$$

i.e. Z_1 is regular as well as tangentially and locally regular at point $\mathbf{z}$;

ii)

$$\mathbf{z} \in WP(Z_1) = PO(Z_1) = PP(Z_1) =$$
$$\{\mathbf{z} \mid \ z_1 + z_2 = -1, \ -1 \leq z_1 \leq 0, \ -1 \leq z_2 \leq 0\},$$
$$\mathbf{z} \in LO(Z_1) = \{(-1, 0)\}, \ \mathbf{z} \notin SE(Z_1) = \emptyset;$$

iii)

$$G_{WP}(Z_1) = G_{PO}(Z_1) = G_{PP}(Z_1) =$$
$$\{\mathbf{d} \mid d_2 = -d_1 - 1, \ -1 \leq d_1\},$$
$$G_{LO}(Z_1) = \{\mathbf{0}\}, \ G_{SE}(Z_1) = \emptyset.$$

Note that iii) is consistent with the results of Theorems 1 through 5.

Fig. 5 represents Z_2 in objective space and $z = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Then we have

i)

$$K_{\mathbf{z}}(Z_2) = T_{\mathbf{z}}(Z_2) = \{\mathbf{z} \mid z_2 \geq -z_1 - \sqrt{2}\},$$
$$D_{\mathbf{z}}(Z_2) = F_{\mathbf{z}}(Z_2) = \{\mathbf{z} \mid z_2 > -z_1 - \sqrt{2}\},$$

i.e. Z_2 is neither regular nor locally regular at point $\mathbf{z}$, but it is tangentially regular;

ii)

$$z \in WP(Z_2) = PO(Z_2) = \{\mathbf{z} \mid \ z_1^2 + z_2^2 = 1, \ -1 \leq z_1 \leq 0, \ -1 \leq z_2 \leq 0\},$$
$$z \in PP(Z_2) = \{\mathbf{z} \mid \ z_1^2 + z_2^2 = 1, \ -1 < z_1 \leq 0, \ -1 < z_2 \leq 0\},$$
$$z \notin LO(Z_2) = \{(-1, 0)\}, \ \mathbf{z} \notin SE(Z_2) = \emptyset;$$

iii)

$$G_{WP}(Z_2) = G_{PO}(Z_2) = G_{PP}(Z_2) = \{\mathbf{d} \mid d_2 = -d_1 - \sqrt{2}\},$$
$$G_{LO}(Z_2) = \emptyset, \ G_{SE}(Z_2) = \emptyset.$$

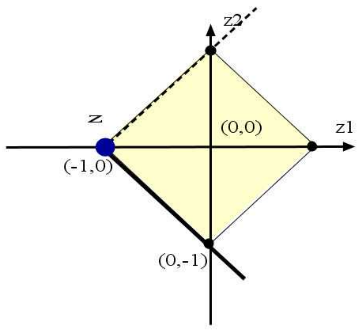

Figure 4. L_1-case

Note that iii) corresponds to the results of Theorems 1 through 5.

Fig. 6 represents Z_3 in objective space and $\mathbf{z} = (-1,-1)$. Then we have

i)

$$F_{\mathbf{z}}(Z_3) = D_{\mathbf{z}}(Z_3) = K_{\mathbf{z}}(Z_3) = T_{\mathbf{z}}(Z_3) =$$
$$\{\mathbf{z} \mid z_2 \geq -1,\ z_1 \geq -1\},$$

i.e. Z_3 is regular as well as tangentially and locally regular at point $\mathbf{z}$;

ii)

$$\mathbf{z} \in WP(Z_3) = \{\mathbf{z} \mid z_2 = -1,\ -1 \leq z_1 \leq 1\} \cup \{\mathbf{z} \mid z_1 = -1,\ -1 \leq z_2 \leq 1\},$$
$$\mathbf{z} \in PO(Z_3) = PP(Z_3) = LO(Z_3) = SE(Z_3) = \{(-1,-1)\};$$

iii)

$$G_{WP}(Z_3) = \{\mathbf{d} \mid d_2 = -1,\ -1 \leq d_1\} \cup \{\mathbf{d} \mid d_1 = -1,\ -1 \leq d_2\} \neq \emptyset,$$
$$G_{PO}(Z_3) = G_{PP}(Z_3) = G_{LO}(Z_3) = G_{SE}(Z_3) = \{\mathbf{0}\} \neq \emptyset.$$

Note that iii) is consistent with the results of Theorems 1 through 5.

The second example illustrates the results in nonconvex case.

Nonconvex case example

Fig. 7 represents Z in objective space and some fixed point $\mathbf{z} \in Z$. Then we have

i) Z is neither regular nor locally regular at point $\mathbf{z}$. However, it is tangentially regular;

ii)

$$\mathbf{z} \in LWP(Z),\ \mathbf{z} \in LPO(Z),\ \mathbf{z} \in LPP(Z),$$
$$\mathbf{z} \notin LLO(Z),\ \mathbf{z} \notin LSE(Z);$$

iii)

$$G_{WP}(Z) = G_{PO}(Z) = G_{PP}(Z) \neq \emptyset,$$
$$G_{LO}(Z) = \emptyset,\ G_{SE}(Z) = \emptyset.$$

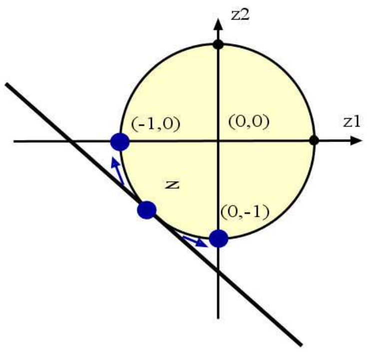

Figure 5. L_2-case

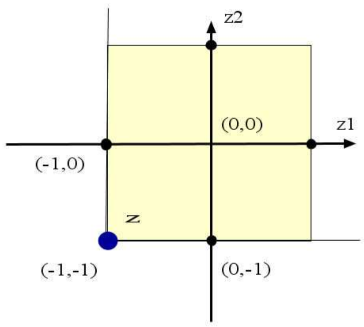

Figure 6. L_∞-case

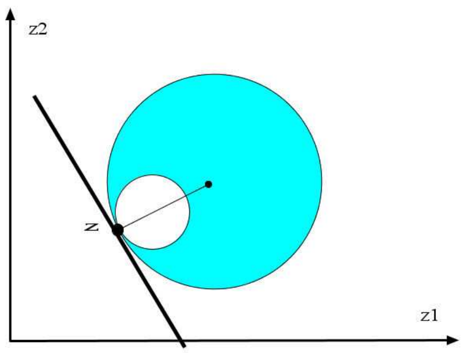

Figure 7. The first example for nonconvex Z

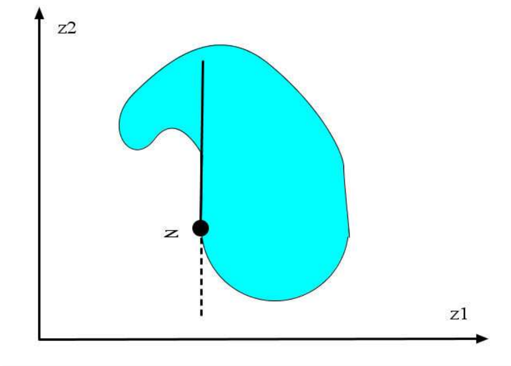

Figure 8. The second example for nonconvex Z

Fig. 8 represents Z in objective space and some fixed point $\mathbf{z} \in Z$. Then we have
i) Z is neither regular nor locally regular at point $\mathbf{z}$. However, it is tangentially regular;
ii)

$$\mathbf{z} \in LWP(Z),\ \mathbf{z} \in LPO(Z),\ \mathbf{z} \in LLO(Z),$$

$$\mathbf{z} \notin LPP(Z),\ \mathbf{z} \notin LSE(Z);$$

iii)

$$G_{WP}(Z) \neq \emptyset,\ G_{PO}(Z) = G_{PP}(Z) = \emptyset,$$

$$G_{LO}(Z) = \{\mathbf{0}\} \neq \emptyset,\ G_{SE}(Z) = \emptyset.$$

Note that in both nonconvex examples iii) is consistent with the results of Theorems 6 through 12.

6. Conclusions

In this paper we introduced and characterized the concept of trade-off directions for five most common optimality principles in multiobjective optimization. We generally followed the approach, initially proposed by Henig and Buchanan (1997), then followed by Lee and Nakayama (1997), as well as Miettinen and Mäkelä (2002, 2003), where trade-off directions are defined via some optimal surface of appropriate cones. The approach of Henig and Buchanan (1997) is independent of the scalarizing function and has only minor presumptions to the problem treated. The cone of trade-off directions is defined via Pareto optimal surface of the tangent cone and the treatment is based on classical tools of convex analysis. The special attention was made to the proper Pareto optimality. To maintain nonconvexity Lee and Nakayama (1997) suggested to use generalized trade-off directions employing nonconvex contingent cone and formulating some essential results assuming differentiability. Some related results are also given in Aubin, Frankowska (2008) and Luc (1989) in more general spaces.

Relaxing the convexity means that we have to analyze small neighborhoods of points instead of the whole set. We derive our results for a general framework imposing some additional local regularity properties to maintain nonconvexity as well as some general regularity properties in convex case for the ”hardest” optimality principles. Only few extra assumptions about the problem itself in nonconvex case are needed to avoid degeneracy in our analysis. Under our approach, we specified necessary and in some cases also sufficient conditions of optimality in terms of corresponding trade-off directions in both convex and nonconvex cases. The results obtained not only summarize and structure some already known facts about trade-off directions but also shed new light on their structural properties, emphasizing some fundamental similarities and differences existing in convex and nonconvex optimization. An interesting topic of further research is to investigate applicability of the proposed concepts in different multiobjective interactive methods (see, e.g., Branke, Deb, Miettinen, Slowinski, 2008).

Now we shortly analyze the similarity and difference between the results in two cases: convex and nonconvex. Here we would like to emphasize two facts about the results. The first fact is that some conditions, which are necessary and sufficient (under some extra assumptions) for optimality in convex case, are transformed into necessary but not sufficient conditions for local optimality in nonconvex case. The loss of sufficiency can be explained by the fact that the above-mentioned conditions use the contingent cone, which may have ”bad” directions towards feasibility. In the case with proper Pareto optimality, tangent regularity is crucial to prove sufficiency. Sufficiency in Pareto case is not guaranteed, but it can be achieved by imposing some regularity rules, which actually create local convexity towards some directions but keep the remaining areas irregular, i.e. nonconvex. The local and tangent regularity is used to prove the sufficiency. To investigate if the the assumptions of tangent regularity and local regularity could be weakened is an interesting direction for continuation of research in this area. Secondly, we noticed that in the case of lexicographic and strong efficiency, the set of generalized trade-off directions is either empty or it contains zero vector only. This reflects the fact that these two optimality principles do not contain non-zero trade-offs, i.e. there is no meaningful compromise between objectives in these cases. Indeed, the lexicographic optimality principle involves sequential optimization, and strong efficiency is a kind of parallel optimization, whose ideas are closer to single objective than to multiple objective optimization. Despite this, non-emptiness of the set of generalized trade-offs is quite informative itself, and therefore generalized trade-offs can be seen as an alternative advanced tool to describe the given optimality conditions for all five optimality principles considered in the paper.

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