Vector space of increments*

by

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Abstract: The article discusses definitions of vector space for variance increment and standard deviation increment, as well as definition of scalar product for variance increment. This justifies using a vector calculus for variance increment and allows for emplyoing vector calculus methods for variance recalculated into variance increment. The paper also presents a practical example of combining images from sector-scan sonar based on comparison made between local increments of variance.

Keywords: incremental arithmetic, vector space, vector calculus, standard deviation increment, variance increment.

1.   Introduction

Defining vectors in space allows for creating the coordinate system, using which one may determine the coordinates of all the points in space relative to a chosen point of reference. Coordinates are numbers by which vectors in the coordinate system should be multiplied so that after addition one obtains a vector defining the translation of the point from the coordinate origin to the point determined by the coordinates. The sets of rational, real and complex numbers are used for the description of vectors.

The basic problem, whose solution will be searched in this paper, is to take into account data inaccuracies in a vector description. These inaccuracies result in inaccuracies of point position in a vector space. This requires the appropriate definitions of elements in vector space: vectors and scalars. It is necessary to find numbers which will include the description of uncertainty and, on the other hand, will meet Abelian group axioms (for the numbers describing vectors), field axioms (for the numbers describing scalars) and vector space axioms. The inaccuracy can be included in additional components of the numbers used for

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the vector description. The use of these components is the subject of this paper and it will be further presented by examples.

Inaccuracy of the position may result from measurement errors, computational errors, data corruption due to lossy compression and many other factors. It is often possible to estimate inaccuracy, what theoretically allows to take it into account in the results obtained on the basis of calculations. In the case of vector calculus, it is necessary to change the notation of vector by introducing numbers together with information about inaccuracy of value determined by a given number. Such numbers must have at least two components since they must represent a value and its inaccuracy at the same time.

As far as vector calculus is concerned, there are two possibilities of introducing numbers defining data inaccuracy: as numbers defining coordinates and vectors, or only vectors. In the former case, these numbers must be scalars in vector calculus and therefore must satisfy all the field axioms. There is only one set of numbers that does so, namely the set of complex numbers. Unfortunately, this is a set of multi-component numbers that satisfy field axioms. For the purpose of addition, the set of vectors must satisfy all the axioms of Abelian group. It is not necessary to define the multiplication of vectors, which considerably facilitates finding the right set of numbers.

The sets of numbers defining data inaccuracy are found in fuzzy arithmetic and interval arithmetic. Nevertheless, a problem arises, namely numbers belonging to the aforementioned methods of arithmetic are in a different group of numbers than those used in vector calculus. Vector is a difference between the coordinates of two points. Hence, numbers defining the vector must be directed numbers, i.e. not defining a given value in a direct way, but just the difference between the value and a given point of reference. Values that define fuzzy arithmetic numbers may be directed numbers but there is some inaccuracy involved - always expressed in an absolute way. Negative inaccuracies are inadmissible. Support of a fuzzy set must always be positive. Just as in the case of interval arithmetic, where the width of interval must always be positive. Negative values are inadmissible.

Axioms of Abelian group for addition are formulated in such a way that in order to satisfy them, there must be a symmetry relative to a neutral element. If any parameter defining the number is positive, there must be a possibility for it to be negative. This stems from the way of defining the opposite number that is different from fuzzy arithmetic and interval arithmetic. If the support of fuzzy number is positive, it should be assumed that it can also be negative, this being in contradiction with fuzzy arithmetic principles. Arithmetic of directed and arithmetic of absolute numbers have various practical applications, which may be noticed while interpreting the result of subtraction. In fuzzy arithmetic, fuzziness is subject to increase as a result of subtraction, and the result "informs" about its inaccuracy. As for arithmetic in accordance with the axioms of Abelian group, the subtraction result will "inform" about difference between inaccuracies of the numbers subtracted, and not about the inaccuracy of the result itself.

As far as fuzzy arithmetic is concerned, researchers attempted to make it conforming with the axioms of the aforementioned group, e.g. Mareš (1977, 1989). He used probability distributions as fuzzy numbers, and then adopted convolution as addition and finally created convolution representation. Nevertheless, a problem arises, namely there is no accordance with fuzzy arithmetic. As a result, the definition of a set of fuzzy numbers was provided (the definition of the opposite element was in accordance with fuzzy arithmetic but not with the axioms of the group, Mareš, 1994 . Similarly, there is a problem with classifying other sets of numbers that satisfy Abelian group axioms (such as, e.g. directed fuzzy numbers proposed by Kosiński and Prokopowicz, 2004 into fuzzy arithmetic.

As far as interval arithmetic is concerned, Kaucher (1973, 1980) suggested extended interval arithmetic which, by definition, satisfies all the axioms of the Abelian group. Therefore, it can be employed for defining vectors in vector spaces.

Since numbers used for defining vectors must be absolute, one must seek for proper sets of numbers. As a rule, such sets are found within a given arithmetic that may be referred to as incremental arithmetic, in the case of which numbers do not define the real value directly, but as a difference (increment) between a given level of reference and the level measured. Unlike fuzzy arithmetic and classical interval arithmetic, the primary aim of incremental arithmetic is not to determine the inaccuracy of the result obtained, but to analyze this inaccuracy in order to make comparisons, examine variability, forecast, etc. Theoretically, it is possible to determine the inaccuracy of results. In practice, this is as a rule pointless as it is much easier to employ fuzzy arithmetic or interval arithmetic. The only exception is vector calculus in the case of which one cannot use absolute numbers for formal reasons (axioms are not satisfied).

This distinction of applications results from the nature of subtraction operations. In a typical subtraction used in fuzzy and interval arithmetic the inaccuracy of information increases. For example, if there is some amount of apples in a big basket, then this amount can be more or less precisely determined on the basis of human estimation. One can assign some inaccuracy to this estimation. Similarly, after taking some apples from the big basket and putting them into a small one, it is possible to estimate the amount of apples in the small basket and to determine inaccuracy of this estimation. One can subtract both estimated amounts to find the amount of apples left in the big basket. However, the inaccuracy estimates should be added, because "the lack of knowledge" concerning apples in the big basket increased due to the summation of "uncertainties" originating from two inaccuracy estimates.

In some cases one can deal with a comparative task. For example, the amount of apples could be estimated using different methods for the same basket. Then, the result of subtraction presents the difference between estimation methods. In this case, when inaccuracies have been estimated each time, then the difference of inaccuracies needs to be calculated. It will enable evaluation of

the correctness of inaccuracies estimation for both methods. The "minus" sign of the difference shows which inaccuracy is greater.

One can also use the subtraction of inaccuracies to predict inaccuracies. For example, if elements of a particular size are produced on a production line, then on the basis of data concerning inaccuracies of production one can determine if the inaccuracy increases or decreases. This makes it possible to predict, as an example, when the error limit of inaccuracies for elements would be exceeded, and to plan an adjustment of production tools.

To describe inaccuracies one can use, in paritcular, the distribution or parameters defining it, like mean value, standard deviation, variance or range. The mean and the variance (standard deviation) are associated, therefore they can be noted as ordered pairs:

$$\forall \eta \in \mathbb{R}, \ \sigma^2 \in \mathbb{R}^+ : \left(\eta; \sigma^2 \right). \tag{1}$$

This pair consists of two numbers; the first of them is the relative one and the second the absolute number. This is the reason why this ordered pair can not be used for the vector description. A vector has to be described by two relative numbers. It is possible to convert the description from absolute to relative numbers according to the method given in Zaremba (1918). Having the given absolute number a one combines it with any absolute number b to make a pair. In such a manner the following ordered pair is created:

$$\forall a, b \in \mathbb{R}^+ : (a; b). \tag{2}$$

As an example, the measurement of temperature presented in Kelvin degrees is performed using absolute numbers. The lowest value is zero, and there are not lower values at all. If the measured temperature of 227.15ºK is given, then one can combine it with any other absolute value, for example with the freezing point of water, which is 237.15ºK. In this way one can obtain the ordered pair $(227.15; 237.15)$ºK. One can take any of these values as the reference point and then, if it is possible to calculate the difference between them, a single signed number can be used instead of the ordered pair. If in the above example the value 237.15ºK is taken as the reference point, then as the result of subtraction the value -10 is created. However, this value is not calculated in relation to the absolute zero, but is relative to the reference point 237.15ºK.

Thanks to the fact that the relative number is written as two absolute numbers, all operations performed on absolute numbers can be transferred to relative numbers. All operations are performed on absolute numbers. One can present addition, subtraction, multiplication and division of relative numbers using arithmetic operations on absolute numbers. For example, addition can be written as follows:

$$\forall a, b, c, d \in \mathbb{R}^+ : \{(a; b) + (c; d) \equiv (a + c; b + d)\}. \tag{3}$$

Moreover, it is possible to define the inverse element for a relative number:

$$- (a; b) = (b; a) \, . \tag{4}$$

The inversion is equivalent to the change of the number sign. This operation is possible only for those numbers, which can have the sign. It is impossible for absolute numbers, because they have no sign. The change of the number sign enables performing subtraction using the addition operation:

$$\forall a, b, c, d \in \mathbb{R}^+ : \{(a; b) - (c; d) \equiv (a; b) + [- (c; d)]\} \, . \tag{5}$$

The so defined subtraction operation is always executable (due to the fact that the addition of absolute numbers can be performed always). This makes difference in comparison with the subtraction of absolute numbers, which is not executable for some cases.

Similarly, one can present the ordered pair $\left(\eta; \sigma^2\right)$ as the relative number:

$$\forall \eta_a, \eta_b \in \mathbb{R} \ \forall \sigma_a^2, \sigma_b^2 \in \mathbb{R}^+ : \left(\left(\eta_a; \sigma_a^2\right); \left(\eta_b; \sigma_b^2\right)\right) \, . \tag{6}$$

The way in which the values η_a, η_b, σ_a^2 and σ_b^2 are calculated, depends on the nature of data. In the case of image processing these values are calculated for images pixels. The value of the "black & white" image pixel depends on the light intensity associated with a relevant light-sensitive element. The most frequent solution is to assign to this element an integer number from the interval 0-255. However, this number is not the measure of light intensity, because in the registration process it is changed during complex operations aimed to ensure a particular visual impression of the whole of the photo. Therefore, pixel values are dimensionless.

In the image processing the average value and the standard deviation are used as so called amplitude features of the image. In such a case the average value is calculated according to the formula (Pratt, 2001):

$$\eta_{j,k} = \frac{1}{W^2} \sum_{l=-w}^{w} \sum_{o=-w}^{w} P_{j+l,k+o} \, , \tag{7}$$

where $W = 2w + 1$, $P_{j+l,k+o}$ is the pixel of an image, for which the average value is calculated, and $\eta_{j,k}$ is this calculated value.

The average value is calculated for the square block of pixel neighbors. The size of this block is W, and for the formula mentioned above it has to be an odd value.

The standard deviation is calculates according to the formula (Pratt, 2001):

$$\sigma_{j,k} = \frac{1}{W} \sqrt{\sum_{l=-w}^{w} \sum_{o=-w}^{w} [P_{j+l,k+o} - \eta_{j+l,k+o}]^2} \, . \tag{8}$$

In such a way two tables are constructed, $\eta_{j,k}$ for average values, and $\sigma_{j,k}$ for standard deviations. The one separate random variable $X_{j,k}$ is associated with every pair of values $(\eta_{j,k}; \sigma_{j,k})$. In image processing this variable is mostly the discrete random variable taking the values $x = 0, 1, \ldots, 255$. Probabilities $P(X_{j,k} = x)$ of occurrence of particular random variable values one can find on the basis of the pixel block assigned to this variable. Usually the main goal of image processing is the analysis or modification of one selected image, and neglecting all the other. Due to this the pixel block is treated as a separate and closed population.

The average value, calculated in the way mentioned above, determines the local level of pixel values, and the standard deviation determines the local variability of an image. They can be used in further calculations, but the basic problem is that many methods require the use of relative numbers. Then one can use the relative number $((\eta_X; \sigma_X^2); (\eta_Y; \sigma_Y^2))$ associated with two random variables X and Y. One of these random variables, for example X, is the random variable associated with some pixel block, and another one is the reference random variable. The reference random variable can be chosen freely, as in the case of selection of the reference point for measurements of temperature.

The ordered pair formula is only some way of representing relative numbers. It determines indirectly some relative number. One can describe the relative number directly as an increment in relation to the reference point, i.e. the difference between both components of that number $((\eta_X; \sigma_X^2); (\eta_Y; \sigma_Y^2))$:

$$((\eta_X; \sigma_X^2); (\eta_Y; \sigma_Y^2)) \equiv (\eta_X; \sigma_X^2) - (\eta_Y; \sigma_Y^2) = (\Delta\eta; \Delta\sigma^2). \qquad (9)$$

The subtraction used above is the subtraction based on the inverse element of the group:

$$f - g \equiv f + (-g), \qquad (10)$$

where-g is the inverse element for g fulfilling the condition:

$$g + (-g) = e, \qquad (11)$$

where e is the neutral element for addition.

In the case of ordered pair $(\eta_Y; \sigma_Y^2)$, the calculated inverse element has the form $(-\eta_Y; \ -\sigma_Y^2)$. $-\sigma_Y^2$ is not variance, therefore the increment of variance $\Delta\sigma^2$ is not variance itself.

The increment is always the difference of two absolute numbers. Therefore, it always concerns two values. The increment of variance represents the difference of two variances, i.e. it illustrates how variances of two random variables differ (and which one is greater). So, the increment of variance concerns not only one random variable, but two random variables. One of them can be treated as the reference variable. The positive value of the increment of variance means that the random variable has greater variance value than the reference variable, and the negative value means the opposite case.

The variance is associated with the average value and, similarly, the increment of variance is associated with the increment of standard deviation. Therefore they also can be treated as the ordered pair:

$$\forall \Delta\eta, \Delta\sigma^2 \in \mathbb{R} : \left(\Delta\eta; \Delta\sigma^2\right). \tag{12}$$

The way of addition of average values and variances corresponds to the behavior of these parameters in the addition of independent random variables. One can assume that the increment of variance is a relative number, and so it can have negative values. The relativity of average value increment and variance increment results in the relativity of the ordered pair that will be treated as a number belonging to set $\mathbb{L}_{\Delta\sigma^2}$. Addition in this set may be defined as follows:

$$\forall a, b \in \mathbb{L}_{\Delta\sigma^2} : \left\{ a + b \equiv \left(\Delta\eta_a + \Delta\eta_b; \Delta\sigma_a^2 + \Delta\sigma_b^2\right) \right\}. \tag{13}$$

Element $(0; 0)$ is neutral, and element $\left(-\Delta\eta; -\Delta\sigma^2\right)$ is the opposite of element $\left(\Delta\eta; \Delta\sigma^2\right)$. The set $\mathbb{L}_{\Delta\sigma^2}$ together with addition forms an Abelian group. Taking the definition of addition into account, it is assumed that variables, interrelated with elements of the set $\mathbb{L}_{\Delta\sigma^2}$, are independent one of another. On the basis of multiplication via multiple addition:

$$na = \underbrace{a + a + \ldots + a}_{n}, \tag{14}$$

for the set $\mathbb{L}_{\Delta\sigma^2}$ one may define multiplication as number increment, i.e. increase in the number of summations Δn:

$$\forall \Delta n \in \mathbb{R}, b \in \mathbb{L}_{\Delta\sigma^2} : \left\{ \Delta n b \equiv \left(\Delta n \Delta\eta_b; \Delta n \Delta\sigma_b^2\right) \right\}. \tag{15}$$

This corresponds to n-time addition of independent random variables.

The above addition can be used to compare pixels of an image. For example, the images can be taken from a still camera. The comparison is aimed to detect changes on the image and, consequently, to detect objects which can appear there. The result of comparison of particular pixels is used for the detection of changed image fragments. A random variable $X_{j,k}$ is associated with one compared pixel and a random variable $Y_{j,k}$ is associated with another, where j and k are pixel coordinates. One can compare average values (to compare brightness levels of pixels) and standard deviations as well. This second comparison enables to compare local spatial changeability of pixels, what in the case of digital cameras means the comparison of object outlines. So, for each pixel average values and standard deviations are calculated.

It can be assumed at the beginning that pixels are compared for totally different fragments of an image. For these fragments there is no dependence between random variables $X_{j,k}$ and $Y_{j,k}$. On the basis of local average values and local

standard deviations ordered pairs $\left(\Delta\eta_{Xj,k}; \Delta\sigma^2_{Xj,k}\right)$ and $\left(\Delta\eta_{Yj,k}; \Delta\sigma^2_{Yj,k}\right)$ are calculated. For $\left(\Delta\eta_{Yj,k}; \Delta\sigma^2_{Yj,k}\right)$ the inverse element $\left(-\Delta\eta_{Yj,k}; -\Delta\sigma^2_{Yj,k}\right)$ is calculated, and then the sum $\left(\Delta\eta_{Xj,k}; \Delta\sigma^2_{Xj,k}\right) + \left(-\Delta\eta_{Yj,k}; -\Delta\sigma^2_{Yj,k}\right)$. This sum has a comparative nature. It determines how two pixel values and their spatial changeability differ, i.e. how object outlines differ.

In the strictly theoretical case of comparison of the same image fragments, pixel values correspond to each other directly. The random variables $X_{j,k}$ and $Y_{j,k}$ are dependent. If the variable $X_{j,k}$ takes some value, then the variable $Y_{j,k}$ takes the same value, the same with standard deviations of variables $X_{j,k}$ and $Y_{j,k}$. If these two images are added, then the average value and standard deviation of the resulted image are doubled. This results from doubling of amplitude of image value changes. This corresponds to the multiplication of random variable by the constant 2.

In practice, even if the scene before the camera does not change, the registered images are not necessarily the same. This can be the result of changes of sunlight intensity. It affects average value levels, which will be greater for the brighter image. It affects also the amplitude of image changeability, which will be greater for brighter images, and therefore standard deviations of brighter images will be greater as well. This differences in average values and standard deviations do not affect the dependency of random variables associated with particular image pixels. In the case of addition of such images average values are added and standard deviations are added as well.

In the case of comparison of camera images with the same fragments, the local average values and local standard deviations are the base for the calculation of ordered pairs $(\Delta\eta_{Xj,k}; \Delta\sigma_{Xj,k})$ and $(\Delta\eta_{Yj,k}; \Delta\sigma_{Yj,k})$. For $(\Delta\eta_{Yj,k}; \Delta\sigma_{Yj,k})$ the inverse element is calculated, and then the sum $(\Delta\eta_{Xj,k}; \Delta\sigma_{Xj,k})$ $+ (-\Delta\eta_{Yj,k}; -\Delta\sigma_{Yj,k})$.

In the above example the interdependence of random variables is not known before the start of calculations. Only the result of calculations determines the obtained interdependence of random variables. For both cases one can determine the similarity of fragments and choose the appropriate final value just after the calculation results are obtained. Therefore, it is necessary to define operations for increments of variance and for increments of standard deviation as well.

The other case to be examined occurs when variables depend on one another. It is necessary then to change the notation. Average value increment $\Delta\eta$ and standard deviation increment $\Delta\sigma$ should be adopted as ordered pair:

$$\forall\Delta\eta, \sigma \in \mathbb{R} : (\Delta\eta; \Delta\sigma). \tag{16}$$

A set of all the ordered pairs will belong to the set $\mathbb{L}_{\Delta\sigma}$. Addition in this set may be expressed as follows:

$$\forall a, b \in \mathbb{L}_{\Delta\sigma} : \{a + b \equiv (\Delta\eta_a + \Delta\eta_b; \Delta\sigma_a + \Delta\sigma_b)\}. \tag{17}$$

Element $(0; 0)$ is neutral, and element $(-\Delta\eta; -\Delta\sigma)$ is the opposite of element $(\Delta\eta; \Delta\sigma)$. The set $\mathbb{L}_{\Delta\sigma}$ together with addition form the Abelian group. On the basis of summation via multiple addition, for the set $\mathbb{L}_{\Delta\sigma}$ one may define multiplication as number increment Δn:

$$\forall \Delta n \in \mathbb{R}, b \in \mathbb{L}_{\Delta\sigma} : \{\Delta nb \equiv (\Delta n \Delta\eta_b; \Delta n \Delta\sigma_b)\}. \tag{18}$$

This corresponds to n-time addition of totally dependent random variables and is equivalent to the multiplication of an average value increment and standard deviation increment by the constant.

The objective of this paper is to define the vector space, where vectors are described by means of ordered pairs: (increment of average value, increment of variance) and (increment of average value, increment of standard deviation). The scalars are quantity increments generalized to the form allowing to take real numbers (values). In the case of ordered pairs (increment of average value, increment of standard deviation) the scalars are constants which the random variables are multiplied by. This results from the nature of vector space. Each vector consists of a specified number of ordered pairs $(\Delta\eta; \Delta\sigma)$. Their number is the dimension of the space. The interdependence of random variables associated with these pairs is meaningless, because there are no arithmetic operations combining them. There is only one exception: calculation of scalar product. Vectors can be added, what corresponds to the addition of corresponding vector components. Additionally, dependence of random variables associated with added vectors is assumed. The multiplication by the scalar n corresponds to n-time addition of the multiplied vector itself.

The usage of vector spaces in an image comparison will be presented to indicate practical applications of the defined spaces.

2. Vector space

As a rule, a vector consists of multiple elements, whose number determines the dimension of space. Therefore, it may be defined as an ordered set of n elements:

$$(a_1; a_2; \ldots; a_n), \tag{19}$$

while $a_1; a_2; \ldots a_n$ may belong to the set $\mathbb{L}_{\Delta\sigma^2}$:

$$\left(\left(\Delta\eta_1; \Delta\sigma_1^2\right); \left(\Delta\eta_2; \Delta\sigma_2^2\right); \ldots; \left(\Delta\eta_n; \Delta\sigma_n^2\right)\right). \tag{20}$$

Such and ordered set of n elements may be denoted $\mathbb{L}_{\Delta\sigma^2 n}$. Analogously, one may define the ordered set of n elements belonging to the set $\mathbb{L}_{\Delta\sigma}$.

Operator of addition in the set $\mathbb{L}_{\Delta\sigma^2 n}$ may be defined in the following way:

$$\forall u, v \in \mathbb{L}_{\Delta\sigma^2 n} : \big\{[u + v] \equiv \big(\big(\Delta\eta_{u1} + \Delta\eta_{v1}; \Delta\sigma_{u1}^2 + \Delta\sigma_{v1}^2\big);$$
$$\big(\Delta\eta_{u2} + \Delta\eta_{v2}; \Delta\sigma_{u2}^2 + \Delta\sigma_{v2}^2\big); \ldots; \big(\Delta\eta_{un} + \Delta\eta_{vn}; \Delta\sigma_{un}^2 + \Delta\sigma_{vn}^2\big)\big)\big\}. \tag{21}$$

Formally speaking, vector space may be defined as a set X over field $(K, +, \cdot)$ if the following conditions are satisfied (Bronsztejn et al., 2004):

1. $(X, +)$ is the Abelian group.
2. A binary operation, assigning the element $c \in X$ to the ordered pair (k, x) where $k \in K$ and $x \in X$, has been defined. This operation may be referred to as the multiplication of a vector by a scalar.
3. Neutral element of multiplication in the field K is also a neutral element of multiplication of vector by scalar. If $e \in K$ is a neutral element of multiplication in the set K, then:

$$\forall x \in X : [ex = x].\tag{22}$$

4. Associative character of multiplication of a vector by scalar:

$$\forall r, s \in K, \forall x \in X : [r(sx) = (rs)x].\tag{23}$$

5. Commutativity of multiplication of a vector by scalar toward addition in the set K:

$$\forall r, s \in K, \forall x \in X : [(r + s)x = rx + sx].\tag{24}$$

6. Commutativity of multiplication of a vector by scalar toward addition in the set X:

$$\forall r \in K, \forall x, y \in X : [r(x + y) = rx + ry].\tag{25}$$

Elements of the set K may be referred to as scalars, and elements of the set X - as vectors.

One may point out the following properties of vector space for the set $\mathbb{L}_{\Delta\sigma^2 n}$:

1. Operation $+$ is associative and commutative in the set $\mathbb{L}_{\Delta\sigma^2 n}$, which stems from associative and commutative character of addition in the set $\mathbb{L}_{\Delta\sigma^2}$. There is a neutral element in the set $\mathbb{L}_{\Delta\sigma^2 n}$:

$$((0;0);(0;0);\ldots;(0;0)),\tag{26}$$

and the inverse of an element:

$$\left(\left(-\Delta\eta_1; -\Delta\sigma_1^2\right); \left(-\Delta\eta_2; -\Delta\sigma_2^2\right); \ldots; \left(-\Delta\eta_n; -\Delta\sigma_n^2\right)\right).\tag{27}$$

Hence, the set $\mathbb{L}_{\Delta\sigma^2 n}$ together with addition form the Abelian group.

2. It can be adopted as binary operation:

$$\forall a \in \mathbb{R}, \forall u \in \mathbb{L}_{\Delta\sigma^2 n} :$$
$$\left[a \cdot u \equiv \left(\left(a\Delta\eta_{u1}; a\Delta\sigma_{u1}^2\right); \left(a\Delta\eta_{u2}; a\Delta\sigma_{u2}^2\right); \ldots; \left(a\Delta\eta_{un}; a\Delta\sigma_{un}^2\right)\right)\right].\tag{28}$$

3. Unity — the neutral element of multiplication in the field $\mathbb{R}$ — is also a neutral element in the multiplication of the vector by scalar:

$$\forall u \in \mathbb{L}_{\Delta\sigma^2 n} :$$
$$1 \cdot a = \left(\left(1\Delta\eta_{u1}; 1\Delta\sigma_{u1}^2 \right) ; \left(1\Delta\eta_{u2}; 1\Delta\sigma_{u2}^2 \right) ; \ldots ; \left(1\Delta\eta_{un}; 1\Delta\sigma_{un}^2 \right) \right) =$$
$$= \left(\left(\Delta\eta_{u1}; \Delta\sigma_{u1}^2 \right) ; \left(\Delta\eta_{u2}; \Delta\sigma_{u2}^2 \right) ; \ldots ; \left(\Delta\eta_{un}; \Delta\sigma_{un}^2 \right) \right) . \quad (29)$$

4. Associative character of multiplication of a vector by scalar:

$$\forall a, b \in \mathbb{R}, \forall u \in \mathbb{L}_{\Delta\sigma^2 n} :$$
$$a\left[bu \right] = a\left[b \cdot \left(\left(\Delta\eta_{u1}; \Delta\sigma_{u1}^2 \right) ; \left(\Delta\eta_{u2}; \Delta\sigma_{u2}^2 \right) ; \ldots ; \left(\Delta\eta_{un}; \Delta\sigma_{un}^2 \right) \right) \right] =$$
$$= a \cdot \left(\left(b\Delta\eta_{u1}; b\Delta\sigma_{u1}^2 \right) ; \left(b\Delta\eta_{u2}; b\Delta\sigma_{u2}^2 \right) ; \ldots ; \left(b\Delta\eta_{un}; b\Delta\sigma_{un}^2 \right) \right) =$$
$$= \left(\left(ab\Delta\eta_{u1}; ab\Delta\sigma_{u1}^2 \right) ; \left(ab\Delta\eta_{u2}; ab\Delta\sigma_{u2}^2 \right) ; \ldots ; \left(ab\Delta\eta_{un}; ab\Delta\sigma_{un}^2 \right) \right) . \quad (30\text{a})$$

$$\forall a, b \in \mathbb{R}, \forall u \in \mathbb{L}_{\Delta\sigma^2 n} :$$
$$\left[ab \right] u = ab \cdot \left(\left(\Delta\eta_{u1}; \Delta\sigma_{u1}^2 \right) ; \left(\Delta\eta_{u2}; \Delta\sigma_{u2}^2 \right) ; \ldots ; \left(\Delta\eta_{un}; \Delta\sigma_{un}^2 \right) \right) =$$
$$= \left(\left(ab\Delta\eta_{u1}; ab\Delta\sigma_{u1}^2 \right) ; \left(ab\Delta\eta_{u2}; ab\Delta\sigma_{u2}^2 \right) ; \ldots ; \left(ab\Delta\eta_{un}; ab\Delta\sigma_{un}^2 \right) \right) . \quad (30\text{b})$$

5. Commutativity of a multiplication of a vector by scalar toward addition in the set $\mathbb{R}$:

$$\forall a, b \in \mathbb{R}, \forall u \in \mathbb{L}_{\Delta\sigma^2 n} : au + bu =$$
$$= \left(\left(a\Delta\eta_{u1}; a\Delta\sigma_{u1}^2 \right) ; \left(a\Delta\eta_{u2}; a\Delta\sigma_{u2}^2 \right) ; \ldots ; \left(a\Delta\eta_{un}; a\Delta\sigma_{un}^2 \right) \right) +$$
$$+ \left(\left(b\Delta\eta_{u1}; b\Delta\sigma_{u1}^2 \right) ; \left(b\Delta\eta_{u2}; b\Delta\sigma_{u2}^2 \right) ; \ldots ; \left(b\Delta\eta_{un}; b\Delta\sigma_{un}^2 \right) \right) =$$
$$= \left(\left(a\Delta\eta_{u1} + b\Delta\eta_{u1}; a\Delta\sigma_{u1}^2 + b\Delta\sigma_{u1}^2 \right) ; \right.$$
$$\left(a\Delta\eta_{u2} + b\Delta\eta_{u2}; a\Delta\sigma_{u2}^2 + b\Delta\sigma_{u2}^2 \right) ; \ldots$$
$$\left. \ldots ; \left(a\Delta\eta_{un} + b\Delta\eta_{un}; a\Delta\sigma_{un}^2 + b\Delta\sigma_{un}^2 \right) \right) . \quad (31\text{a})$$

$$\forall a, b \in \mathbb{R}, \forall u \in \mathbb{L}_{\Delta\sigma^2 n} : \left[a + b \right] u =$$
$$= \left(\left(\left[a + b \right] \Delta\eta_{u1}; \left[a + b \right] \Delta\sigma_{u1}^2 \right) ; \left(\left[a + b \right] \Delta\eta_{u2}; \left[a + b \right] \Delta\sigma_{u2}^2 \right) ; \ldots \right.$$
$$\left. \ldots ; \left(\left[a + b \right] \Delta\eta_{un}; \left[a + b \right] \Delta\sigma_{un}^2 \right) \right) =$$
$$= \left(\left(a\Delta\eta_{u1} + b\Delta\eta_{u1}; a\Delta\sigma_{u1}^2 + b\Delta\sigma_{u1}^2 \right) ; \right.$$
$$\left(a\Delta\eta_{u2} + b\Delta\eta_{u2}; a\Delta\sigma_{u2}^2 + b\Delta\sigma_{u2}^2 \right) ; \ldots$$
$$\left. \ldots ; \left(a\Delta\eta_{un} + b\Delta\eta_{un}; a\Delta\sigma_{un}^2 + b\Delta\sigma_{un}^2 \right) \right) . \quad (31\text{b})$$

6. Commutativity of multiplication of a vector by scalar toward addition in the set $\mathbb{L}_{\Delta\sigma^2 n}$:

$\forall a \in \mathbb{R}, \forall u, v \in \mathbb{L}_{\Delta\sigma^2 n} : au + av =$
$$= \left(\left(a\Delta\eta_{u1}; a\Delta\sigma_{u1}^2\right); \left(a\Delta\eta_{u2}; a\Delta\sigma_{u2}^2\right); \ldots; \left(a\Delta\eta_{un}; a\Delta\sigma_{un}^2\right)\right) +$$
$$+ \left(\left(a\Delta\eta_{v1}; a\Delta\sigma_{v1}^2\right); \left(a\Delta\eta_{v2}; a\Delta\sigma_{v2}^2\right); \ldots; \left(a\Delta\eta_{vn}; a\Delta\sigma_{vn}^2\right)\right) =$$
$$= \left(\left(a\Delta\eta_{u1} + a\Delta\eta_{v1}; a\Delta\sigma_{u1}^2 + a\Delta\sigma_{v1}^2\right);\right.$$
$$\left(a\Delta\eta_{u2} + a\Delta\eta_{v2}; a\Delta\sigma_{u2}^2 + a\Delta\sigma_{v2}^2\right); \ldots$$
$$\left.\ldots; \left(a\Delta\eta_{un} + a\Delta\eta_{vn}; a\Delta\sigma_{un}^2 + a\Delta\sigma_{vn}^2\right)\right). \quad (32a)$$

$\forall a \in \mathbb{R}, \forall u, v \in \mathbb{L}_{\Delta\sigma^2 n} : a\left[u + v\right] =$
$$= a\left(\left(\Delta\eta_{u1} + \Delta\eta_{v1}; \Delta\sigma_{u1}^2 + \Delta\sigma_{v1}^2\right); \left(\Delta\eta_{u2} + \Delta\eta_{v2}; \Delta\sigma_{u2}^2 + \Delta\sigma_{v2}^2\right); \ldots\right.$$
$$\left.\ldots; \left(\Delta\eta_{un} + \Delta\eta_{vn}; \Delta\sigma_{un}^2 + \Delta\sigma_{vn}^2\right)\right) =$$
$$= \left(\left(a\Delta\eta_{u1} + a\Delta\eta_{v1}; a\Delta\sigma_{u1}^2 + a\Delta\sigma_{v1}^2\right);\right.$$
$$\left(a\Delta\eta_{u2} + a\Delta\eta_{v2}; a\Delta\sigma_{u2}^2 + a\Delta\sigma_{v2}^2\right); \ldots$$
$$\left.\ldots; \left(a\Delta\eta_{un} + a\Delta\eta_{vn}; a\Delta\sigma_{un}^2 + a\Delta\sigma_{vn}^2\right)\right). \quad (32b)$$

The set $\mathbb{L}_{\Delta\sigma^2 n}$ over the field of real numbers $\mathbb{R}$ is a vector space.

It is also plausible to define vector space on the basis of the set $\mathbb{L}_{\Delta\sigma}$:

$$\left(\left(\Delta\eta_1; \Delta\sigma_1\right); \left(\Delta\eta_2; \Delta\sigma_2\right); \ldots; \left(\Delta\eta_n; \Delta\sigma_n\right)\right). \quad (33)$$

A complete set of all such possible n elements will be referred to as $\mathbb{L}_{\Delta\sigma n}$ from now on. Addition in this set may be defined in the following way:

$$\forall u, v \in \mathbb{L}_{\Delta\sigma n} : \{[u + v] \equiv \left(\left(\Delta\eta_{u1} + \Delta\eta_{v1}; \Delta\sigma_{u1} + \Delta\sigma_{v1}\right);\right.$$
$$\left.\left(\Delta\eta_{u2} + \Delta\eta_{v2}; \Delta\sigma_{u2} + \Delta\sigma_{v2}\right); \ldots; \left(\Delta\eta_{un} + \Delta\eta_{vn}; \Delta\sigma_{un} + \Delta\sigma_{vn}\right)\right)\}.$$
$$(34)$$

One can demonstrate that the set $\mathbb{L}_{\Delta\sigma n}$ over the field of real numbers is a vector space. Proof is analogical to that concerning the set $\mathbb{L}_{\Delta\sigma^2 n}$. The binary operation is as follows:

$$\forall a \in \mathbb{R}, \forall u \in \mathbb{L}_{\Delta\sigma n} :$$
$$[a \cdot u \equiv \left(\left(a\Delta\eta_{u1}; a\Delta\sigma_{u1}\right); \left(a\Delta\eta_{u2}; a\Delta\sigma_{u2}\right); \ldots; \left(a\Delta\eta_{un}; a\Delta\sigma_{un}\right)\right)]. \quad (35)$$

3.   Scalar product in vector space of increments

In vector space, vectors may be subject to addition, extension and reduction, due to which they can be used for creating the coordinate system in which a

given (delineated) set of vectors allows to determine other vectors on the basis of scalars. Nevertheless, one cannot carry out limitless operations in this system. It is not possible to change the coordinate system once it has been chosen. It is always imposed in advance and it is not plausible to convert coordinates into another coordinate system. Such a mechanism is introduced by the definition of scalar product, which opens up the possibilities of carrying out operations in vector space. Scalar product is a part of the definition of pre-Hilbert space.

Vector space X over the field $(K, +, \cdot)$ is also called pre-Hilbert space if it includes a function that assigns element $\alpha \in K$ to each pair of elements (x, y) that are members of the set X, and satisfying the following conditions (Bronsztejn et al., 2004):

1. Scalar product of vector projected onto itself cannot be negative:

$$\forall x \in X : [(x, x) \geq 0] . \tag{36a}$$

2. Scalar product of vector projected onto itself may equal 0 only for neutral element of addition in the set of vectors:

$$[(x, x) = 0] \equiv [x \text{ is zero element}] . \tag{36b}$$

3. Factoring the constant out the scalar product:

$$\forall x, y \in X, \forall \alpha \in K : [(\alpha x, y) = \alpha(x, y)] . \tag{36c}$$

4. Commutativity of scalar product toward addition in the set X:

$$\forall x, y, z \in X : [(x + y, z) = (x, z) + (y, z)] . \tag{36d}$$

5. Conjugate symmetry:

$$\forall x, y \in X : [(x, y) = (y, x)^*] , \tag{36e}$$

where * denotes conjugate number.

This function is called scalar product in pre-Hilbert space. If - with the use of scalar product - in pre-Hilbert space one defines a norm expressed by the following formula (Bronsztejn et al., 2004):

$$\|x\| = \sqrt{(x, x)} , \tag{37}$$

then this space is referred to as unitary space.

On the basis of scalar product, one may define a formula that allows to calculate the absolute value of a vector along another one (Nermend, 2009):

$$c = \frac{(x, y)}{(y, y)} . \tag{38}$$

Coefficient c is referred to as a component of vector A along vector B, and the entire process that involves finding of this coefficient — projection.

A method for calculating scalar product must be selected in such a way so that vector component determined on this basis enables one to define how many times a vector must be laid off to obtain another one. Therefore, a vector must be reconstructed with the use of another vector in a correct manner. However, such reconstructions may differ depending on the way of interpreting the multiplication of the vector by scalar. In vector space created over the set $\mathbb{L}_{\Delta\sigma^2 n}$, one should strive for the right reconstruction of both average value increment and variance increment. In practice, when a vector is projected onto another one, it is generally impossible to reconstruct both parameters properly. Nonetheless, one may determine their significance, and hence scalar product together with weight should be defined:

$$\forall u,v \in \mathbb{L}_{\Delta\sigma^2 n} : \left\{ (u,v)_w \equiv \sum_{k=1}^{N} \left[w\Delta\eta_{u_k}\Delta\eta_{v_k} + (1-w)\,\Delta\sigma^2_{u_k}\Delta\sigma^2_{v_k} \right] \right\} \;, \; (39)$$

where N is a number of elements in the sets of n - elements that make up vectors u and v, and w — weight ascribed to the increment of average value.

It is accepted that the value of weight ranges from 0 to 1, where 1 implies that variance increment is insignificant, whereas 0 — that increment of average value is such. The result of scalar product calculations is the scalar, corresponding to the relation of the objects associated with two vectors. The greater is its value, the more similar are both objects and the greater are values describing them. The scalar is not the variance nor the average value; it is only some coefficient. Therefore the variances in the formula for scalar calculations are not treated as variances themselves, but only as some coefficients allowing to calculate another coefficient — the scalar. Scalar product is characterized by the following:

1. Scalar product of vector projected onto itself cannot be negative:

$$(a,a)_w = \sum_{k=1}^{N} \left[w\Delta\eta^2_{a_k} + (1-w)\left(\Delta\sigma^2_{a_k}\right)^2 \right] \;. \quad (40)$$

   Square of average value increment as well as square of variance increment are always non-negative, hence scalar product of vector projected onto itself will never be negative.

2. Scalar product of vector projected onto itself can equal zero only for neutral element of addition in the set of vectors. Square of average value increment as well as variance increment are always non-negative, hence scalar product equals zero only for vector that is a neutral element of addition in the set of vectors. The only exception to the rule is a situation when the weight equals 0 or 1. In the former case, scalar product equals zero for all the vectors with zero variance increments and any increment of average value. As for the latter case, it is the other way round: scalar product equals zero for all the vectors with zero increment of average value and any variance increment.

3. Factoring the constant out of the scalar product:

$$\alpha \left(a, b\right)_w = \alpha \sum_{k=1}^{N} \left[w \Delta \eta_{a_k} \Delta \eta_{b_k} + \left(1 - w\right) \Delta \sigma_{a_k}^2 \Delta \sigma_{b_k}^2\right] =$$

$$\sum_{k=1}^{N} \left[\alpha w \Delta \eta_{a_k} \Delta \eta_{b_k} + \alpha \left(1 - w\right) \Delta \sigma_{a_k}^2 \Delta \sigma_{b_k}^2\right], \quad (41a)$$

whereas:

$$\left(\alpha a, b\right)_w = \sum_{k=1}^{N} \left[\alpha w \Delta \eta_{a_k} \Delta \eta_{b_k} + \alpha \left(1 - w\right) \Delta \sigma_{a_k}^2 \Delta \sigma_{b_k}^2\right]. \quad (41b)$$

Multiplication by a constant is understood as multiple addition. Multiplying the constant α by vector leads to a α-times greater increment of average value as well as increment of variance. Therefore, $\alpha \left(a, b\right)_w = \left(\alpha a, b\right)_w$.

4. Commutativity of multiplication of the vector by scalar toward addition:

$$\left(a, b\right)_w + \left(a, c\right)_w = \sum_{k=1}^{N} \left[w \Delta \eta_{a_k} \Delta \eta_{b_k} + \left(1 - w\right) \Delta \sigma_{a_k}^2 \Delta \sigma_{b_k}^2\right] +$$

$$\sum_{k=1}^{N} \left[w \Delta \eta_{a_k} \Delta \eta_{c_k} + \left(1 - w\right) \Delta \sigma_{a_k}^2 \Delta \sigma_{c_k}^2\right], \quad (42a)$$

hence:

$$\left(a, b\right)_w + \left(a, c\right)_w =$$

$$= \sum_{k=1}^{N} \left[w \Delta \eta_{a_k} \Delta \eta_{b_k} + \Delta \eta_{a_k} \Delta \eta_{c_k}\right.$$

$$\left. + \left(1 - w\right) \Delta \sigma_{a_k}^2 \Delta \sigma_{b_k}^2 + \Delta \sigma_{a_k}^2 \Delta \sigma_{c_k}^2\right] =$$

$$= \sum_{k=1}^{N} \left[w \Delta \eta_{a_k} \left(\Delta \eta_{b_k} + \Delta \eta_{c_k}\right) + \left(1 - w\right) \Delta \sigma_{a_k}^2 \left(\Delta \sigma_{b_k}^2 + \Delta \sigma_{c_k}^2\right)\right] =$$

$$= \left(a, b + c\right)_w .$$

$$(42b)$$

5. Conjugate symmetry. Increments of average values and variances are not conjugate numbers. Hence, only common symmetry should be examined here. Symmetry of scalar product under consideration stems from the symmetry of multiplication in the set of real numbers.

For w (ranging from 0 to 1) the aforementioned scalar product satisfies all the axioms of pre-Hilbert space (except for $w = 0$ and $w = 1$). The two cases do

not satisfy the axiom that assumes a non-zero scalar product of vector projected onto itself, not being neutral elements of addition in the set of vectors. Scalar products for $w = 1$ and $w = 0$ are very interesting cases as they allow to calculate projection coefficient in such a way so that average value increment or variance increment is reconstructed best. Thanks to this they complement one another. This is of profound importance since scalar is a one-component number and so projection coefficient does not allow for reconstructing a vector considering properly the average value increment and the variance increment at the same time. Therefore, it may often be useful to calculate not one but two projection coefficients — one for average value increment ($w = 1$) and the other one for variance increment ($w = 0$).

Scalar product, defined just as above, enables calculation of the projection of vector a onto vector b, which may be expressed in the following way:

$$c = \frac{(a,b)_w}{(b,b)_w} = \frac{\sum_{k=1}^{N}\left[w\Delta\eta_{a_k}\Delta\eta_{b_k} + (1-w)\Delta\sigma^2_{a_k}\Delta\sigma^2_{b_k}\right]}{\sum_{k=1}^{N}\left[w\Delta\eta^2_{b_k} + (1-w)\left(\Delta\sigma^2_{b_k}\right)^2\right]}. \tag{43}$$

In this case, variance increment may be treated as a full element of space related to its own dimension. The projection of a vector onto another is made in the same way as in the case of 'normal' spaces. Variance increment is treated as a coordinate, which is shown in Fig. 1a.

Formula (43) may be expressed in a different way:

$$c = \frac{w\sum_{k=1}^{N}\Delta\eta_{a_k}\Delta\eta_{b_k} + (1-w)\sum_{k=1}^{N}\Delta\sigma^2_{a_k}\Delta\sigma^2_{b_k}}{\sum_{k=1}^{N}\left[w\Delta\eta^2_{b_k} + (1-w)\left(\Delta\sigma^2_{b_k}\right)^2\right]} =$$

$$= \frac{w\sum_{k=1}^{N}\Delta\eta_{a_k}\Delta\eta_{b_k}}{\sum_{k=1}^{N}\left[w\Delta\eta^2_{b_k} + (1-w)\left(\Delta\sigma^2_{b_k}\right)^2\right]} + \frac{(1-w)\sum_{k=1}^{N}\Delta\sigma^2_{a_k}\Delta\sigma^2_{b_k}}{\sum_{k=1}^{N}\left[w\Delta\eta^2_{b_k} + (1-w)\left(\sigma^2_{b_k}\right)^2\right]}. \tag{44}$$

Therefore, coefficient c is a sum of two projection coefficients: projection of the vector made up of coordinates of vector a and relating to increments of
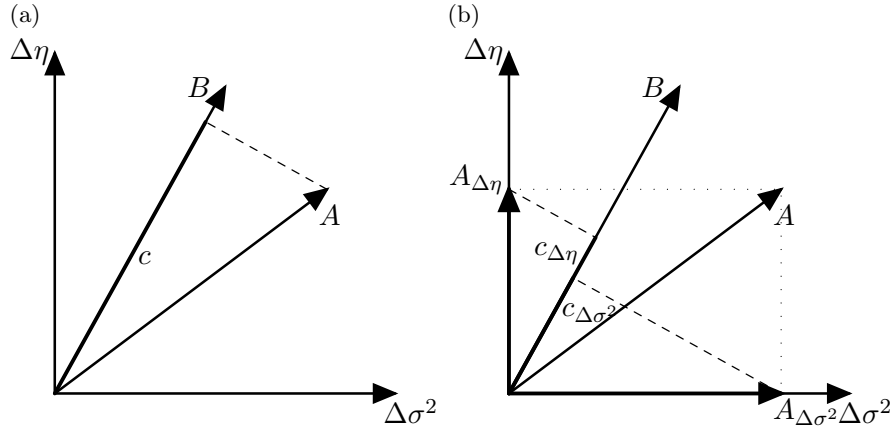
Figure 1. Projection of vectors defined with the use of average value and variance increments: a) direct calculation of projection coefficient; b) calculation of projection coefficients for average value increment and variance increment of vector a, respectively

average values onto vector b:

$$c_{\Delta\eta} = \frac{w \sum\limits_{k=1}^{N} \Delta\eta_{a_k} \Delta\eta_{b_k}}{\sum\limits_{k=1}^{N} \left[w\Delta\eta_{b_k}^2 + (1-w)\left(\Delta\sigma_{b_k}^2\right)^2 \right]} \ , \tag{45a}$$

as well as the projection of vector created on the basis of coordinates of vector a and relating to increments of variance onto vector b:

$$c_{\Delta\sigma^2} = \frac{(1-w) \sum\limits_{k=1}^{N} \Delta\sigma_{a_k}^2 \Delta\sigma_{b_k}^2}{\sum\limits_{k=1}^{N} \left[w\Delta\eta_{b_k}^2 + (1-w)\left(\Delta\sigma_{b_k}^2\right)^2 \right]} \ . \tag{45b}$$

Coefficients of projection $c_{\Delta\eta}$ and $c_{\Delta\sigma^2}$ enable examination of increments of average values and variances relating to vector a individually while projecting. To do so, vector a is divided into two vectors: $a_{\Delta\eta}$ (with variance increments equal 0) and $a_{\Delta\sigma^2}$ (with increments of average values equal 0). Vector a is a sum of the two vectors. Analogously, vector b can be divided into vectors $b_{\Delta\eta}$ and $b_{\Delta\sigma^2}$. It is possible to determine scalar product and projection coefficient only for pairs of vectors $a_{\Delta\eta}$, $b_{\Delta\eta}$ and $a_{\Delta\sigma^2}$, $b_{\Delta\sigma^2}$. Such a procedure is of major

importance, particularly in calculations that compare variance increments, i.e. variance increment may, for instance, provide an extra piece of information allowing to identify object examined. Values of scalar product for pairs of vectors $a_{\Delta\eta}$, $b_{\Delta\sigma^2}$ and $a_{\Delta\sigma^2}$, $b_{\Delta\eta}$ will always equal zero and hence there is no point in calculating them.

On the basis of scalar product $(a,a)_w$ one may determine the absolute value of a vector:

$$\|a\|_w = \sqrt{(a,a)_w} = \sqrt{\sum_{k=1}^{N} \left[w\Delta\eta_{a_k}^2 + (w-1)\Delta\sigma_{a_k}^4 \right]} \ . \tag{46}$$

For $w \in [0,1]$ the absolute value of the vector is a norm. It may be used for comparing vectors. Scalar products for which $w = 1$ and $w = 0$ allow to compare the part of vector relating to average value increment and the part relating to variance increment.

It is not always important to reconstruct a vector paying attention to variance increment. Hence, scalar product $(a,b)_1$ can be used for this purpose. However, the result obtained should be supplemented with a parameter defining the potential scatter. Yet, it is not possible to use scalar product $(a,b)_0$ since the projection based on this product determines by what number the vector (which is subject to projection aimed at reconstructing variance increment properly) should be multiplied, which is not tantamount to determining the scatter for average value increment.

Attention should be paid to the fact that variance increments calculated in such a way relate only to a vector which is subject to projection. Therefore, it is necessary to determine variance increments relating to a vector that is being projected as well. Projection onto one vector is the simplest operation that allows to determine coordinates in a new coordinate system. In this case, the coordinate system is one-dimensional and is a vector onto which the projection is made. In computer graphics, vector transformation is used interchangeably with operation involving the change of the coordinate system. For example, it is possible to rotate a vector or (instead) calculate the coordinates of the vector in a new coordinate system rotated relative to the former system of coordinates. As far as vector space relating to the set $\mathbb{L}_{\Delta\sigma^2 n}$ is concerned, the results will be the same as for average value increment, yet different for variance increment. In the case of vector transformation, the projected vector is subject to transformation, which allows to calculate variance increments relating to this vector.

Any vector a may be transformed on the basis of the following formula (Karaśkiewicz, 1974):

$$x_a = \mathscr{D}a \ , \tag{47}$$

where $\mathscr{D}$ is a square matrix with size equal the number of space dimensions, or

(Karaśkiewicz, 1974):

$$x_{ai} = \sum_{k=1}^{N} d_{ik} a_k \ ,$$

(48)

where d_{ik} is an element of $\mathscr{D}$.

With the use of matrix $\mathscr{D}$ it is possible to determine coordinates of the vector x_a on the basis of coordinates of vector a, which enables one to transform both average value increment and increment of variance of vector a into the direction of vector b. There is a relationship between vector x_a and vector a expressed in the formula (47). However, vector x_a may also be determined on the basis of the projection of vector a onto vector b. Replacing x_a with its coordinates calculated on the basis of the projection, the formula (47) becomes:

$$cb = \mathscr{D}a \ ,$$

(49)

hence:

$$\frac{(a,b)_1}{(b,b)_1} b = \mathscr{D}a \ .$$

(50)

The formula for determining matrix $\mathscr{D}$ cannot take any general form for any scalar product. Nevertheless, matrix $\mathscr{D}$ may be determined for particular scalar products. For the scalar product $(a,b)_1$ such formula will be as follows:

$$\frac{\sum\limits_{i=1}^{N} \eta_{a_i} \eta_{b_i}}{\sum\limits_{j=1}^{\frac{N}{2}} \eta_{b_j}^2} \eta_{b_k} = \sum_{i=1}^{N} d_{k,i} \eta_{a_i} \ ,$$

(51)

which may also be expressed as:

$$\sum_{i=1}^{N} \eta_{a_i} \frac{\eta_{b_i} \eta_{b_k}}{\sum\limits_{j=1}^{N} \eta_{b_j}^2} - \sum_{i=1}^{N} d_{k,i} \eta_{a_i} = 0 \ .$$

(52)

Factoring out all η_{a_i}, we obtain the following:

$$\sum_{i=1}^{N} \eta_{a_i} \left(\frac{\eta_{b_i} \eta_{b_k}}{\sum\limits_{j=1}^{N} \eta_{b_j}^2} - d_{k,i} \right) = 0 \ .$$

(53)

It should be emphasized that matrix $\mathscr{D}$ transforms any vector a into a vector with the same direction as that of vector b. Thus, values of this matrix depend only on vector b and so the above formula should be true for any a. Hence, all the factors in brackets must always equal 0. The value of matrix $\mathscr{D}$ may be calculated with the use of the following formula:

$$d_{k,i} = \frac{\eta_{b_i}\eta_{b_k}}{\sum\limits_{j=1}^{N}\eta_{b_j}^2} \ . \tag{54}$$

Values of matrix $\mathscr{D}$ are calculated only for average value increments. Nevertheless, one should bear in mind that, by assumption, variance increment follows average value increment. Therefore, if average value increment is multiplied by some value, variance increment should be multiplied by this value as well. Hence, vector X may be treated as a sum of two vectors $X_{\eta A}$ and $X_{\sigma^2 A}$ the coordinates of which may be calculated as follows:

$$X_{\eta A} = \mathscr{D}A_{\eta} \ , \tag{55a}$$

and:

$$X_{\sigma^2 A} = \mathscr{D}A_{\sigma^2} \ . \tag{55b}$$

Once variance increments have been determined, it is necessary to reduce them to one value. Similar procedures are adopted for the projected vector and for the vector which is subject to projection. One should use the absolute value of vector $\|b'\|_0$. It determines the absolute value of variance defined with the use of increments. In this way, one obtains variance increment relating to the absolute value of vector, and not to projection coefficient. This coefficient is expressed in units measuring the absolute value of the vector that is subject to projection, which may be expressed as:

$$c = \frac{\|b'\|_1}{\|b\|_1} \ . \tag{56}$$

where $\|b'\|_1$ is the absolute value relating to average value increment reconstructed onto the projected vector.

Variance increment for coefficient c may be calculated by replacing the absolute value of average value increment of the reconstructed vector with the absolute value of variance increment in the aforementioned formula:

$$\Delta\sigma_c^2 = \frac{\|b'\|_0}{\|b\|_1} \ . \tag{57}$$

Due to the fact that $\Delta\sigma_c^2$ is calculated for two vectors (the projected one and the one that is subject to projection) we obtain two values $\Delta\sigma_c^2$ that should be summed.

4. Practical applications in image processing

The presented example will use the cross-correlation function values calculated in the vector space for the analysis of local standard deviations of an image. These standard deviations constitute two-dimensional table of standard deviations (or variances). This table is treated in image processing as an image. For instance, Fourier transform of local variances was carried out in Heizmann (2005) and wavelet transform of local standard deviations was presented in Truong, Dorai and Venkatesh (2000). From the vector calculus point of view these transforms can be calculated via the transformation of the coordinates system, and this transformation, due to its definition, can not be performed for variances and standard deviations. These types of operations can be performed for increments of variances and increments of standard deviations only.

For formal reasons, every operation in vector space involving variance and standard deviation must be carried out on their respective increments. If calculations are aimed at comparing variances, the methodology of calculations will not differ from classical methods much. In the case of simple methods for comparing the entire bit-mapped images or their fragments, using local increments of variance will not as a rule produce good results. This is due to the fact that effective field including a useful piece of information (contour) is very small compared to the entire image. Furthermore, the variance of surface of various colours (texture) is rather slight compared with the variance of contour, which makes it impossible to compare images by the character of their surface unless the variance of contour is weakened or eliminated.

Nevertheless, only images created by means of the reflection of light have such a character. In the case of longer waves (e.g. sound wave) contours of objects are not so sharp. Transition from the image that reflects wave to a greater extent to the image that reflects it to a lesser degree is much smoother. Therefore, contours of the images, whose pixels are local variances are practically invisible (Fig. 2). In this case, variance mainly provides information concerning the character of surface that reflects the wave, which allows to employ simple methods for comparing the variance increments in order to solve major problems with such images.

The automation of mosaic creation for the image from sector-scan sonar may be quoted as an example of comparing variance increments in practice. Sector-scan sonar, unlike side-scan one, is lowered onto the bottom where rotating head transmits sound beam that - once it rebounds from the bottom - is used for creating the image. Yet, unlike echo sounder, it records the strength of signal reflected, and not the time from transmission to reception, due to which the image shows objects that reflect sound wave both to a lesser and greater extent. Sector-scan sonar may be used for supervising diver's work, seeking sunk objects and people, and for inspection of technical objects such as bridges. Since the distance from objects registered is small and insensitive to the movement of water and ship, the side-scan sonar is better, yet it is more laborious to use it.
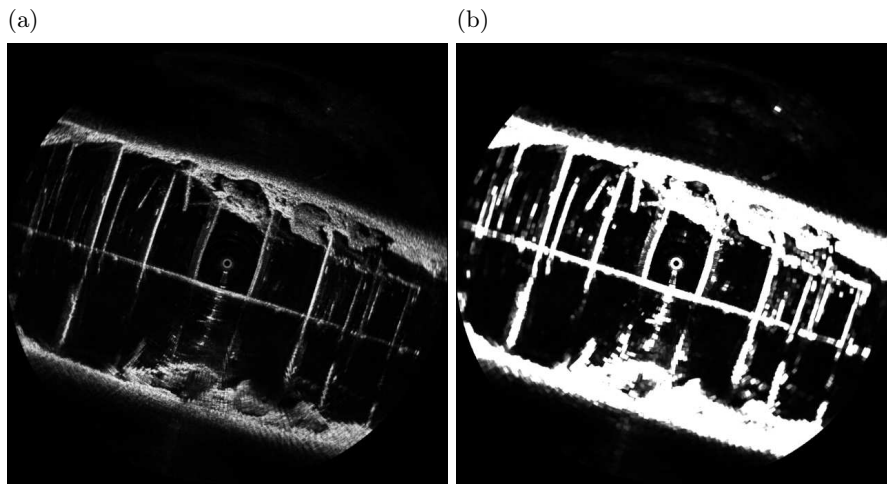
(a) (b)

Figure 2. Local variances for a sonar image: a) original image (source: Nawigator XXI, Maritime Univeristy of Szczecin); b) computed local variances

The sector-scan sonar has been employed in the military for a very long time, yet it has been in commercial use only since recently. Hence, there is a number of problems with creating and processing of image that have not been solved for this sonar yet (no solutions were published or implemented by commercial programmes). One of such problems is related to combining several images from the sonar that is located in different places at the bottom. It is often the case while conducting underwater inspection of bridges, wharves and other objects when it is necessary to locate sonar in different places at the bottom to obtain a more clear image. Once soundings have been made, images are put together manually into one big image. The automation of this action is hindered by the fact that one does not know the precise location of the sonar. This location can be determined to a centimetre accuracy with the use of information derived from RTK. However, a current may move the sonar along an uneven bottom and a ship (if the sonar is lowered from a vessel) may be subject to movement as well, and therefore the position of the sonar may change by several metres compared to the initial one.

Every image recorded by the sonar has its scale resulting from the scope of sounding adopted. The position relative to the North is also known, which stems from the fact that sonar of this type usually has built-in compass and thus, even if it rotates while being lowered, this has no effect on its reading of the North. Its approximate position is known as well. All the data is saved in a file with sonar image (except for the position that is being recorded only when GPS or RTK is connected).

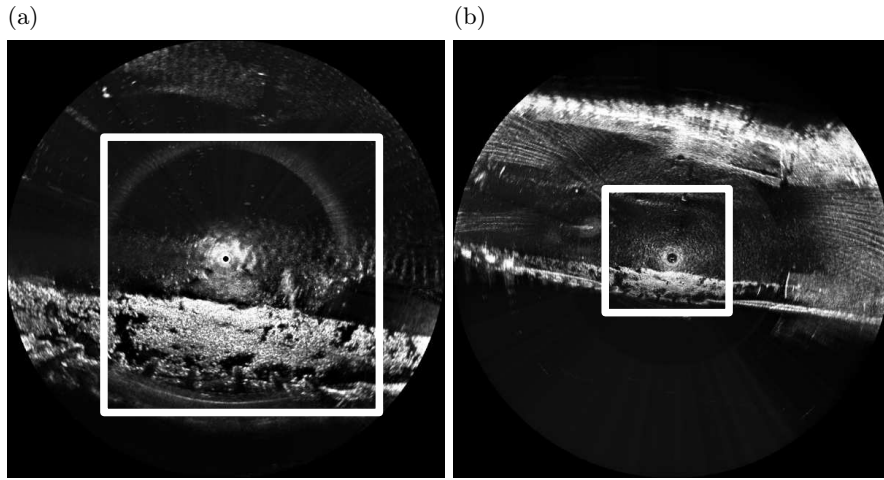Fig. 3 shows two images from sector-scan sonar presenting the same bottom

(a) (b)

Figure 3. Images of the same fragment of the bottom obtained from the sonar placed in different places of the bottom (source: Nawigator XXI, Maritime University of Szczecin)

regions produced from different positions and operating range of sonar. On the basis of data derived from GPS, fragments of the image, that should overlap, were chosen and marked with white rectangle. Local variances were calculated for these fragments. The average value of variance was used as the reference point to determine the increments of variances. Thus, increments were calculated as a difference between variance and average value of variance. Local increments of variance, calculated in such a way for both fragments of sonar image, were treated as two vectors, a and b. Further calculations make use of scalar product $(a, b)_0$ and hence increments of average values are not significant.

In the image processing the cross-correlation function is calculated according to the following formula (Gonzalez and Wood, 2002; Jähne, 2004):

$$k_{j,k} = \frac{1}{S} \sum_{l=1}^{L} \sum_{m=1}^{M} a_{l,m}^* b_{j+l,k+m} \ , \tag{58}$$

where $S = LM$, a and b are compared images (in the concerned case they are the tables of variances), * marks the conjugate number, k — table of the cross-correlation function values, and L, M — size of an image.

It is assumed, to ensure the range of the cross-correlation function values between minus one and plus one, that the S value is:

$$S = \sqrt{\sum_{l=1}^{L} \sum_{m=1}^{M} a_{l,m}^2} \sqrt{\sum_{l=1}^{L} \sum_{m=1}^{M} b_{l,m}^2}. \tag{59}$$

The value of cross-correlation function, calculated for one element of the table k, with S given by the formula (59), can be interpreted as the projection of one unitary vector onto another. The division by S corresponds to the calculation of the unitary vectors a' and b' on the basis of the vectors a and b. The sum of products from the formula (58) is used to determine the coefficient of projection of vector a' onto vector b'.

After determination of table k the projection coefficient with the greatest value was selected. For this coefficient the value of the cyclic horizontal and vertical shift for the image fragment was read out. These values determine the mutual shift of combined images.
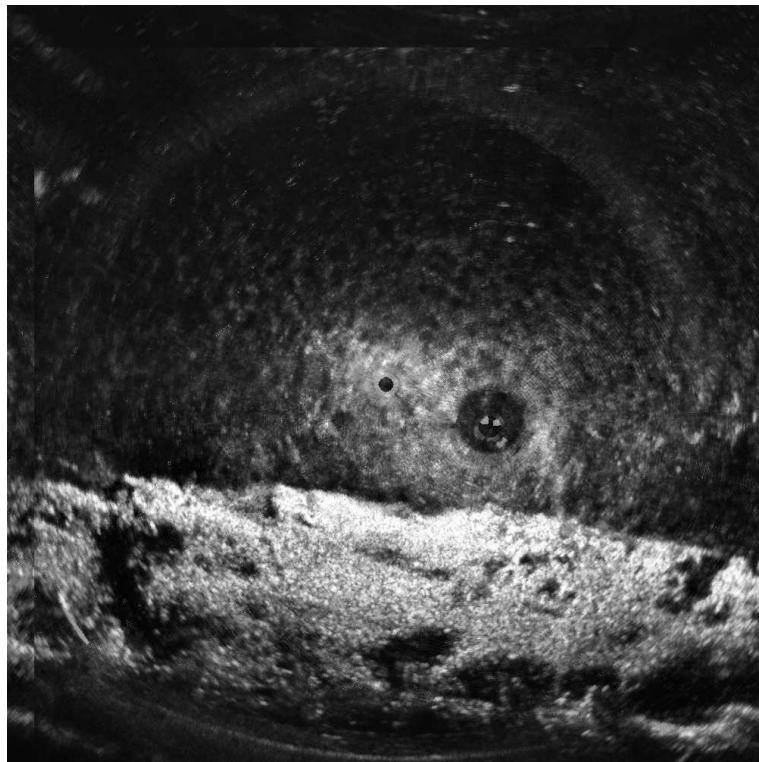

Figure 4. Two sonar images overlapped on each other

The aforementioned procedure for determining the shift between images is a classical one. Nevertheless, in this case it was employed for calculating the shift between local increments of variance. For formal reasons, such action is not plausible with local variance. Fig. 4 shows the images combined. They underwent overlapping by calculating their average value. The combined image

shows two circles that determine the location of the sonar when both images are being recorded.

5. Conclusion

The article presents definitions of vector space of increments that allow to conduct operations on increments of variance and standard deviation. Other types of increments (e.g. interval increments) can also be defined. The present paper focused mainly on variance increment. The definition of vector space and scalar product justifies using vector calculus for variance increment. At the same time one can compare increments of variances, and determine the variance of the result (i.e. its possible inaccuracy) as well. The procedure for defining inaccuracy of calculation results requires a different attitude compared to the classical vector calculus. In order to put it into practice, it is essential to conduct further research. As for comparing the increments of variance, the procedure is very similar to the existing methods involving vector calculus, which was illustrated with the example of combining the images from sector-scan sonar.

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