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A modified ranking approach for solving fuzzy critical path problems with LR flat fuzzy numbers*

by

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Abstract: There are several fuzzy critical path methods for solving fuzzy critical path problems in which ranking approaches are used for comparing fuzzy numbers. In this paper, it is shown that if the existing ranking approaches are used for solving such fuzzy critical path problems in which duration times of activities are represented by LR flat fuzzy numbers, then more than one fuzzy numbers, representing the fuzzy project completion time, are obtained and a new ranking approach for comparing LR flat fuzzy numbers is proposed. Also, it is proved that if the proposed ranking approach is used for solving fuzzy critical path problems then a unique fuzzy number, representing the fuzzy project completion time, is obtained.

Keywords: fuzzy critical path problems, ranking function, LR fuzzy numbers, LR flat fuzzy numbers

1. Introduction

The theory of fuzzy sets was first introduced by Zadeh (1965). Since then, the theory of fuzzy sets has been applied in many fields such as pattern recognition, control theory, management sciences, picture processing etc. In many applications of fuzzy set theory to decision making, there is need to select that one from a collection of possible solution which is the best.

In the selection process a need arises of comparing fuzzy numbers. Since fuzzy numbers are represented by possibility distributions, they can overlap with each other, and so it is difficult to determine clearly whether one fuzzy number is larger or smaller than the other.

To the task of comparing fuzzy numbers, many authors (Jain, 1976; Yager, 1978; Murakami, Maeda and Imamura, 1983; Liou and Wang, 1992; Choobineh and Li, 1993; Cheng, 1998; Kwang and Lee, 1999; Yao and Wu, 2000; Modarres

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and Nezhad, 2001; Chu and Tsao, 2002; Abbasbandy and Asady, 2006; Wang, Yang, Xu and Chin, 2006; Asady and Zendehnam, 2007; Wang and Lee, 2008; Zhao and Liu, 2008; Abbasbandy and Hajjari, 2009; Ramli and Mohamad, 2009; Farhadinia, 2009; Kumar, Singh, Kaur and Kaur, 2011a; Kumar, Singh, Kaur and Kaur, 2011b) have proposed ranking approaches. Until now, no unified ranking approach has arisen for comparing fuzzy numbers.

Several authors (Nasution, 1994; Yao and Lin, 2000; Chanas and Zielinski, 2001; Chen and Chang, 2001; Lin, 2001, 2002; Lin and Yao, 2003; Liu, 2003; Liang and Han, 2004; Han, Chung and Liang, 2006; Chen, 2007; Chen and Hsueh, 2008; Shankar, Sireesha and Rao, 2010) have proposed fuzzy critical path methods based on different ranking approaches.

This paper is organized as follows: in Section 2, some basic definitions and arithmetic operations on LR flat fuzzy numbers are presented. In Section 3, some existing ranking approaches for comparing fuzzy numbers are presented. Shortcomings of existing ranking approaches are discussed in Section 4. In Section 5, a new ranking approach, obtained by modifying an existing ranking approach, is proposed for comparing LR flat fuzzy numbers. The validity of the proposed ranking approach is discussed in Section 6. Advantages of proposed ranking approach over existing ranking approaches are discussed in Section 7. In Section 8, conclusion and future work are presented.

2. Preliminaries

In the literature (Dubois, 1980) it is pointed out that the computational efforts required to solve a fuzzy linear programming problem can be reduced, if decision makers express their data using LR flat fuzzy numbers. So, LR flat fuzzy numbers are frequently used to increase computational efficiency without limiting the generality beyond the acceptable limits and to facilitate the ease of acquisition of data to solve real life problems.

In this section, some basic definitions and arithmetic operations between two LR flat fuzzy numbers are presented.

2.1. Basic definitions

In this section, some basic definitions are presented (Dubois, 1980).

DEFINITION 2.1 *A function* $L : [0, \infty) \to [0, 1]$ *(or* $R : [0, \infty) \to [0, 1]$*) is said to be reference function of fuzzy number iff*

(i) $L(x) = L(-x)(or R(x) = R(-x))$
(ii) $L(0) = 1(or R(0) = 1)$
(iii) $L(or R)$ is non $-$ increasing $[0, \infty)$.

DEFINITION 2.2 *A fuzzy number $\widetilde{A}$ defined on the universal set of real numbers $\mathbb{R}$, denoted as $(m, \alpha, \beta)_{LR}$, is said to be an LR fuzzy number if its membership function $\mu_{\widetilde{A}}(x)$ is given by*

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}), & \text{for } x \leq m,\ \alpha > 0 \\ R(\frac{x-m}{\beta}), & \text{for } x \geq m,\ \beta > 0. \end{cases}$$

DEFINITION 2.3 *A fuzzy number $\widetilde{A}$ defined on the universal set of real numbers $\mathbb{R}$, denoted as $(m, n, \alpha, \beta)_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{\widetilde{A}}(x)$ is given by*

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}), & \text{for } x \leq m,\ \alpha > 0 \\ R(\frac{x-n}{\beta}), & \text{for } x \geq n,\ \beta > 0 \\ 1, & \text{otherwise} \end{cases}.$$

DEFINITION 2.4 *Let $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ be an LR flat fuzzy number and λ be a real number in the interval $[0, 1]$. Then, the crisp set $A_\lambda = \{x \in X : \mu_{\widetilde{A}}(x) \geq \lambda\} = [m - \alpha L^{-1}(\lambda), n + \beta R^{-1}(\lambda)]$, is said to be λ-cut of $\widetilde{A}$.*

DEFINITION 2.5 *Let $\widetilde{A}$ be an LR or LR flat fuzzy number. Then, the crisp set $Supp(\widetilde{A}) = \{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) > 0\}$ is said to be the support of $\widetilde{A}$.*

DEFINITION 2.6 *Two LR flat fuzzy numbers $\widetilde{A_1} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{A_2} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are said to be equal i.e., $\widetilde{A_1} = \widetilde{A_2}$, iff $m_1 = m_2, n_1 = n_2, \alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$*

2.2. Arithmetic operations

In this section, some arithmetic operations between two LR flat fuzzy numbers are presented (Dubois, 1980).
Let $\widetilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two LR flat fuzzy numbers. Then,

(i) $\widetilde{A}_1 \oplus \widetilde{A}_2 = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$

(ii) $\lambda \tilde{A}_1 = \begin{cases} (\lambda m_1, \lambda n_1, \lambda \alpha_1, \lambda \beta_1)_{LR} & \lambda \geq 0 \\ (\lambda n_1, \lambda m_1, -\lambda \beta_1, -\lambda \alpha_1)_{RL} & \lambda \leq 0 \end{cases}.$

REMARK 1 *The arithmetic operations, presented in Section 2.2, can also be used for LR fuzzy numbers by putting $m_1 = n_1$ and $m_2 = n_2$.*

3. Comparison of fuzzy numbers

An efficient approach for comparing the fuzzy numbers is by the use of a ranking function $\Re : F(\mathbb{R}) \rightarrow \mathbb{R}$, where $F(\mathbb{R})$ is a set of fuzzy numbers defined on set of real numbers, which maps each fuzzy number into the real line, where a natural order exists.
Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy numbers. Then,

(i)$\widetilde{A} \succ \widetilde{B}$ if $\Re(\widetilde{A}) > \Re(\widetilde{B})$
(ii)$\widetilde{A} \approx \widetilde{B}$ if $\Re(\widetilde{A}) = \Re(\widetilde{B})$
(iii)$\widetilde{A} \succeq \widetilde{B}$ if $\Re(\widetilde{A}) \geq \Re(\widetilde{B})$.

3.1. Some ranking formulae for comparing fuzzy numbers

Ranking formulae used in some existing ranking approaches (Yager, 1978; Murakami, Maeda and Imamura, 1983; Liou and Wang, 1992; Cheng, 1998; Yao and Wu, 2000; Chu and Tsao, 2002; Asady and Zendehnam, 2007; Zhao and Liu, 2008; Abbasbandy and Hajjari, 2009) for comparing LR fuzzy numbers and LR flat fuzzy numbers are shown in Table 1.

3.2. The Wang and Lee ranking approach

In this section, ranking approach of Wang and Lee (2008) for comparing LR flat fuzzy numbers is presented. Let $\widetilde{A} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{B} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two LR flat fuzzy numbers. Then, use the following steps to compare $\widetilde{A}$ and $\widetilde{B}$:

Step 1 Find $x_{\widetilde{A}} = \frac{\int_{m_1-\alpha_1}^{m_1} xL(\frac{m_1-x}{\alpha_1})dx+\int_{m_1}^{n_1} xdx+\int_{n_1}^{n_1+\beta_1} xR(\frac{x-n_1}{\beta_1})dx}{\int_{m_1-\alpha_1}^{m_1} L(\frac{m_1-x}{\alpha_1})dx+\int_{m_1}^{n_1} dx+\int_{n_1}^{n_1+\beta_1} R(\frac{x-n_1}{\beta_1})dx}$ and
$x_{\widetilde{B}} = \frac{\int_{m_2-\alpha_2}^{m_2} xL(\frac{m_2-x}{\alpha_2})dx+\int_{m_2}^{n_2} xdx+\int_{n_2}^{n_2+\beta_2} xR(\frac{x-n_2}{\beta_2})dx}{\int_{m_2-\alpha_2}^{m_2} L(\frac{m_2-x}{\alpha_2})dx+\int_{m_2}^{n_2} dx+\int_{n_2}^{n_2+\beta_2} R(\frac{x-n_2}{\beta_2})dx}$.

Case (i) If $x_{\widetilde{A}} > x_{\widetilde{B}}$, then $\widetilde{A} \succ \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $x_{\widetilde{A}} < x_{\widetilde{B}}$, then $\widetilde{A} \prec \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $x_{\widetilde{A}} = x_{\widetilde{B}}$, then go to Step 2.

Step 2 Find $y_{\widetilde{A}} = \frac{\int_0^1 (y(m_1-\alpha_1 L^{-1}(y)))dy+\int_0^1 (y(n_1+\beta_1 R^{-1}(y)))dy}{\int_0^1 (m_1-\alpha_1 L^{-1}(y))dy+\int_0^1 (n_1+\beta_1 R^{-1}(y))dy}$ and
$y_{\widetilde{B}} = \frac{\int_0^1 (y(m_2-\alpha_2 L^{-1}(y)))dy+\int_0^1 (y(n_2+\beta_2 R^{-1}(y)))dy}{\int_0^1 (m_2-\alpha_2 L^{-1}(y))dy+\int_0^1 (n_2+\beta_2 R^{-1}(y))dy}$.

Case (i) If $y_{\widetilde{A}} > y_{\widetilde{B}}$, then $\widetilde{A} \succ \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Table 1: Ranking formulae used in some existing ranking approaches

Ranking approaches	Ranking formulae for an LR fuzzy number $\widetilde{A}=(m,\alpha,\beta)_{LR}$	Ranking formulae for an LR flat fuzzy number $\widetilde{A}=(m,n,\alpha,\beta)_{LR}$
Yager (1978)	$\Re(\widetilde{A})=\frac{\int_{m-\alpha}^{m} xL(\frac{m-x}{\alpha})dx+\int_{m}^{m+\beta} xR(\frac{x-m}{\beta})dx}{\int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx+\int_{m}^{m+\beta} R(\frac{x-m}{\beta})dx}$	$\Re(\widetilde{A})=\frac{\int_{m-\alpha}^{m} xL(\frac{m-x}{\alpha})dx+\int_{m}^{n} xdx+\int_{n}^{n+\beta} xR(\frac{x-n}{\beta})dx}{\int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx+\int_{m}^{n} dx+\int_{n}^{n+\beta} R(\frac{x-n}{\beta})dx}$
Murakami, Maeda and Imamura (1983)	$\Re(\widetilde{A})=x_{\widetilde{A}}$ or $y_{\widetilde{A}}$ where, $x_{\widetilde{A}}=\frac{\int_{m-\alpha}^{m} xL(\frac{m-x}{\alpha})dx+\int_{m}^{m+\beta} xR(\frac{x-m}{\beta})dx}{\int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx+\int_{m}^{m+\beta} R(\frac{x-m}{\beta})dx}$, $y_{\widetilde{A}}=\frac{\int_0^1 (y(m-\alpha L^{-1}(y)))dy+\int_0^1 (y(m+\beta R^{-1}(y)))dy}{\int_0^1 (m-\alpha L^{-1}(y))dy+\int_0^1 (m+\beta R^{-1}(y))dy}$	$\Re(\widetilde{A})=x_{\widetilde{A}}$ or $y_{\widetilde{A}}$ where, $x_{\widetilde{A}}=\frac{\int_{m-\alpha}^{m} xL(\frac{m-x}{\alpha})dx+\int_{m}^{n} xdx+\int_{n}^{n+\beta} xR(\frac{x-n}{\beta})dx}{\int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx+\int_{m}^{n} dx+\int_{n}^{n+\beta} R(\frac{x-n}{\beta})dx}$, $y_{\widetilde{A}}=\frac{\int_0^1 (y(m-\alpha L^{-1}(y)))dy+\int_0^1 (y(n+\beta R^{-1}(y)))dy}{\int_0^1 (m-\alpha L^{-1}(y))dy+\int_0^1 (n+\beta R^{-1}(y))dy}$
Liou and Wang (1992)	$\Re(\widetilde{A})=\frac{1}{2}[\int_0^1 (m-\alpha L^{-1}(y))dy+\int_0^1 (m+\beta R^{-1}(y))dy]$	$\Re(\widetilde{A})=\frac{1}{2}[\int_0^1 (m-\alpha L^{-1}(y))dy+\int_0^1 (n+\beta R^{-1}(y))dy]$
Cheng (1998)	$\Re(\widetilde{A})=\sqrt{x_{\widetilde{A}}^2+y_{\widetilde{A}}^2}$ where, $x_{\widetilde{A}}=\frac{\int_{m-\alpha}^{m} xL(\frac{m-x}{\alpha})dx+\int_{m}^{m+\beta} xR(\frac{x-m}{\beta})dx}{\int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx+\int_{m}^{m+\beta} R(\frac{x-m}{\beta})dx}$, $y_{\widetilde{A}}=\frac{\int_0^1 (y(m-\alpha L^{-1}(y)))dy+\int_0^1 (y(m+\beta R^{-1}(y)))dy}{\int_0^1 (m-\alpha L^{-1}(y))dy+\int_0^1 (m+\beta R^{-1}(y))dy}$	$\Re(\widetilde{A})=\sqrt{x_{\widetilde{A}}^2+y_{\widetilde{A}}^2}$ where, $x_{\widetilde{A}}=\frac{\int_{m-\alpha}^{m} xL(\frac{m-x}{\alpha})dx+\int_{m}^{n} xdx+\int_{n}^{n+\beta} xR(\frac{x-n}{\beta})dx}{\int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx+\int_{m}^{n} dx+\int_{n}^{n+\beta} R(\frac{x-n}{\beta})dx}$, $y_{\widetilde{A}}=\frac{\int_0^1 (y(m-\alpha L^{-1}(y)))dy+\int_0^1 (y(n+\beta R^{-1}(y)))dy}{\int_0^1 (m-\alpha L^{-1}(y))dy+\int_0^1 (n+\beta R^{-1}(y))dy}$
Yao and Wu (2000)	$\Re(\widetilde{A})=\frac{1}{2}\int_0^1 [D_L(y)+D_R(y)]dy$, where, $D_L(y)=m-\alpha L^{-1}(y)$ and $D_R(y)=m+\beta R^{-1}(y)$	$\Re(\widetilde{A})=\frac{1}{2}\int_0^1 [D_L(y)+D_R(y)]dy$, where, $D_L(y)=m-\alpha L^{-1}(y)$ and $D_R(y)=n+\beta R^{-1}(y)$
Chu and Tsao (2002)	$\Re(\widetilde{A})=x_{\widetilde{A}}.y_{\widetilde{A}}$ where, $x_{\widetilde{A}}=\frac{\int_{m-\alpha}^{m} xL(\frac{m-x}{\alpha})dx+\int_{m}^{m+\beta} xR(\frac{x-m}{\beta})dx}{\int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx+\int_{m}^{m+\beta} R(\frac{x-m}{\beta})dx}$, $y_{\widetilde{A}}=\frac{\int_0^1 (y(m-\alpha L^{-1}(y)))dy+\int_0^1 (y(m+\beta R^{-1}(y)))dy}{\int_0^1 (m-\alpha L^{-1}(y))dy+\int_0^1 (m+\beta R^{-1}(y))dy}$	$\Re(\widetilde{A})=x_{\widetilde{A}}.y_{\widetilde{A}}$ where, $x_{\widetilde{A}}=\frac{\int_{m-\alpha}^{m} xL(\frac{m-x}{\alpha})dx+\int_{m}^{n} xdx+\int_{n}^{n+\beta} xR(\frac{x-n}{\beta})dx}{\int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx+\int_{m}^{n} dx+\int_{n}^{n+\beta} R(\frac{x-n}{\beta})dx}$, $y_{\widetilde{A}}=\frac{\int_0^1 (y(m-\alpha L^{-1}(y)))dy+\int_0^1 (y(n+\beta R^{-1}(y)))dy}{\int_0^1 (m-\alpha L^{-1}(y))dy+\int_0^1 (n+\beta R^{-1}(y))dy}$
Asady and Zendehnam (2007)	$\Re(\widetilde{A})=\frac{1}{2}\int_0^1 (\underline{A}(r)+\overline{A}(r))dr$, where, $\underline{A}(r)=m-\alpha L^{-1}(r)$ and $\overline{A}(r)=m+\beta R^{-1}(r)$	$\Re(\widetilde{A})=\frac{1}{2}\int_0^1 (\underline{A}(r)+\overline{A}(r))dr$, where, $\underline{A}(r)=m-\alpha L^{-1}(r)$ and $\overline{A}(r)=n+\beta R^{-1}(r)$
Zhao and Liu (2008)	$\Re(\widetilde{A})=\frac{1}{2}\mu_{\widetilde{A}}(x)[\lvert x_{\widetilde{A}}\rvert+\lvert y_{\widetilde{A}}\rvert+\lvert x_{\widetilde{A}}.y_{\widetilde{A}}\rvert]$ where, $x_{\widetilde{A}}=\frac{\int_{m-\alpha}^{m} xL(\frac{m-x}{\alpha})dx+\int_{m}^{m+\beta} xR(\frac{x-m}{\beta})dx}{\int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx+\int_{m}^{m+\beta} R(\frac{x-m}{\beta})dx}$, $y_{\widetilde{A}}=\frac{\int_0^1 y[(m+\beta R^{-1}(y))-(m-\alpha L^{-1}(y))]dy}{\int_0^1 [(m+\beta R^{-1}(y))-(m-\alpha L^{-1}(y))]dy}$, $\mu_{\widetilde{A}}(x)=\begin{cases} 1 & x\geq 0 \\ -1 & x<0 \end{cases}$	$\Re(\widetilde{A})=\frac{1}{2}\mu_{\widetilde{A}}(x)[\lvert x_{\widetilde{A}}\rvert+\lvert y_{\widetilde{A}}\rvert+\lvert x_{\widetilde{A}}.y_{\widetilde{A}}\rvert]$ where, $x_{\widetilde{A}}=\frac{\int_{m-\alpha}^{m} xL(\frac{m-x}{\alpha})dx+\int_{m}^{n} xdx+\int_{n}^{n+\beta} xR(\frac{x-n}{\beta})dx}{\int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx+\int_{m}^{n} dx+\int_{n}^{n+\beta} R(\frac{x-n}{\beta})dx}$, $y_{\widetilde{A}}=\frac{\int_0^1 y[(n+\beta R^{-1}(y))-(m-\alpha L^{-1}(y))]dy}{\int_0^1 [(n+\beta R^{-1}(y))-(m-\alpha L^{-1}(y))]dy}$, $\mu_{\widetilde{A}}(x)=\begin{cases} 1 & x\geq 0 \\ -1 & x<0 \end{cases}$
Abbasbandy and Hajjari (2009)	$\Re(\widetilde{A})=\frac{1}{2}[\int_0^1 ((m-\alpha L^{-1}(y))+(m+\beta R^{-1}(y))+2m)dy]$	$\Re(\widetilde{A})=\frac{1}{2}[\int_0^1 ((m-\alpha L^{-1}(y))+(n+\beta R^{-1}(y))+m+n)dy]$

Case (ii) If $y_{\widetilde{A}} < y_{\widetilde{B}}$, then $\widetilde{A} \prec \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $y_{\widetilde{A}} = y_{\widetilde{B}}$ then $\widetilde{A} = \widetilde{B}$.

3.3. The Farhadinia ranking approach

In this section, the ranking approach of Farhadinia (2009) for comparing LR flat fuzzy numbers is presented.
Let $\widetilde{A} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{B} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two LR flat fuzzy numbers. Then, use the following steps to compare $\widetilde{A}$ and $\widetilde{B}$:

Step 1 Find $C(\widetilde{A})= \inf\{x \in \text{Supp}(\widetilde{A}) : \mu_{\widetilde{A}}(x) = 1\} = m_1$ and
$C(\widetilde{B}) = \inf\{x \in \text{Supp}(\widetilde{B}) : \mu_{\widetilde{B}}(x) = 1\} = m_2$

Case (i) If $C(\widetilde{A}) > C(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $C(\widetilde{A}) < C(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $C(\widetilde{A}) = C(\widetilde{B})$ then go to Step 2.

Step 2 Find $L(\widetilde{A}) = \inf \text{Supp}(\widetilde{A}) = m_1 - \alpha_1$ and
$L(\widetilde{B}) = \inf \text{Supp}(\widetilde{B}) = m_2 - \alpha_2$

Case (i) If $L(\widetilde{A}) > L(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $L(\widetilde{A}) < L(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $L(\widetilde{A}) = L(\widetilde{B})$ then go to Step 3.

Step 3 Find $W(\widetilde{A}) = |\text{Supp}(\widetilde{A})| = n_1 - m_1 + \alpha_1 + \beta_1$ and
$W(\widetilde{B}) = |\text{Supp}(\widetilde{B})| = n_2 - m_2 + \alpha_2 + \beta_2$

Case (i) If $W(\widetilde{A}) > W(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $W(\widetilde{A}) < W(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $W(\widetilde{A}) = W(\widetilde{B})$ then go to Step 4.

Step 4 Find $S(\widetilde{A}) = \int \mu_{\widetilde{A}}(x)dx = n_1 - m_1 + \alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda + \beta_1 \int_0^1 R^{-1}(\lambda)d\lambda$
and $S(\widetilde{B}) = \int \mu_{\widetilde{B}}(x)dx = n_2 - m_2 + \alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda + \beta_2 \int_0^1 R^{-1}(\lambda)d\lambda$

Case (i) If $S(\widetilde{A}) > S(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $S(\widetilde{A}) < S(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,

$$\text{maximum } \{\widetilde{A}, \widetilde{B}\} = \widetilde{B} \text{ and minimum } \{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$$

Case (iii) If $S(\widetilde{A}) = S(\widetilde{B})$ then $\widetilde{A} = \widetilde{B}$.

REMARK 2 *For an LR flat fuzzy number* $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$
$S(\widetilde{A}) = \int \mu_{\widetilde{A}}(x)dx = \int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx + \int_{m}^{n} dx + \int_{n}^{n+\beta} R(\frac{x-n}{\beta})dx = n - m + \alpha \int_0^1 L^{-1}(\lambda)d\lambda + \beta \int_0^1 R^{-1}(\lambda)d\lambda.$

4. Shortcomings of existing ranking approaches

There are several fuzzy critical path methods in which ranking approaches are used for comparing fuzzy numbers. In this section, it is shown that the results of fuzzy critical path problem, chosen in Example 4.1, obtained by using the different existing fuzzy critical path methods (Nasution, 1994; Yao and Lin, 2000; Chanas and Zielinski, 2001; Chen and Chang, 2001; Lin, 2001, 2002; Lin and Yao, 2003; Liu, 2003; Liang and Han, 2004; Han, Chung and Liang, 2006; Chen, 2007; Chen and Hsueh, 2008; Shankar, Sireesha and Rao, 2010) with different existing ranking approaches (Yager, 1978; Murakami, Maeda and Imamura, 1983; Liou and Wang, 1992; Cheng, 1998; Yao and Wu, 2000; Chu and Tsao, 2002; Asady and Zendehnam, 2007; Wang and Lee, 2008; Zhao and Liu, 2008; Abbasbandy and Hajjari, 2009) are not appropriate.

EXAMPLE 4.1 *Find the fuzzy critical path and maximum total fuzzy completion time of the project, shown in Fig. 1, in which the fuzzy duration of activity* (i, j) *is represented by LR fuzzy number* $\widetilde{t_{ij}}$ *with* $L(x) = R(x) = max\{0, 1 - x\}$ *as follows:*
$\widetilde{t_{12}} = (1, 0.5, 0.5)_{LR}$, $\widetilde{t_{23}} = (1, 1, 1)_{LR}$, $\widetilde{t_{24}} = (2, 0.5, 0.5)_{LR}$, $\widetilde{t_{34}} = (1, 0.5, 0.5)_{LR}$.

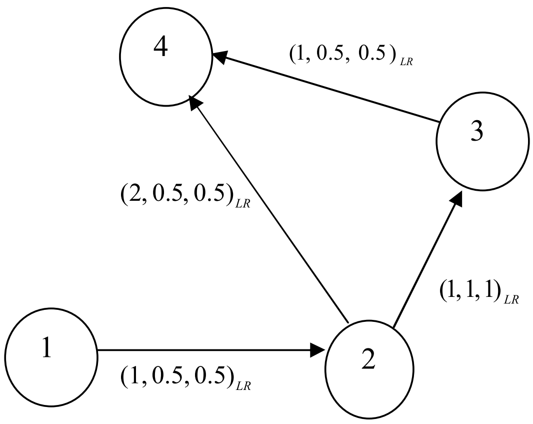

Figure 1. Project network of Example 4.1

Table 2: Results of Example 4.1 obtained by using existing fuzzy critical path methods with existing ranking approaches

Existing ranking approaches	Results of fuzzy critical path problem obtained by using different existing fuzzy critical path methods (Nasution, 1994; Yao and Lin, 2000; Chanas and Zielinski, 2001; Chen and Chang, 2001; Lin, 2001, 2002; Lin and Yao, 2003; Liu, 2003; Liang and Han, 2004; Han, Chung and Liang, 2006; Chen, 2007; Chen and Hsueh, 2008; Shankar, Sireesha and Rao, 2010)	
	Example 4.1	
	Fuzzy critical path	Maximum total fuzzy project completion time ($\widetilde{T}$)
Yager (1978)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ $1 \Rightarrow 2 \Rightarrow 4$	$(3, 2, 2)_{LR}$; $\Re(\widetilde{T}) = 3$ $(3, 1, 1)_{LR}$; $\Re(\widetilde{T}) = 3$
Murakami, Maeda and Imamura (1983)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ $1 \Rightarrow 2 \Rightarrow 4$	$(3, 2, 2)_{LR}$, $x_{\widetilde{T}} = 3$, $y_{\widetilde{T}} = 0.5$ $(3, 1, 1)_{LR}$, $x_{\widetilde{T}} = 3$, $y_{\widetilde{T}} = 0.5$
Liou and Wang (1992)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ $1 \Rightarrow 3$	$(3, 2, 2)_{LR}$, $\Re(\widetilde{T}) = 3$ $(3, 1, 1)_{LR}$, $\Re(\widetilde{T}) = 3$
Cheng (1998)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ $1 \Rightarrow 2 \Rightarrow 4$	$(3, 2, 2)_{LR}$; $\Re(\widetilde{T}) = 3.041$ $(3, 1, 1)_{LR}$; $\Re(\widetilde{T}) = 3.041$
Yao and Wu (2000)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ $1 \Rightarrow 2 \Rightarrow 4$	$(3, 2, 2)_{LR}$; $\Re(\widetilde{T}) = 3$ $(3, 1, 1)_{LR}$; $\Re(\widetilde{T}) = 3$
Chu and Tsao (2002)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ $1 \Rightarrow 2 \Rightarrow 4$	$(3, 2, 2)_{LR}$; $\Re(\widetilde{T}) = 1.5$ $(3, 1, 1)_{LR}$; $\Re(\widetilde{T}) = 1.5$
Asady and Zendehnam (2007)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ $1 \Rightarrow 2 \Rightarrow 4$	$(3, 2, 2)_{LR}$; $\Re(\widetilde{T}) = 3$ $(3, 1, 1)_{LR}$; $\Re(\widetilde{T}) = 3$
Wang and Lee (2008)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ $1 \Rightarrow 2 \Rightarrow 4$	$(3, 2, 2)_{LR}$, $x_{\widetilde{T}} = 3$, $y_{\widetilde{T}} = 0.5$ $(3, 1, 1)_{LR}$, $x_{\widetilde{T}} = 3$, $y_{\widetilde{T}} = 0.5$
Zhao and Liu (2008)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ $1 \Rightarrow 2 \Rightarrow 4$	$(3, 2, 2)_{LR}$; $\Re(\widetilde{T}) = 2.167$ $(3, 1, 1)_{LR}$; $\Re(\widetilde{T}) = 2.167$
Abbasbandy and Hajjari (2009)	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ $1 \Rightarrow 2 \Rightarrow 4$	$(3, 2, 2)_{LR}$; $\Re(\widetilde{T}) = 6$ $(3, 1, 1)_{LR}$; $\Re(\widetilde{T}) = 6$
Farhadinia (2009)	$1 \Rightarrow 2 \Rightarrow 4$	$(3, 1, 1)_{LR}$, $L(\widetilde{T}) = 2$

4.1. Results for the chosen problem

The results for the fuzzy critical path problem of Example 4.1, obtained by using different existing fuzzy critical path methods (Nasution, 1994; Yao and Lin, 2000; Chanas and Zielinski, 2001; Chen and Chang, 2001; Lin, 2001, 2002; Lin and Yao, 2003; Liu, 2003; Liang and Han, 2004; Han, Chung and Liang, 2006; Chen, 2007; Chen and Hsueh, 2008; Shankar, Sireesha and Rao, 2010) with different existing ranking approaches (Yager, 1978; Murakami, Maeda and Imamura, 1983; Liou and Wang, 1992; Cheng, 1998; Yao and Wu, 2000; Chu and Tsao, 2002; Asady and Zendehnam, 2007; Wang and Lee, 2008; Zhao and Liu, 2008; Abbasbandy and Hajjari, 2009; Farhadinia, 2009) are shown in Table 2.

It is obvious from the results, shown in Table 2, that on solving the fuzzy critical path problem, chosen in Example 4.1, by applying the different existing fuzzy critical path methods with existing ranking approaches more than one fuzzy critical paths are obtained and the maximum total fuzzy project completion times, corresponding to different fuzzy critical paths are different so their physical interpretation will also be different. But, in the literature (Taha, 2003) it is pointed out that if on solving project network problem there exist more than one critical path then the maximum total completion time of the project should be same corresponding to all the critical paths. So, the results obtained by using existing fuzzy critical path methods here referred to, with existing ranking approaches shown in Table 2, are not appropriate.

Although the results of the fuzzy critical path problem, chosen in Example 4.1, obtained by using the existing fuzzy critical path methods, here quoted, with the existing ranking approach of Farhadinia (2009) are appropriate, but on the basis of numerical example it is not possible to say that the results of all fuzzy critical path problems, obtained by using existing fuzzy critical path methods with this ranking approach will always be appropriate. So, in the next section general proofs are introduced to show that if existing fuzzy critical path methods with the ranking approach of Farhadinia (2009) are used to solve such fuzzy critical path problems in which the parameters are represented by LR fuzzy numbers, then always a unique fuzzy number, representing the maximum total fuzzy project completion time, will be obtained, but if the existing fuzzy critical path methods are used with the ranking approach of Farhadinia (2009) to find fuzzy optimal solution of such fuzzy critical path problems in which the parameters are represented by LR flat fuzzy numbers, then more than one fuzzy number, representing the maximum total fuzzy completion time for the same project, may be obtained, which is not appropriate.

4.2. The shortcomings of the Farhadinia ranking approach

Although in the ranking approach of Farhadinia (2009) it is claimed that if $\widetilde{A}$ and $\widetilde{B}$ are two LR flat fuzzy numbers such that (i) $C(\widetilde{A}) = C(\widetilde{B})$, (ii) $L(\widetilde{A}) = L(\widetilde{B})$, (iii) $W(\widetilde{A}) = W(\widetilde{B})$, (iv) $S(\widetilde{A}) = S(\widetilde{B})$ then $\widetilde{A} = \widetilde{B}$, but in Proposition 4.1 it is shown that if $\widetilde{A}$ and $\widetilde{B}$ are two LR fuzzy numbers such that (i) $C(\widetilde{A}) = C(\widetilde{B})$, (ii) $L(\widetilde{A}) = L(\widetilde{B})$, (iii) $W(\widetilde{A}) = W(\widetilde{B})$, (iv) $S(\widetilde{A}) = S(\widetilde{B})$ then $\widetilde{A} = \widetilde{B}$, and in Proposition 4.2 it is shown that if $\widetilde{A}$ and $\widetilde{B}$ are two LR flat fuzzy numbers such that (i) $C(\widetilde{A}) = C(\widetilde{B})$, (ii) $L(\widetilde{A}) = L(\widetilde{B})$, (iii) $W(\widetilde{A}) = W(\widetilde{B})$, (iv) $S(\widetilde{A}) = S(\widetilde{B})$ then $\widetilde{A} \neq \widetilde{B}$ i.e., if $\widetilde{A}$ and $\widetilde{B}$ are two LR fuzzy numbers such that $\widetilde{A} \neq \widetilde{B}$ then by using the ranking approach of Farhadinia (2009) it can be checked that $\widetilde{A} \succ \widetilde{B}$ or $\widetilde{A} \prec \widetilde{B}$ but if $\widetilde{A}$ and $\widetilde{B}$ are two LR flat fuzzy numbers such that $\widetilde{A} \neq \widetilde{B}$ then by using the same ranking approach it can not be checked that $\widetilde{A} \succ \widetilde{B}$ or $\widetilde{A} \prec \widetilde{B}$ and hence the existing fuzzy critical path methods, considered here, can be used with the ranking approach of Farhadinia (2009) to find the appropriate results of such fuzzy critical path problems in which all the parameters are represented by LR fuzzy numbers, but can not be used for finding the results of such fuzzy critical path problems in which all the parameters are represented by LR flat fuzzy numbers.

PROPOSITION 4.1 *Let* $\widetilde{A} = (m_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{B} = (m_2, \alpha_2, \beta_2)_{LR}$ *be two LR fuzzy numbers such that*
(i) $C(\widetilde{A}) = C(\widetilde{B})$ *(ii)* $L(\widetilde{A}) = L(\widetilde{B})$ *(iii)* $W(\widetilde{A}) = W(\widetilde{B})$ *(iv)* $S(\widetilde{A}) = S(\widetilde{B})$.
Then, $\widetilde{A} = \widetilde{B}$.

Proof.

$$
\begin{aligned}
&(\text{i})(\widetilde{A}) = C(\widetilde{B}) \\
&\quad \Rightarrow m_1 = m_2 && (1)\\
&(\text{ii})L(\widetilde{A}) = L(\widetilde{B}) \\
&\quad \Rightarrow m_1 - \alpha_1 = m_2 - \alpha_2 && (2)\\
&(\text{iii})W(\widetilde{A}) = W(\widetilde{B}) \\
&\quad \Rightarrow \alpha_1 + \beta_1 = \alpha_2 + \beta_2 && (3)
\end{aligned}
$$

On solving (1), (2) and (3)
$m_1 = m_2 \alpha_1 = \alpha_2 and \beta_1 = \beta_2$
i.e., $\widetilde{A} = \widetilde{B}$.

PROPOSITION 4.2 *Let $\widetilde{A} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{B} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two LR flat fuzzy numbers such that*
(i) $C(\widetilde{A}) = C(\widetilde{B})$ (ii) $L(\widetilde{A}) = L(\widetilde{B})$ (iii) $W(\widetilde{A}) = W(\widetilde{B})$ (iv) $S(\widetilde{A}) = S(\widetilde{B})$.
Then, $\widetilde{A} \neq \widetilde{B}$.

Proof.

$$
\begin{aligned}
&(\text{i})C(\widetilde{A}) = C(\widetilde{B}) \\
&\quad \Rightarrow m_1 = m_2 && (4)\\
&(\text{ii})L(\widetilde{A}) = L(\widetilde{B}) \\
&\quad \Rightarrow m_1 - \alpha_1 = m_2 - \alpha_2 && (5)\\
&(\text{iii})W(\widetilde{A}) = W(\widetilde{B}) \\
&\quad \Rightarrow n_1 - m_1 + \alpha_1 + \beta_1 = n_2 - m_2 + \alpha_2 + \beta_2 \\
&(\text{iv})S(\widetilde{A}) = S(\widetilde{B}) && (6)\\
&\quad \Rightarrow n_1 - m_1 + \alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda + \beta_1 \int_0^1 R^{-1}(\lambda)d\lambda = n_2 - m_2 \\
&\quad +\alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda \\
&\qquad +\beta_2 \int_0^1 R^{-1}(\lambda)d\lambda. && (7)
\end{aligned}
$$

On solving (4), (5), (6) and (7) we find m_1, n_1, α_1 and β_1 are not equal to m_2, n_2, α_2 and β_2, respectively, so $\widetilde{A}$ is not equal to $\widetilde{B}$. ∎

4.2.1. Illustrative examples

To illustrate the shortcomings of the ranking approach by Farhadinia (2009), the results of the fuzzy critical path problem, chosen in Example 4.2, obtained by using existing fuzzy critical path methods with the ranking approach of Farhadinia (2009) are shown in Table 4.

EXAMPLE 4.2 *Find the fuzzy critical path and maximum total fuzzy completion time of the project, shown in Fig. 2, in which the fuzzy time duration of activity* (i, j) *is represented by LR flat fuzzy numbers* $\widetilde{D_{ij}}$ *with* $L(x) = max\{0, 1 - x^2\}$ *and* $R(x) = e^{-|x|}$ *as shown in Table 3.*

Table 3: Fuzzy normal time for each activity

Activity	Fuzzy duration ($\widetilde{D_{ij}}$) (days)
(1, 2)	$(11, 13, 1, 1)_{LR}$
(1, 3)	$(19, 20, 3, 5)_{LR}$
(2, 3)	$(8, 9, 2, 2)_{LR}$
(2, 4)	$(14, 15, 3, 8)_{LR}$
(3, 4)	$(6, 10, 1, 2)_{LR}$

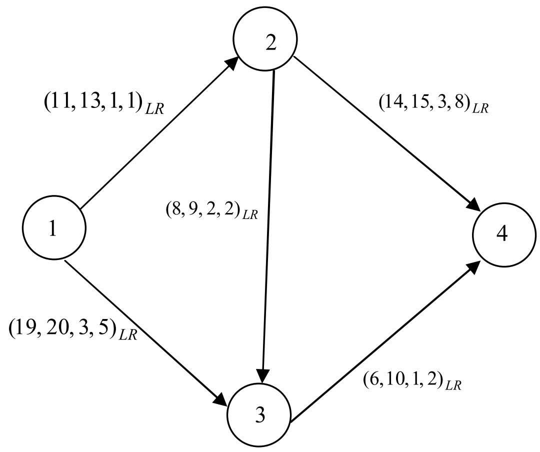

Figure 2. Project network of the Example 4.2

Table 4: Results of Example 4.2 obtained by using existing fuzzy critical path methods with Farhadinia ranking approach

Example	Results of fuzzy critical path problem obtained by using different existing fuzzy critical methods (Nasution, 1994; Yao and Lin, 2000; Chanas and Zielinski, 2001; Chen and Chang, 2001; Lin 2001, 2002; Lin and Yao, 2003; Liu, 2003; Liang and Han, 2004; Han, Chung and Liang, 2006; Chen, 2007; Chen and Hsueh, 2008; Shankar, Sireesha and Rao, 2010) with the ranking approach of Farhadinia (2009)	
	Fuzzy critical path	Maximum total fuzzy project completion time ($\widetilde{T}$)
4.2	$1 \Rightarrow 3 \Rightarrow 4$	$(25, 30, 4, 7)_{LR}$, $S(\widetilde{T}) = 14.667$
	$1 \Rightarrow 2 \Rightarrow 4$	$(25, 28, 4, 9)_{LR}$, $S(\widetilde{T}) = 14.667$
	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$	$(25, 32, 4, 5)_{LR}$, $S(\widetilde{T}) = 14.667$

It is obvious from the results, shown in Table 4, that on solving the fuzzy critical path problem, chosen in Example 4.2, by using the existing fuzzy critical path methods with the ranking approach of Farhadinia (2009) three different LR flat fuzzy numbers $(25, 28, 4, 9)_{LR}$, $(25, 30, 4, 7)_{LR}$ and $(25, 32, 4, 5)_{LR}$, representing the maximum total fuzzy completion time of the same project, are obtained, which is not appropriate.

REMARK 3 *In Section 4, shortcomings of an important ranking approach are pointed out. The same shortcoming can also be found in other existing ranking approaches, which are not discussed in this paper.*

5. Proposed ranking approach

On the basis of the results, discussed in Section 4, it can be concluded that none of the existing ranking approaches, considered here, with the existing fuzzy critical path methods, also referred here, can be used to find the appropriate results of such fuzzy critical path problems in which all the parameters are represented by LR flat fuzzy numbers.

In this section, to overcome the shortcomings of the existing ranking approaches, a new ranking approach is proposed for comparing LR flat fuzzy numbers by modifying the parameters of the ranking approach from Farhadinia (2009).

Let $\widetilde{A} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\widetilde{B} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be two LR flat fuzzy numbers. Then, use the following steps to compare $\widetilde{A}$ and $\widetilde{B}$:

Step 1 Find $C^M(\widetilde{A}) = \inf\{x \in \mathrm{Supp}(\widetilde{A}) : \mu_{\widetilde{A}}(x) = 1\} = m_1$ and
$C^M(\widetilde{B}) = \inf\{x \in \mathrm{Supp}(\widetilde{B}) : \mu_{\widetilde{B}}(x) = 1\} = m_2$

Case (i) If $C^M(\widetilde{A}) > C^M(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $C^M(\widetilde{A}) < C^M(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $C^M(\widetilde{A}) = C^M(\widetilde{B})$ then go to Step 2.

Step 2 Find $L^M(\widetilde{A}) = m_1 - \alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda$ and
$L^M(\widetilde{B}) = m_2 - \alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda$

Case (i) If $L^M(\widetilde{A}) > L^M(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $L^M(\widetilde{A}) < L^M(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $L^M(\widetilde{A}) = L^M(\widetilde{B})$ then go to Step 3.

Step 3 Find $W^M(\widetilde{A}) = n_1 - m_1 + 2\alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda + 2\beta_1 \int_0^1 R^{-1}(\lambda)d\lambda$ and
$W^M(\widetilde{B}) = n_2 - m_2 + 2\alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda + 2\beta_2 \int_0^1 R^{-1}(\lambda)d\lambda$

Case (i) If $W^M(\widetilde{A}) > W^M(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $W^M(\widetilde{A}) < W^M(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $W^M(\widetilde{A}) = W^M(\widetilde{B})$ then go to Step 4.

Step 4 Find $S^M(\widetilde{A}) = \int \mu_{\widetilde{A}}(x)dx = n_1 - m_1 + \alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda + \beta_1 \int_0^1 R^{-1}(\lambda)d\lambda$
and $S^M(\widetilde{B}) = \int \mu_{\widetilde{B}}(x)dx = n_2 - m_2 + \alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda + \beta_2 \int_0^1 R^{-1}(\lambda)d\lambda$

Case (i) If $S^M(\widetilde{A}) > S^M(\widetilde{B})$ then $\widetilde{A} \succ \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$

Case (ii) If $S^M(\widetilde{A}) < S^M(\widetilde{B})$ then $\widetilde{A} \prec \widetilde{B}$ i.e.,
maximum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{B}$ and minimum $\{\widetilde{A}, \widetilde{B}\} = \widetilde{A}$

Case (iii) If $S^M(\widetilde{A}) = S^M(\widetilde{B})$ then $\widetilde{A} = \widetilde{B}$.

5.1. Efficiency of the proposed ranking approach

To show the efficiency of the proposed ranking approach, the ordering of three sets of LR flat fuzzy numbers, obtained by using the ranking approach of Farhadinia (2009) and the proposed ranking approach, are shown in Table 5.

REMARK 4 *The proposed ranking approach can also be used for comparing LR fuzzy numbers by putting* $m = n$.

6. Validity of the proposed ranking approach

In Section 4, it was proved that if $\widetilde{A}$ and $\widetilde{B}$ are two LR fuzzy numbers such that (i) $C(\widetilde{A}) = C(\widetilde{B})$, (ii) $L(\widetilde{A}) = L(\widetilde{B})$, (iii) $W(\widetilde{A}) = W(\widetilde{B})$, (iv) $S(\widetilde{A}) =$

Table 5: Ordering of LR flat fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ obtained by using existing and proposed ranking approaches

Sets of fuzzy numbers	$\widetilde{A}$	$\widetilde{B}$	$L(x)$ and $R(x)$	Ordering of $\widetilde{A}$ and $\widetilde{B}$ obtained by using existing ranking approach (Farhadinia, 2009)	Ordering of $\widetilde{A}$ and $\widetilde{B}$ obtained by using proposed ranking approach
Set 1	$(10,15,2,4)_{LR}$	$(10,16,2,3)_{LR}$	$L(x)=e^{-\lvert x\rvert}$ and $R(x)=e^{-\lvert x\rvert}$	$C(\widetilde{A})=C(\widetilde{B})=10$ $L(\widetilde{A})=L(\widetilde{B})=8$ $W(\widetilde{A})=W(\widetilde{B})=11$ $S(\widetilde{A})=S(\widetilde{B})=11$ i.e., $\widetilde{A}=\widetilde{B}$	$C^M(\widetilde{A})=C^M(\widetilde{B})=10$ $L^M(\widetilde{A})=L^M(\widetilde{B})=8$ $W^M(\widetilde{A})=17$, $W^M(\widetilde{B})=16$ i.e., $\widetilde{A}\succ\widetilde{B}$
Set 2	$(5,7,1,3)_{LR}$	$(5,8,1,2)_{LR}$	$L(x)=\max\{0,1-x^2\}$ and $R(x)=e^{-\lvert x\rvert}$	$C(\widetilde{A})=C(\widetilde{B})=5$ $L(\widetilde{A})=L(\widetilde{B})=4$ $W(\widetilde{A})=W(\widetilde{B})=6$ $S(\widetilde{A})=S(\widetilde{B})=5.67$ i.e., $\widetilde{A}=\widetilde{B}$	$C^M(\widetilde{A})=C^M(\widetilde{B})=5$ $L^M(\widetilde{A})=L^M(\widetilde{B})=4.33$ $W^M(\widetilde{A})=9.33$, $W^M(\widetilde{B})=8.33$ i.e., $\widetilde{A}\succ\widetilde{B}$
Set 3	$(16,20,4,5)_{LR}$	$(16,19,4,6)_{LR}$	$L(x)=\max\{0,1-x^4\}$ and $R(x)=e^{-\lvert x\rvert}$	$C(\widetilde{A})=C(\widetilde{B})=16$ $L(\widetilde{A})=L(\widetilde{B})=12$ $W(\widetilde{A})=W(\widetilde{B})=13$ $S(\widetilde{A})=S(\widetilde{B})=12.2$ i.e., $\widetilde{A}=\widetilde{B}$	$C^M(\widetilde{A})=C^M(\widetilde{B})=16$ $L^M(\widetilde{A})=L^M(\widetilde{B})=12.8$ $W^M(\widetilde{A})=20.4$, $W^M(\widetilde{B})=21.4$ i.e., $\widetilde{A}\prec\widetilde{B}$

$S(\widetilde{B})$, then $\widetilde{A} = \widetilde{B}$, but if $\widetilde{A}$ and $\widetilde{B}$ are two LR flat fuzzy numbers such that (i) $C(\widetilde{A}) = C(\widetilde{B})$, (ii) $L(\widetilde{A}) = L(\widetilde{B})$, (iii) $W(\widetilde{A}) = W(\widetilde{B})$, (iv) $S(\widetilde{A}) = S(\widetilde{B})$, then $\widetilde{A} \neq \widetilde{B}$. Due to the same reason, on applying the existing fuzzy critical path methods with the ranking approach of Farhadinia (2009) for solving such fuzzy critical path problems in which all the parameters are represented by LR fuzzy numbers a unique fuzzy number, representing the maximum total fuzzy project completion time, is obtained, while on applying the existing fuzzy critical path methods with the approach of Farhadinia (2009) for solving such fuzzy critical path problems in which all the parameters are represented by LR flat fuzzy numbers, more than one fuzzy number, representing the maximum total fuzzy project completion time, are obtained. In this section, to prove that by using the existing fuzzy critical path methods, considered here, with the proposed ranking approach, always a unique LR flat fuzzy number, representing the maximum total fuzzy project completion time, will be obtained, it is proved that if $\widetilde{A}$ and $\widetilde{B}$ are either LR fuzzy numbers or LR flat fuzzy numbers such that (i) $C^M(\widetilde{A}) = C^M(\widetilde{B})$, (ii) $L^M(\widetilde{A}) = L^M(\widetilde{B})$, (iii) $W^M(\widetilde{A}) = W^M(\widetilde{B})$, (iv) $S^M(\widetilde{A}) = S^M(\widetilde{B})$, then $\widetilde{A} = \widetilde{B}$.

PROPOSITION 6.1 *Let* $\widetilde{A} = (m_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{B} = (m_2, \alpha_2, \beta_2)_{LR}$ *be two LR fuzzy numbers such that*
(i) $C^M(\widetilde{A}) = C^M(\widetilde{B})$ *(ii)* $L^M(\widetilde{A}) = L^M(\widetilde{B})$ *(iii)* $W^M(\widetilde{A}) = W^M(\widetilde{B})$ *(iv)* $S^M(\widetilde{A}) = S^M(\widetilde{B})$. *Then,* $\widetilde{A} = \widetilde{B}$.

Proof.

$$\text{(i)}C^M(\widetilde{A}) = C^M(\widetilde{B})$$
$$\Rightarrow m_1 = m_2 \tag{8}$$
$$\text{(ii)}L^M(\widetilde{A}) = L^M(\widetilde{B})$$
$$\Rightarrow m_1 - \alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda = m_2 - \alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda \tag{9}$$
$$\text{(iii)}W^M(\widetilde{A}) = W^M(\widetilde{B})$$
$$\Rightarrow 2\alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda + 2\beta_1 \int_0^1 R^{-1}(\lambda)d\lambda$$
$$= 2\alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda$$
$$+2\beta_2 \int_0^1 R^{-1}(\lambda)d\lambda \tag{10}$$
$$\text{(iv)}S^M(\widetilde{A}) = S^M(\widetilde{B})$$

$$\Rightarrow \alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda + \beta_1 \int_0^1 R^{-1}(\lambda)d\lambda$$
$$= \alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda + \beta_2 \int_0^1 R^{-1}(\lambda)d\lambda \quad (11)$$

On solving (8), (9), (10) and (11)
$m_1 = m_2$, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$
i.e., $\widetilde{A} = \widetilde{B}$.

REMARK 5 *For an LR fuzzy number* $\widetilde{A} = (m, \alpha, \beta)_{LR}$, $S^M(\widetilde{A}) = \int \mu_{\widetilde{A}}(x)dx = \int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx + \int_m^{m+\beta} R(\frac{x-m}{\beta})dx = \alpha \int_0^1 L^{-1}(\lambda)d\lambda + \beta \int_0^1 R^{-1}(\lambda)d\lambda$

PROPOSITION 6.2 *Let* $\widetilde{A} = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ *and* $\widetilde{B} = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ *be two LR flat fuzzy numbers such that*
(i) $C^M(\widetilde{A}) = C^M(\widetilde{B})$ *(ii)* $L^M(\widetilde{A}) = L^M(\widetilde{B})$ *(iii)* $W^M(\widetilde{A}) = W^M(\widetilde{B})$
(iv) $S^M(\widetilde{A}) = S^M(\widetilde{B})$. *Then,* $\widetilde{A} = \widetilde{B}$.

Proof.

(i)$C^M(\widetilde{A}) = C^M(\widetilde{B})$
$$\Rightarrow m_1 = m_2 \quad (12)$$
(ii)$L^M(\widetilde{A}) = L^M(\widetilde{B})$
$$\Rightarrow m_1 - \alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda = m_2 - \alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda \quad (13)$$
(iii)$W^M(\widetilde{A}) = W^M(\widetilde{B})$
(iv)$S^M(\widetilde{A}) = S^M(\widetilde{B})$
$$\Rightarrow n_1 - m_1 + 2\alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda + 2\beta_1 \int_0^1 R^{-1}(\lambda)d\lambda$$
$$= n_2 - m_2 + 2\alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda$$
$$+2\beta_2 \int_0^1 R^{-1}(\lambda)d\lambda \quad (14)$$

$$\Rightarrow n_1 - m_1 + \alpha_1 \int_0^1 L^{-1}(\lambda)d\lambda + \beta_1 \int_0^1 R^{-1}(\lambda)d\lambda$$
$$= n_2 - m_2 + \alpha_2 \int_0^1 L^{-1}(\lambda)d\lambda$$
$$+\beta_2 \int_0^1 R^{-1}(\lambda)d\lambda \quad (15)$$

On solving (12), (13), (14) and (15)
$m_1 = m_2$, $n_1 = n_2$, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ i.e., $\widetilde{A} = \widetilde{B}$. ■

REMARK 6 *For an LR flat fuzzy number* $\widetilde{A} = (m, n, \alpha, \beta)_{LR}$ $S^M(\widetilde{A}) = \int \mu_{\widetilde{A}}(x)dx = \int_{m-\alpha}^{m} L(\frac{m-x}{\alpha})dx + \int_m^n dx + \int_n^{n+\beta} R(\frac{x-n}{\beta})dx = n - m + \alpha \int_0^1 L^{-1}(\lambda)d\lambda + \beta \int_0^1 R^{-1}(\lambda)d\lambda$

7. Advantage of the proposed ranking approach over the existing ranking approaches

The main advantage of the proposed ranking approach over the existing ones, referred to several times over in this papaer, is that when solving the fuzzy critical path problems by using the different existing fuzzy critical path methods, also quoted here, with the ranking approaches mentioned, more than one fuzzy number, representing the maximum total fuzzy project completion time, is obtained so that there will be different interpretations for the maximum total fuzzy project completion time of the same project, which is not appropriate. At the same time, by using the different existing fuzzy critical path methods with the proposed ranking approach a unique fuzzy number, representing the maximum total fuzzy project completion time, is obtained. So, there will be a unique interpretation of maximum total fuzzy completion time of a project.

To show the advantage of the proposed ranking approach over existing ranking approach, the results of fuzzy critical path problem of Example 4.2, obtained by using different existing fuzzy critical path methods with the proposed ranking approach are shown in Table 6.

Table 6: Results of Example 4.2 obtained by using existing fuzzy critical path methods with proposed ranking approach

Example	Results for the fuzzy critical path problem obtained by using existing fuzzy critical path methods with the proposed ranking approach	
	Fuzzy critical path	Maximum total fuzzy project completion time ($\widetilde{T}$)
4.2	$1 \Rightarrow 2 \Rightarrow 4$	$(25, 28, 4, 9)_{LR}$, $W^M(\widetilde{T}) = 26.33$

It is obvious from the results, shown in Tables 2 and 4 that if the fuzzy critical path problem, chosen in Example 4.1, is solved by using the existing fuzzy critical path methods with the ranking approach of Farhadinia (2009), then a unique fuzzy number, representing the maximum total fuzzy project completion time, is obtained, but if the fuzzy critical path problem, chosen in Example 4.2 is solved by using the existing fuzzy critical path methods with the ranking approach of Farhadinia (2009), then more than one fuzzy number, representing the maximum total fuzzy project completion time, is obtained, which is not appropriate. The results, shown in Table 6, indicate that on solving the chosen fuzzy critical path problem by using the existing fuzzy critical path methods with the proposed ranking approach a unique fuzzy number, representing the maximum total fuzzy project completion time, is obtained, which is appropriate.

8. Conclusion and future work

Some fuzzy critical path problems have been used to show that it is better not to use the existing fuzzy critical path methods with existing ranking approaches for solving the fuzzy critical path problems, and a new ranking approach, obtained by modifying an existing ranking approach, is proposed for comparing LR flat fuzzy numbers.

On the basis of the presented study it can be concluded that it is better to use the existing fuzzy critical path methods with the proposed ranking approach instead of the existing fuzzy critical path methods with existing ranking approaches, referred to throughout this paper, for solving the fuzzy critical path problems.

In the future it will be attempted to obtain the appropriate results for all such problems in which the existing ranking approaches are used for comparing LR flat fuzzy numbers by using the proposed ranking approach.

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References

Abbasbandy, S. and Asady, B. (2006) Ranking of fuzzy numbers by sign distance. *Information Sciences* **176**, 2405–2416.

Abbasbandy, S. and Hajjari, T. (2009) A new approach for ranking of LR flat fuzzy numbers. *Computers and Mathematics with Applications* **57**, 413–419.

Asady, B. and Zendehnam, A. (2007) Ranking fuzzy numbers by distance minimization. *Applied Mathematical Modelling* **31**, 2589–2598.

Chanas, S. and Zielinski, P. (2001) Critical path analysis in the network with fuzzy activity times. *Fuzzy Sets and Systems* **122**, 195–204.

Chen, S.P. (2007) Analysis of critical paths in a project network with fuzzy activity times. *European Journal of Operational Research* **183**, 442–459.

Chen, S.M. and Chang, T.H. (2001) Finding mutiple possible critical paths using fuzzy PERT. *IEEE Transactions on Systems, Man and Cybernetics-Part B: Cybernetics* **31**, 930–937.

Chen, S.P. and Hsueh, Y.J. (2008) A simple approach to fuzzy critical path analysis in project networks. *Applied Mathematical Modelling* **32**, 1289–1297.

Cheng, C.H. (1998) A new ranking approach for ranking fuzzy numbers by distance method. *Fuzzy Sets and Systems* **95**, 307–317.

Choobineh, F. and Li, H. (1993) An index for ordering fuzzy numbers. *Fuzzy Sets and Systems* **54, 287–294.**

Chu, T.C. and Tsao, C.T. (2002) Ranking fuzzy numbers with an area between the centroid point and original point. *Computers and Mathematics with Applications* **43**, 111–117.

Dubois, D. and Prade, H. (1980) *Fuzzy Sets and Systems: Theory and Applications.* Academic Press, New York.

Farhadinia, B. (2009) Ranking fuzzy numbers based on lexicographical ordering. *World Academy of Science, Engineering and Technology* **57**, 1029–1032.

Han, T.C., Chung, C.C. and Liang, G.S. (2006) Application of fuzzy critical path method to airport's cargo ground operation systems. *Journal of Marine Science and Technology* **14**, 139–146.

Jain, R. (1976) Decision-making in the presence of fuzzy variables. *IEEE Transactions on Systems, Man and Cybernetics* **6**, 698–703.

Kumar, A., Singh, P., Kaur, A. and Kaur, P. (2011a) A new approach for ranking non-normal p-norm trapezoidal fuzzy numbers. *Computers and Mathematics with Applications* **61**, 881–887.

Kumar, A., Singh, P., Kaur, P. and Kaur, A. (2011b) RM approach for ranking of $L - R$ type generalized fuzzy numbers. *Soft Computing* **15**, 1373–1381.

Kwang, H.C. and Lee, J.H. (1999) A method for ranking fuzzy numbers and its application to decision making. *IEEE Transactions on Fuzzy Systems* **7**, 677–685.

Liang, G.S. and Han, T.C. (2004) Fuzzy critical path for project network. *Information and Management Sciences* **15**, 29–40.

Lin, F.T. (2001) Critical path method in activity networks with fuzzy activities durarion times. *Proceedings IEEE International Conference on Systems, Man and Cybernetics* **2**, 1155–1160.

Lin, F.T. (2002) Fuzzy critical path method based on statistical data. *Pro-*

ceedings IEEE International Conference on Fuzzy Systems **2**. IEEEE, 1245–1250.

LIN, F.T. and YAO, J.S. (2003) Fuzzy critical path method based on signed-distance ranking and statistical confidence-interval estimates. *Journal of Supercomputing* **24**, 305–325.

LIOU, T.S. and WANG, M.J. (1992) Ranking fuzzy numbers with integral value. *Fuzzy Sets and Systems* **50**, 247–255.

LIU, S.T. (2003) Fuzzy activity times in critical path and project crashing problems. *Cybernetics and Systems* **34**, 161–172.

MODARRES, M. and NEZHAD, S. (2001) Ranking fuzzy numbers by preference ratio. *Fuzzy Sets and Systems* **118**, 429–436.

MURAKAMI, S., MAEDA, S. and IMAMURA, S. (1983) Fuzzy decision analysis on the development of centralized regional energy control system. *Proceedings IFAC Symposium on Fuzzy Information, Knowledge Representation and Decision Analysis*, Marseilles, France, 353–358.

NASUTION, S.H. (1994) Fuzzy critical path method. *IEEE Transactions on Systems, Man and Cybernetics* **24**, 48–57.

RAMLI, N. and MOHAMAD, D. (2009) A comparative analysis of centroid methods in ranking fuzzy numbers. *European Journal of Scientific Research* **28**, 492–501.

SHANKAR, N.R., SIREESHA, V. and RAO, P.B. (2010) An analytical method for finding critical path in a fuzzy project network. *International Journal of Contemporary Mathematical Sciences* **5**, 953–962.

TAHA, H.A. (2003) *Operations Research: An Introduction.* Prentice-Hall, New Jersey.

WANG, Y.J. and LEE, H.S. (2008) The revised method of ranking fuzzy numbers with an area between the centroid and original points. *Computers and Mathematics with Applications* **55**, 2033–2042.

WANG, Y.M., YANG, J.B., XU, D.L. and CHIN, K.S. (2006) On the centroids of fuzzy numbers. *Fuzzy Sets and Systems* **157**, 919–926.

YAGER, R.R. (1978) Ranking fuzzy subsets over the unit interval. *Proceedings* 17^{th} *IEEE International Conference on Decision and Control.* IEEEE, 1435–1437.

YAO, J.S. and LIN, F.T. (2000) Fuzzy critical path method based on signed distance ranking of fuzzy numbers. *IEEE Transactions on Systems, Man and Cybernetics-Part A: Systems and Humans* **30**, 76–82.

YAO, J. and WU, K. (2000) Ranking fuzzy numbers based on decomposition principle and signed distance. *Fuzzy Sets and Systems* **116**, 275–288.

ZADEH, L.A. (1965) Fuzzy sets. *Information and Control* **8**, 338–353.

ZHAO, J. AND LIU, Q.S. (2008) Ranking fuzzy numbers based on the centroids of fuzzy numbers. *Fuzzy Systems and Mathematics* **22**, 142–146.