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Deriving topological concepts for fuzzy regions: from properties to definitions*

by

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Abstract: Fuzzy regions are a concept allowing the uncertain or imprecise spatial data to be represented. Many geographic or spatial data are prone to uncertainty or imprecision and while such data can be represented, it is still necessary to be able to perform basic tasks one expects to perform in a geographic context. A number of operations have been considered in the past; in this contribution the topological concepts of fuzzy regions will be examined. For this purpose, appropriate definitions for the boundary, interior and exterior of fuzzy regions will be developed. These definitions can then be applied in an extension of the 9-intersection model.

Keywords: fuzzy regions, fuzzy topological concepts, fuzzy toplogy.

1. Introduction

When dealing with geographically related data, one concludes that many of the data are prone to either imprecision or uncertainty. This can be due to the fact that the data are inherently imprecise or uncertain, or that it is impossible to make an accurate assessment (for whatever reason: too costly, too complicated, etc.). While a lot of theoretical work has been done, currently used practical models have no or limited provisions to represent the uncertainty; this tends to lead to less realistic models: areas where a feature can be present are often over-estimated to be sure to include all possible cases, ignoring the fact that in some locations the feature may be far less possible to occur.

There are different approaches to representing spatial spatial uncertainty or imprecision. An intuitive approach is to define the region by means of two

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boundaries: an inner boundary and an outer boundary. Points within the inner boundary are said to completely belong to the region, points outside the outer are said not to belong to the region, and the points in between both boundaries make up some grey area. This approach was adopted by Clementini (Clementini and Di Felice, 1996; Clementini, 2004) and Cohn and Gotts. This approach allows for an interesting analysis but, as points in the grey area are not differentiated further, the applicability is somewhat limited. Another approach to extend tradition regions with uncertainty or imprecision is to define a buffer region around the area. This approach has been used by us in Verstraete, Van Der Cruyssen and De Caluwe (2000) and by Du et al. (2005a), and applies fuzzy set theory to assign to points different degrees of membership for belonging to the region. The buffer is used to define an area around the region, in which points are commonly given decreasing degrees of membership. While this approach allows for a more fuzzy modelling of the data, it is also not without flaws. The most noticeable problem is the fact that it does not yield a closed system for simple set operations: the intersection or union of two such regions is not necessarily representable as a new region. Other approaches in literature are the Fuzzy Minimum Bounding Rectangles (Petry, Robinson and Cobb, 2005) and the Realm/ROSE approach (Schneider, 1996; Schneider and Pauly, 2007).

In this contribution, attention will go to the representation model for fuzzy regions which we first presented in Verstraete et al. (2004). There, a fuzzy region is defined as a fuzzy set over a two-dimensional domain, without imposing further limitations on the distribution of membership grades. In Verstraete et al. (2008), a methodology for the topological aspects of such regions was considered using the nine-intersection model and simple ad-hoc concepts of boundary, interior and exterior definitions. The definitions of those concepts were fairly simple and it became obvious they exhibited some shortcomings; in this contribution, the search for better definitions for these topological concepts is presented. From the shortcomings of the previous definitions, required and desired properties are deduced, and used to uncover a number of definitions that will be put forward and compared against one another. With these new definitions, the same methodology for the topological study (Verstraete et al., 2008) can be applied.

The presented concept will concern fuzzy regions defined as fuzzy sets over the two-dimensional domain, thus carrying a veristic interpretation. Since submitting this article, we have extended the model with an additional level of uncertainty that carries a possibilistic interpretation. This additional level can be used for the notion of candidate regions (and thus candidate boundaries). So far, the development of that model has taught us that the use of candidate regions makes most operations result in a number of possible outcomes - each annotated with a possibility degree - and this will also be the case for the topology. The new model also unifies the representation of regions (veristic) and points (possibilistic), which will also mean that the topology between different types of objects will also result from it. At this point, we are concerned with the definitions of the concepts we need in order to derive the topology between fuzzy regions that only have the veristic level.

As the model uses fuzzy set theory, some definitions and notations will be presented first, mainly to provide a reference for the notations used further on.

1.1. Fuzzy set theory

1.1.1. Definitions

Fuzzy set theory is an extension of traditional set theory, where a value in the range $[0, 1]$ is associated with each element of the set. This value is the membership grade; a value 0 indicates that the element does not belong to the set.

DEFINITION 1 (FUZZY SET) *Consider the universe U. A* fuzzy set *$\tilde{A}$ is a set, with an associated membership function $\mu_{\tilde{A}}$, defined by*

$$\begin{aligned} \mu_{\tilde{A}} : U &\rightarrow [0,1] \\ x &\mapsto \mu_{\tilde{A}}(x). \end{aligned}$$

The membership function associates a membership grade $\mu_{\tilde{A}}(x)$ to each element x. Elements that are not part of the set are considered to have membership grade 0. A fuzzy set $\tilde{A}$ with a finite number of elements x, y and z, can be denoted as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), (y, \mu_{\tilde{A}}(y)), (z, \mu_{\tilde{A}}(z))\}.$$

Membership grades can carry one of three interpretations: they can be interpreted as degrees of membership (i.e. a veristic interpretation), degrees of possibility (i.e. a possibilistic interpretation), and degrees of truth. It was proven in Dubois and Prade (1999) that other interpretations are equivalent to one of these three. In this concept, the interpretation as degrees of membership is meaningful for fuzzy regions (all points belong to a region, but to a different extent), whereas the interpretation as degrees of possibility can be used to represent fuzzy points (only one point is the "real" one, the region represents all possible candidates). In this contribution, we will only consider the model to be used to represent fuzzy regions, thus using the veristic interpretation.

1.1.2. Operations

When working with fuzzy sets, it can often be required to derive crisp sets based on the fuzzy set at hand. For this purpose, the α-cut is introduced: it removes all aspects of fuzziness and reverts the fuzzy argument to a crisp set. The α-cut of a fuzzy set is a crisp set containing all the elements of the fuzzy set for which a constraint it satisfied: a *strong α-cut* contains the elements with membership grades greater than a given α; a *weak α-cut* contains the elements with membership grades greater than or equal a given α, as illustrated in Fig. 1.

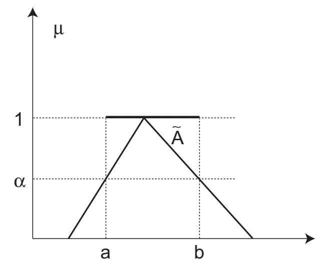


Figure 1. Example of the α-cut of a fuzzy region $\tilde{A}$.

DEFINITION 2 (STRONG α-CUT) *The* strong α-cut *is defined as:*

$$\tilde{A}_{\overline{\alpha}} = \{x \mid \mu_{\tilde{A}}(x) > \alpha\}.$$

A special case of a strong α-cut is the *support*; this is the strong α-cut with threshold 0. This is an important α-cut, as it contains all the elements that belong to some extent to the fuzzy set.

DEFINITION 3 (WEAK α-CUT) *The* weak α-cut *is defined as:*

$$\tilde{A}_{\alpha} = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\}.$$

Similarly to the strong α-cut, the weak α-cut has a special case, now for a threshold 1. This α-cut is called the *core*, and returns all the elements that fully belong (membership grade 1) to the given fuzzy set.

DEFINITION 4 (HEIGHT) *The* height *of a fuzzy set is the highest membership grade in a fuzzy set. Thus, the height of a fuzzy set $\tilde{A}$ is (Dubois and Prade, 2000):*

$$height(\tilde{A}) = \sup_{x}(\mu_{\tilde{A}}(x))$$

where sup is the supremum.

Quite often it is interesting to have at least one element in a fuzzy set with membership grade 1, as such fuzzy sets satisfy additional properties that can be of importance. Such a fuzzy set is called *normed.* The choice of working with normalized fuzzy sets or not depends on a lot of factors, but the presented model supports both approaches.

1.2. Related work

Many authors have described the expected behaviour of imprecise or uncertain regions. In the following subsections, different approaches, for which topology has been studied, are briefly mentioned and matched against the model that is continued in this contribution.

1.2.1. Broad boundary model and Egg-yolk model

The broad boundary model was presented in Clementini and Di Felice (1996); it extends the model for regions (in an entity based model) using a geometrical approach. The concept extends the boundary of a region: instead of a single polyline representing the boundary, two lines are used, called *inner* and *outer* boundary. Inside the inner boundary are all the points that certainly belong to the region (or the points that completely belong to the region - the broad boundary model makes no distinction between imprecision and uncertainty); outside the outer boundary are the points that do not belong to the region. In between remains the *broad boundary*, containing points that may or may not belong to the region (or that belong partly to the region). No other information concerning points in the broad boundary is provided, but the assumption is made that the extension of the boundary is smaller than the inner boundary (this is called the *small boundaries assumption*, which limits the uncertainty of a region). The interior (inside the inner boundary) and exterior (outside the outer boundary) of a region with broad boundary are open sets, whereas the boundary itself is a closed set.

The egg-yolk approach presented by Cohn and Gotts (Cohn and Gotts, 1994; Gotts and Cohn, 1996) describes a similar concept, but now using a logical approach: the model also makes use of two crisp boundaries: an inner boundary (called the *yolk*) inside which points belong to the region and an outer boundary (called the *egg*) outside which points do not belong to the region. The main difference is in the definition of the boundary, which in the egg-yolk model is an open set.

For both models, topology is described based on the intersection of the different components that play a part: interior, exterior and boundary. The described topology for both models differs slightly: Clementini has 44 cases, whereas Cohn and Gotts have 46 cases. Technically, these differences are minor and mainly due to assumptions (e.g. small boundaries assumption made by Clementini) and the small difference in definitions, the interesting aspect is that both the geometrical and the logical approach lead to similar conclusions.

Our model can be seen as a generalization of these models, in the sense that it provides for more information regarding points inside the broad boundary. In both approaches mentioned, such points are considered similarly, regardless of the fact that they may be closer to the inner or closer to the outer boundary. Similar to Clementini's approach, we also will use the 9-intersection model. This requires appropriate definitions for the different topological concepts.

1.2.2. Buffering regions

Several authors define a fuzzy region by fuzzifying the boundary of a crisp region. Du et al. (2005a) map a trapezoidal function over the crisp border and use this to derive the topological notions of boundary, interior and exterior.

This approach is similar in concept to what we presented in Verstraete, Van Der Cruyssen and De Caluwe (2000), where we also allowed for non-linear decreasing membership functions by introducing the concept of a shape function: the shape function allowed for non-linear changes of membership grades within the fuzzy boundary. Du et al. (2005b) present a way of defining the topology of such regions; for this purpose, new concepts for interior, exterior and boundary are defined. All these are fuzzy sets over the two-dimensional domain, with membership functions derived from the membership function of the initially extended region. Using these new concepts, a 9-intersection matrix is constructed. Due to the membership grades of the points, however, the elements of this intersection matrix are not in $\{0, 1\}$, but in $[0, 1]$, which makes interpreting the matrix more challenging. As a concept for fuzzification of a region, it is a fairly intuitive technique to grasp and to work with in simple situations; but it poses problems to define many operations. Du, Qin and Wang (2008) introduce the concept of simple regions with broad boundaries (SBBRs for short), defined as in Du et al. (2005a), and perform a topological case study. For this, appropriate definitions for interior, exterior and boundary, and the 9-intersection model using a 4-tuple representation are used. The definitions are derived from trapezoidal function; those for for interior and exterior are quite straight-forward, the definition for the boundary is more open to argumentation. Considering the inner and outer boundaries of two regions, a 4-tuple containing the relation between both inner boundaries, the relation between both outer boundaries, and the relations between the inner boundary of one region and the outer boundary of the other region can be used to represent the topology. The result aims to provide an easier classification of intersection cases and is matched against the aforementioned egg-yolk approach.

The biggest critique against the model Du uses is that the set operations are not closed: the intersection or union of two such regions can not always be represented as such a region (e.g. the intersection of two regions with differing membership functions, or the intersection of two regions that only have overlapping boundaries cannot be represented as a region).

In the model developed by us and used in this contribution, quite a different approach is taken as we started with the premises that basic set operations need to be closed. To describe the topology of our model, fuzzy regions for boundary, interior and exterior will also be derived from the distribution of the membership grades in the fuzzy regions. Given the fact that our model also uses intersection matrices with element in the range $[0, 1]$, they look quite similar; however due to the fact that more arbitrary (and realistic) distributions for the membership grades are possible, the topology cases become more complicated.

In Petry, Robinson and Cobb (2005), the authors use the smallest umscribing rectangle and largest inscribed rectangle as means to approximate a crisp boundary. While it works for simple regions, it offers the user no freedom regarding the distribution of the membership grades. For more complex regions, the largest inscribed rectangle is not necessarily unique, and rectangles may not be the best approximation of oddly shaped regions.

1.2.3. Rough set approach

Beauboeuf and Petry (2001, 2007) present a topology model for vague regions in a rough set approach. The authors adopt the RCC calculus by replacing the elements in the crisp 9-intersection model by appropriate definitions for regions defined as rough sets. A rough set is defined in a non-empty universe by means of an *indiscernability relation* or equivalence relation. This indiscernability indicates how different elements of the universe relate to one another, in particular, if they can be considered similar. Using this relation, an upper and a lower approximation of a region can be defined. The combination of upper and lower approximation yields the rough region. In the intersection matrix, appropriate definitions for the 9 intersection are considered, using the lower approximation for interior, the difference between upper and lower approximation for boundary and the difference between the universe and the upper approximation for exterior. This approach is quite close to the aforementioned egg-yolk method, and yields 46 intersection cases just like that model.

While the adopted approach of modifying the intersection matrix elements is similar to our approach, in our approach fuzzy set theory is used. This has the advantage of not having to determine the equivalence relation between each two elements in the universe (which might not be possible for all types of data or all types of universes).

1.2.4. Extending the existing topological concepts

In Schockaert, De Cock and Cornelis (2008), an interpretation of the existing intersection matrix for application with fuzzy regions is considered. The paper generalizes 11 relations from the RCC calculus by fuzzifying the C-relation to work on fuzzy sets (using complement, t-norms and t-conorms). In the paper, no assumptions or models for fuzzy regions are under consideration, nor is any model put forward. From a theoretical point of view, it yields a number of interesting results; especially as conclusions are drawn regarding transitivity of fuzzy topologies. The approach, however, is situated at quite a different level compared to the approach presented in this contribution: the authors perform a pure theoretical analysis, whereas we start from a proposed model for fuzzy regions, and work our way up to study the topology aspects of these regions. The conclusions from both approaches should still be put together for full verification, but a preliminary study has not revealed any contradictions thus far.

1.2.5. Qualified topological relations

Bejaoui et al. (2008) consider objects with a vague shape consisting of several crisp objects; ranging from *broad point*, over a *line with a vague shape* to a *region* *with a broad boundary.* From this assumption, a minimal and maximal extent of each vague shape can be defined, and this results in a crisp object. For each minimal and maximal extent, topological invariants of interior and boundary are defined. Using these concepts, in an approach not dissimilar to the egg-yolk model, topological relations are considered and clustered.

1.2.6. A probabilistic approach

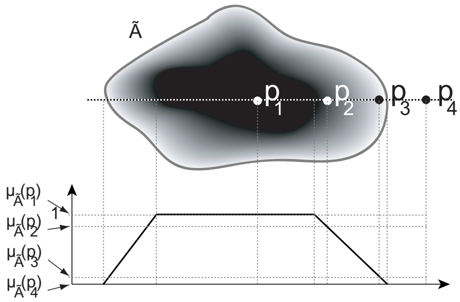
Winter (2000) presents a probabilistic approach to deal with uncertainty and imprecision. To describe topology, the approach uses the minimal and maximal distance between regions to derive what the author calls a *morphological distance*. This measurement provides quantitative information sensitive to location imprecision of the regions, which are then abstracted to range classes to provide topological distinctions. Conceptually, this approach attempts to assign one of the topological relations to two regions, rather than providing a multitude of options. For the given example (comparing two regions on different data sets), this is an appropriate method, but not applicable to fuzzy regions in general as it is not always possible to label their topology in a single case.

2. The fuzzy region model

2.1. Basic model

In geographical information systems, regions are often represented in a vector model by means of an outline, which, in turn, is represented by a polygon (possibly, the region can have holes). The region is then defined as all the points located inside this polygon. In order to define the concept of fuzzy regions, a different point of view is necessary: rather than considering a region to consist of *all the points inside a given polygon*, consider it as *a set of locations*, more specifically all the locations that are inside the polygon that is used to represents it. Once a region is considered as a set of locations, it is a small step to extend a region to a fuzzy region: a fuzzy set of locations (where each location is represented by a point and each location has a membership grade associated). The membership grades for regions are interpreted in a veristic way: all locations belong to the region, but some more than others. As mentioned before, a possibilistic interpretation can be used to represent fuzzy points. As this contribution concerns topology of fuzzy regions, fuzzy points extend beyond the scope of this contribution. The underlying approach to determine the position of fuzzy points in relation to fuzzy regions is similar to the presented approach for regions, though.

Some also would consider a probabilistic interpretation; interpreting the value associated with each point as the probability that this location belongs to the region. This opens up interesting properties (e.g. the area where the value equals 1 would be certain). Our model, however, is based on fuzzy set theory, and the probabilistic interpretation is not one of the three interpretations commonly considered for fuzzy sets (Dubois and Prade, 1999). Consequently, while this interpretation may be possible, it may at the same time prevent compatibility with fuzzy set theory. This interpretation could be used for the representation of uncertain regions, but for that purpose we opted to use level-2 fuzzy sets (Verstraete, 2011a), in order to maintain a guaranteed compatibility with fuzzy set theory.



$$\mu_{\tilde{A}}(p_1) = 1$$
$$\mu_{\tilde{A}}(p_2) = 0.8$$
$$\mu_{\tilde{A}}(p_3) = 0.1$$
$$\mu_{\tilde{A}}(p_4) = 0$$
...

Figure 2. A fuzzy region, for illustration purposes the fuzzy region is delimitted by a grey line. The membership grades for points belonging to the region are shaded, ranging from black (membership grade 1) to white (membership grade 0). A cross-section shows how the membership grades along the drawn line evolve. On the right are possible membership grades for the points illustrated.

Consider $A \subseteq U$ the set of all the points that belong to the region (this is a crisp set). The crisp set A is then generalized to a fuzzy set $\tilde{A}$, defined as follows.

Definition 5 (Fuzzy region) *A* fuzzy region $\tilde{A}$ *is defined as:*

$$\tilde{A} = \{(p, \mu_{\tilde{A}}(p)) \mid p \in U, \mu_{\tilde{A}}(p) > 0\}$$

where

$$\begin{aligned} \mu_{\tilde{A}} : U &\rightarrow [0,1] \\ p &\mapsto \mu_{\tilde{A}}(p). \end{aligned}$$

Here, U is the universe of all locations p; the membership grade $\mu_{\tilde{A}}(p)$ expresses the extent to which p belongs to the fuzzy region.

In Fig. 2, an example of a fuzzy region is shown. This example exhibits a simple behaviour, where membership grades decrease from the inside towards the outside, although the model allows for more complex distributions of membership grades.

With each element of the fuzzy set, a membership grade is associated; it has a veristic interpretation (Dubois and Prade, 2000), indicating the degree to which this element belongs to the fuzzy region.

The main difference between fuzzy regions and more common fuzzy sets, is that the domain of the fuzzy region U is in itself a two dimensional domain (theoretically $\mathbb{R}^2$, but normally limited to a region of interest). For clarity, the notation $\tilde{A}$ will be used for fuzzy regions (and fuzzy sets in general), whereas A will be used for crisp regions (and crisp sets). The above definition is the theoretical definition for fuzzy regions. In this form, it is not well suited for implementation; for implementation purposes, we refer to the models derived from it as elaborated in Verstraete, Hallez and De Tré (2006), and Verstraete et al. (2007). In this definition, each point point is a basic element. The definition has been refined in Verstraete (2010a) using the concept of the powerset $\wp$ (the set of all subsets of a given set) to allow for elements of the fuzzy region to be grouped together, yielding the definition:

$$\tilde{R} = \{(P, \mu_{\tilde{R}}(P)) | P \in \wp(\mathbb{R}^2) \wedge \forall P_1, P_2 : P_1 \cap P_2 = \emptyset\}. \tag{1}$$

Note that it is required that no two elements of the fuzzy region overlap: the intersection between any two elements should be empty. A point can only be considered to belong to the region once, even if it is to a membership grade less than 1.

2.2. Level-2 fuzzy region

The concept of the fuzzy powerset extension was first introduced in Verstraete (2011a) and is similar to the previous extension: a fuzzy region will now be defined as a fuzzy set of fuzzy sets, which is achieved using the fuzzy powerset. The fuzzy powerset $\tilde{\wp}$ of a set A is the set of of all fuzzy sets over the given set A,

$$\tilde{\wp}(A) = \left\{\tilde{X} | \forall x : \mu_{\tilde{x}}(x) > 0 \Rightarrow x \in A\right\}. \tag{2}$$

By using the $\tilde{\wp}(\mathbb{R}^2)$ as the domain for the fuzzy region, a region with fuzzy subregions can be defined. Using the fuzzy powerset, it is possible to define a fuzzy region similarly as has been done with the powerset.

Definition 6 (Level-2 fuzzy region)

$$\tilde{R} = \{(\tilde{R}', \mu_{\tilde{R}}(\tilde{R}')) | \tilde{R}' \in \tilde{\wp}(\mathbb{R}^2)\}. \tag{3}$$

The membership function is defined as:

$$\begin{aligned} \mu_{\tilde{R}} : \tilde{\wp}(\mathbb{R}^2) &\mapsto [0,1] \\ \tilde{R}' &\rightarrow \mu_{\tilde{R}}(\tilde{R}'). \end{aligned}$$

The elements of the fuzzy region $\tilde{R}$ are fuzzy regions as per Definition 5; an important difference with the previous extension is that it is now allowed for different subregions to share elements. The definition comprises what is referred to as a level-2 fuzzy set: a fuzzy set defined over a fuzzy domain (Gottwald, 1979; Klir and Yuan, 1995) and is named accordingly. This concept is not to be confused with a type-2 fuzzy set (Mendel, 2001), which is a fuzzy set defined over a crisp domain but where the membership grades are fuzzy sets.

By using fuzzy regions (in the original definition) as basic elements, we effectively obtain two membership grades for every point p belonging to a subregion $\tilde{R}'$: the first membership grade is associated directly with the point within the basic element ($\mu_{\tilde{R}'}(p)$), whereas the second membership grade is associated with the subregion ($\mu_{\tilde{R}}(\tilde{R}')$) and thus indirectly associated with each of its points.

2.2.1. Operations

Several operators have been defined (Verstraete et al., 2004) and even preliminary topology considerations have been made (Verstraete, 2010b). Additionally, more practical models have been derived to facilitate the representation of fuzzy regions in geographic systems; mainly imposing some limitations on the distribution of membership grades to yield more manageable representations using existing data concepts (bitmaps, Verstraete, Hallez and De Tré, 2006, or triangular networks, Verstraete, De Tré and Hallez, 2002). For the level-2 fuzzy regions, some operations have already been developed (Verstraete, 2011b, 2012).

In this contribution, the topological aspects of basic fuzzy regions will be considered in detail, with the main focus on optimizing definitions for boundary, interior and exterior. The approach will use our theoretical model in combination with an extension of the 9-intersection model. The fact that the definition of the regions deviates from the classical approach (region delimited by a boundary) poses a problem for topological definitions, where the concepts of boundary, interior and exterior are required. For a fuzzy region, it makes sense for these concepts to be fuzzy entities themselves, but which properties should they have and can we derive appropriate definitions for our fuzzy region model without imposing additional limitations on the model?

The level-2 fuzzy region model is still in early development. Consequently, in this contribution, only the topological aspects for the basic fuzzy regions defined by means of the original definition will be considered. The early development stages of the level-2 fuzzy regions have shown that the operations follow those of the basic regions very closely; after all, level-2 fuzzy regions are an extension of basic fuzzy regions that still is compatible. The main impact of moving to the level-2 fuzzy regions is that there can be multiple candidate regions, which are also reflected in the output and which may need some aggregation afterwards. However, up to that last aggregation/interpretation step, the operations and thus topology should be similar: each candidate region is a basic fuzzy region, the topological concepts for level-2 fuzzy regions will be the same as those for basic fuzzy regions, just with multiple candidates.

3. Topology concepts

3.1. Introduction

3.1.1. Crisp regions

For a crisp region, the concepts of boundary, interior and exterior are pretty straightforward, even though their purely mathematical definitions may feel a bit abstract. Informally, the boundary of a region R contains those points around which no subregion of R can be found that fully encloses the point and completely belongs to R; which is a mathematical way of saying that it contains those points that are really on the edge of the region. The interior is then defined as the set of points that belong to the region, but not to the boundary. The exterior simply comprises of all the points that do not belong to the region or its boundary.

3.1.2. Broad boundary regions

As a first step towards introduction imprecision or uncertainty in regions, Clementini and Di Felice (1994) defined the concept of broad boundary regions. In this concept the region was basically delimited by two boundaries: inner and outer boundary. The combination of these two made up the broad boundary: the entire region enclosed between the inner and outer boundary is considered to be the broad boundary. While it extends on the crisp region model, it still uses very crisp concepts: all points either belong to the region, to the boundary or to none of those. As it plays with an intuitive notion, it is interesting to have this model as special case of the fuzzy region model, just like the crisp model is a special case.

The egg-yolk model (Cohn and Gotts, 1994) uses a similar representation. The results are quite similar to the broad boundary model, even though the authors use a different approach and different assumptions.

3.1.3. 9-intersection model

One approach to model the topology between crisp regions, is the 9-intersection model, see Egenhofer and Sharma (1993) . It uses the concepts of interior (points considered to be inside the region, denoted $\cdot^\circ$), exterior (points considered outside the region, denoted $\cdot^-$) and boundary (denoted $\partial\cdot$), then considers every possible intersection between these concepts for both regions. This yields a total of 9 possible intersections, commonly grouped in the matrix shown below:

$$\begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^- \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^\circ & A^- \cap \partial B & A^- \cap B^- \end{pmatrix} \tag{4}$$

By assigning each matrix element 0 if the intersection is empty, and 1 if the intersection is not empty, $2^9 = 512$ matrices are possible. Depending on imposed restrictions on the regions (e.g. presence of holes), only a subset of the 512 relations is valid. For crisp regions without holes and no disconnected parts in a two-dimensional space $\mathbb{R}^2$, only eight such intersection matrices are meaningful, yielding the relations: disjoint, contains, inside, equal, meet, covers, coveredBy and overlap; illustrated in Fig. 3. An alternative way of describing topology is by

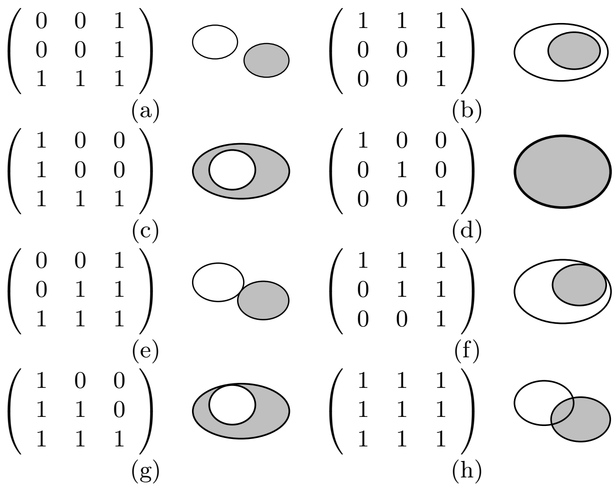


Figure 3. Topological relations for crisp regions: disjoint (a), contains (b), inside (c), equal (d), meet (e), covers (f), coveredBy (g) and overlap (h), with their intersection matrices.

means of the RCC calculus, but as the nine-intersection model lends itself easier toward an extension in a qualitative approach, we opted for the 9-intersection model. In this paper, several options for appropriate definitions for boundary, interior and exterior will be discussed. These definitions can then be used to define a similar intersection matrix which will then be further investigated.

4. Topology concepts for fuzzy regions

4.1. Introduction to fuzzy concepts

In Verstraete et al. (2008) and Verstraete (2010b), the topology of fuzzy regions was considered using simple, intuitive definitions for the topology concepts. These definitions had some shortcomings, which led to the development of improved definitions. The initial definitions are introduced here, so they can be used to explain the shortcomings.

4.1.1. Boundary

To come to an appropriate definition of the topological concepts for fuzzy regions, first let us consider a fuzzy region $\tilde{A}$, consisting of an inner section where points have membership grades 1 and gradually decreasing membership grades in a concentric way. We will give an intuitive definition as a starting point, and use the shortcomings of this definition as a guideline to derive a better definition. Consider only those points for which $\mu_{\tilde{R}}(p) < 1$. It can be argued that points for which $\mu_{\tilde{R}}(p) > 0.5$ belong more to the region than not, and similarly that points for which $\mu_{\tilde{R}}(p) < 0.5$ belong more to the outside of the region than to the region itself. This makes points with membership grade $\mu_{\tilde{R}}(p) = 0.5$ seem like good candidates for perfect boundary points, with points with greater differences from 0.5 receiving lower membership grades. This can be achieved as follows:

Definition 7

$$\Delta\tilde{R} = \bigcup_{\alpha\in]0,1]} \{(p, 2(0.5 - |0.5 - \alpha|))|p \in \partial\tilde{R}_\alpha\}. \tag{5}$$

By definition of the α cut, $\tilde{R}_\alpha$ is a crisp region. As such, it is possible to consider the traditional boundary concept of a crisp region, denoted $\partial\tilde{R}_\alpha$. This definition can be rewritten to introduce the notation of the membership function as

$$\Delta\tilde{R} = \{(p, \mu_{\Delta\tilde{R}}(p)\}$$

where

$$\begin{aligned} \mu_{\Delta\tilde{R}} : U &\rightarrow [0,1] \\ p &\mapsto \inf_\alpha\{2(0.5 - |0.5 - \alpha|))|p \in \partial\tilde{R}_\alpha\}. \end{aligned}$$

This definition is similar to the definition Du et al. (2005a) applied for fuzzy regions defined by means of a fuzzy buffer (Verstraete, Van Der Cruyssen and De Caluwe, 2000; Du et al., 2005a). It will now be used as a starting point to search for properties that are either mandatory or interesting for a fuzzy boundary; based on these properties a new definition will be derived. In Verstraete et al. (2008) and Verstraete (2010b), a slightly different definition was used to compensate for a major shortcoming (it is incompatible with crisp regions) of the above definition.

4.1.2. Interior and exterior

Following the same reasoning as above, we can consider an intuitive definition for the interior and exterior. If the points with membership grade 0.5 are considered to be the *best* points of the boundary, then it also makes sense to consider the points with a membership grade less than 0.5 not to belong to the interior. Defining the interior based on the above boundary definition is a fairly straightforward matter.

Definition 8 (interior $\tilde{R}^\circ$ of fuzzy region $\tilde{R}$)

$$\tilde{R}^\circ = \{(p, \mu_{\tilde{R}^\circ}(p)\}$$

where

$$\begin{array}{rcl} \mu_{\tilde{R}^\circ} : U & \to & [0,1] \\ p & \mapsto & \begin{cases} 0 & \mu_{\tilde{R}}(p) \leq 0.5 \\ 1 - \mu_{\Delta\tilde{R}}(p) & \textit{elsewhere.} \end{cases} \end{array}$$

The exterior can be considered in a completely analogous manner.

Definition 9 (exterior $\tilde{R}^-$ of fuzzy region $\tilde{R}$)

$$\tilde{R}^- = \{(p, \mu_{\tilde{R}^-}(p)\}$$

where

$$\begin{array}{rcl} \mu_{\tilde{R}^-} : U & \to & [0,1] \\ p & \mapsto & \begin{cases} 0 & \mu_{\tilde{R}}(p) \geq 0.5 \\ 1 - \mu_{\Delta\tilde{R}}(p) & \textit{elsewhere.} \end{cases} \end{array}$$

4.2. Examples used

To examine the definitions, a number of specific sample regions will be used. These are examples chosen to be more or less representative for situations that can be expected in the general case (either for the entire region, or for some part of the region):

- crisp region A, represented as a fuzzy region $\tilde{A}$ (Fig. 4a)
- broad boundary region B, represented as $\tilde{B}$ (Fig. 4b)
- fuzzy region $\tilde{C}_1$, with continuously strictly decreasing membership grades from the core outward and fuzzy region $\tilde{C}_2$, with continuously strictly decreasing membership grades from the core outward, but with lines of equal membership grades not forming closed lines (Fig. 4c)
- fuzzy region $\tilde{D}_1$, with decreasing membership grades in $\{0.2, 0.4, 0.6, 0.8, 1\}$ and $\tilde{D}_2$, with decreasing membership grades in $\{0.2, 0.4, 0.6, 0.8, 1\}$ but with lines of equal membership not forming closed lines (Fig. 4d)
- fuzzy region $\tilde{E}$, with non-decreasing membership grades from the inside out: the membership grades first decrease to 0.4, then increase again to 0.6, to finally decrease to 0 (Fig. 4e)

Regions $\tilde{A}$ and $\tilde{B}$ are interesting to illustrate the compatibility with crisp and with simple models. How the broad boundary region can be represented may differ in definitions, and the intuitiveness and workability of its representation as a fuzzy region is of interest; membership grade distributions will therefore depend on the definitions. Regions $\tilde{C}_1$ and $\tilde{D}_1$ are straightforward examples

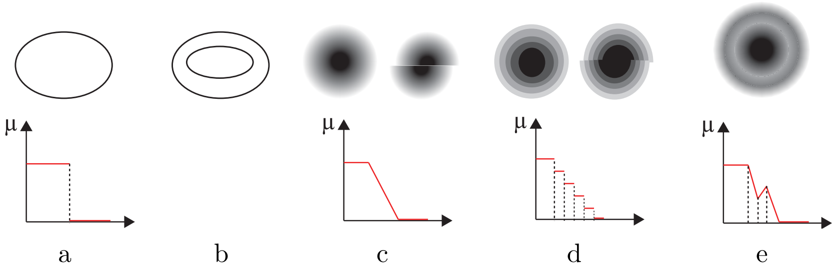


Figure 4. Graphical illustration of the example regions, the top row shows a drawing of the region with grey scales to indicate membership grades, the bottom row shows a cross section from the core outward. (a) crisp region $\tilde{A}$, (b)broad boundary region $\tilde{B}$, (c) continues decreasing membership grades (with and without spatial discontinuity, $\tilde{C}2$ and $\tilde{C}1$), (d) discontinuous membership grades (with and without spatial discontinuity, $\tilde{D}2$ and $\tilde{D}1$), (e) non-decreasing membership grades, $\tilde{E}$

of regions that are likely to be quite common, where the latter is used to illustrate discontinuities in the membership grade. Regions $\tilde{C}_2$ and $\tilde{D}_2$ are both regions that are used to verify how the definitions handle regions with spatial discontinuities. Region $\tilde{E}$ finally is used to see how *strange* membership grade distributions are dealt with.

4.2.1. Deriving required properties

In the above sections, we have introduced some fairly intuitive definitions. We will now use these so see on what points they fail and what properties are desirable for the fuzzy topological concepts.

A first and major point of criticism to the definition of the boundary is that it yields an empty boundary when applied to crisp regions (e.g. the example region $\tilde{A}$). Surely, any extension of an existing model should be compatible with the existing model. A second point of criticism to this is that in our fuzzy region model there currently is no a priori requirement for a fuzzy region to contain points with membership grade 0.5, in which case no points in $\Delta\tilde{R}$ will have membership grade 1. While it is strictly speaking not necessary for fuzzy sets to be normalized (Klir and Yuan, 1995), the downside here is that it can be the result even when working with only normalized regions. This also appears counter-intuitive: a region which has membership grades in $\{0.2, 0.4, 0.6, 0.8, 1\}$, such as examples $\tilde{D}1$ and $\tilde{D}_2$, can be nicely defined, but would yield 0.8 as the highest membership grade in the boundary and no single part that is the border to the full extent (Fig. 5c). Finally, there is a more tricky problem that can surface as well. Consider the examples $\tilde{C}_2$ and $\tilde{D}_2$: membership grades are decreasing from the inside out, but points with membership grade 0.5 do

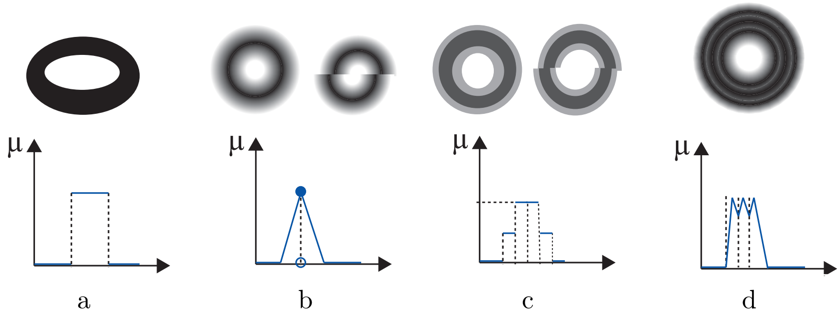


Figure 5. Graphical illustration of the boundaries of the example regions, using definition 12. The top row shows a drawing of the region with grey scales to indicate membership grades, the bottom row a cross section from the core outward. (a) crisp region, (b) broad boundary region, the cross section may depend on on how the region is modelled to match various definitions, (c) continuous decreasing membership grades (with and without spatial discontinuity), (d) discontinuous membership grades (with and without spatial discontinuity, (e) non-decreasing membership grade

not form a closed line. The boundary will yield a normalized fuzzy set, but the α level of the boundary at 1 is not a closed line, as illustrated by the example region $\tilde{C}_2$ (Fig. 5b). It should be noted that it is possible to define regions with membership grade distributions such that there are α levels at which the boundary does not form a closed line. In general, this should not lead to problems (it usually means the region is oddly defined to begin with), but one should not expect it to happen to the core of the boundary. The definition does make it easy to represent broad boundary regions: by assigning all points of the broad boundary the membership grade 0.5, these points will become the boundary of the region (Fig. 5a). For the last example of a region with a non-decreasing membership grade, the result on Fig. 5d was to be expected; even though it is a bit odd to have multiple concentric cores in the boundary region.

Trying to solve these issues by imposing restrictions to the definition of fuzzy regions themselves may limit the concept too much, so the attention will go towards improving the definition of the boundary concept. Using the above observations, the following properties are deemed important:

1. no concept (boundary, interior and exterior) should ever be empty for non-empty regions
2. when dealing with normalized regions, the concepts should be normalized
3. the concepts should not depend on the presence (or absence) of any particular membership grade
4. there should be a closed line for *relevant* alpha cuts.

Apart from properties that will be required, additional interesting properties

- while not mandatory - will play a part in comparison of different definitions:

1. logical connection between neighbouring points inside the boundary
2. a relation between $\partial\tilde{R}_\alpha$ and $(\Delta\tilde{R})_\alpha$
3. ability to represent broad boundary regions.

In this contribution, the above definition will be modified, but a different approach will also be considered. This will yield a number of candidate definitions which will then be examined further.

5. Membership grade derived concepts

Here, we will continue with Definition 7, section 4.1.1, and improve on it to derive a better candidate for the definition of the boundary.

5.1. Boundary

To overcome the first criticism, it suffices to take into account different α levels: these are crisp regions and their boundary is found using the definition for crisp boundaries. Not only does this modification solve the issue for crisp regions, it also guarantees that if a fuzzy region has a crisp boundary at some part, it will have boundary points for that part as well.

Definition 10

$$\begin{aligned} \mu_{\Delta\tilde{R}} : U &\to [0,1] \\ p &\mapsto \max(\sup\{\alpha | p \in \partial\tilde{R}_\alpha\}, 2(0.5 - |0.5 - \mu_{\tilde{R}}(p)|)). \end{aligned}$$

This is also the definition we used used for the case study in Verstraete (2010b). This adaptation solves the issues that occur in the crisp case, but it still exhibits some strange results in the general case. For discontinuities in the range $[0, 0.5]$, the results poses no surprises: points p with membership grade $\mu_{\tilde{R}}(p) = 0.2$ that belong to $\partial\tilde{R}_{0.2}$ are given membership grade $\mu_{\Delta\tilde{R}} = 0.4$, and points that are more *inside* the region but also have membership grade $\mu_{\tilde{A}} = 0.2$ are also given the membership grade $\mu_{\Delta\tilde{R}} = 0.4$. Discontinuities in the range $[0.5, 1]$ pose more issues: points p with membership grade $\mu_{\tilde{R}}(p) = 0.8$ that belong to $\partial\tilde{R}_{0.8}$ are given membership grade $\mu_{\Delta\tilde{R}} = 0.8$; other points that also have membership grade $\mu_{\tilde{A}} = 0.8$ are actually given the membership grade $\mu_{\Delta\tilde{R}} = 0.4$.

5.2. Interior and exterior

Regardless of the changes to the definition of the boundary, the previous definitions for the interior and exterior still satisfy the properties put forward. Both these definitions are similar and will use the new boundary definition.

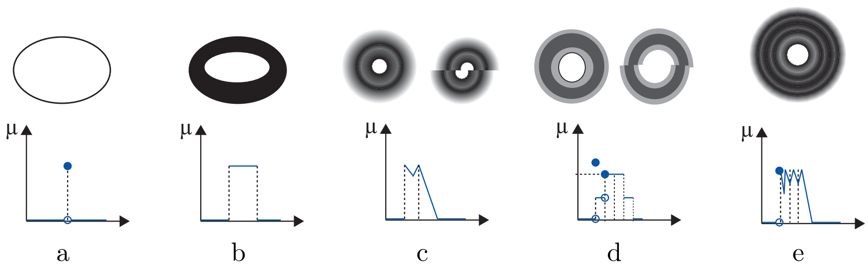


Figure 6. Graphical illustration of the boundaries of the example regions, using Definition 10. The top row shows a drawing of the region with grey scales to indicate membership grades, the bottom row a cross section from the core outward. (a) crisp region, (b) broad boundary region, the cross section may depend on how the region is modelled to match various definitions, (c) continuous decreasing membership grades (with and without spatial discontinuity), (d) discontinuous membership grades (with and without spatial discontinuity, (e) non-decreasing membership grade

5.3. Digression

5.3.1. Examples

We will now look in to the performance of these definitions with the examples from Section 4.2.

Crisp region A, represented as $\tilde{A}$ The definition is such that it reverts back to the crisp definition: the only points in the boundary will be the points that are in $\partial\tilde{A}_\alpha$, for $\alpha = 1$, Fig. 6a. The interior and exterior are also exactly the same as in the crisp definition.

Broad boundary region B, represented as $\tilde{B}$ The first question is how to represent a broad boundary region; the most obvious approach is to assign points in the inner broad boundary the membership grade 1, and points in between inner and outer boundary the membership grade 0.5. The definitions revert back to the definitions for boundary, interior and exterior of broad boundary regions, Fig. 6b.

Fuzzy region $\tilde{C}_1$, with continuously strictly decreasing membership grades from the core outward Here things get interesting. All the points with membership grades in $]0, 0.666]$ belong to the boundary to the extent determined by the formula $2(0.5 - |0.5 - \mu_{\tilde{R}}(p)|)$. Points with membership grades in $]0.666, 1[$ will get the same membership grades as in the original region, which provides for a local minimum in 0.666, Fig. 6c. Furthermore, the points in $\partial\tilde{A}_\alpha$, for $\alpha = 1$ belong to this extent to the boundary.

Fuzzy region $\tilde{C_2}$, with continuously strictly decreasing membership grades from the core outward, but no closed line The boundary looks similar to the above boundary, the lines at each α-level are closed.

Fuzzy region $\tilde{D_1}$, with decreasing membership grades $\in \{0.2, 0.4, 0.6, 0.8, 1\}$. For this region, the boundary looks as in the figure. Points, for which $\mu_{D_1}(p) \in]0, 0.5[$ belong to the boundary to some extent greater than the extent to which they belong to the region. Points with $\mu_{D_1}(p) \in]0.5, 1[$ that are not part of the boundary of an α level also belong to the boundary, but now to a lesser extent than the extent to which they belong to the boundary. Finally, points for which $\mu_{D_1}(p) \in]0, 0.5[$ that are part of the boundary of some α level belong to the boundary to the same extent as to the region, Fig. 6d.

Fuzzy region $\tilde{D_2}$, with decreasing membership grades $\in \{0.2, 0.4, 0.6, 0.8, 1\}$. The boundary will look similar to the boundary of $\tilde{D_1}$.

Fuzzy region $\tilde{E}$ The boundary will have a similar shape as before, but with a change similar as was observed for region $\tilde{C_1}$: there will be an additional local minimum, Fig. 6e.

5.3.2. Properties

Required properties The definitions will never yield an empty boundary for a non-empty region: as soon as there are points, there will be points that are part of the boundary of some α level. The same reasoning can be applied for points with membership grade 1, so as soon as there are points with membership grade 1 in the region, there will be such points in the boundary. As the boundary always contains the points of the boundary of core of the fuzzy region, the core of the fuzzy boundary will form a closed line around the core of the region. There is no dependency of specific membership grades, so all the mandatory properties are satisfied.

Desired properties Apart from properties that will be required, additional interesting properties - not being mandatory - will play a part in the comparison of the different definitions. Here, the strange membership grades around the points p for which $\mu_{\tilde{C_1}}(p) = 0.666$ stand out: intuitively, there is no reason for this to be a local minimum. It also occurs in region $\tilde{E}$. As a result of this, the definition fails on the first property. There is a connection between the α levels of the boundary and the boundaries of the same α levels: $(\Delta\tilde{R})_\alpha$ encloses $\partial\tilde{R}_\alpha$. Finally, broad boundary regions can be easily represented, as shown in the examples.

5.3.3. Summary

This approach improves a lot on the intuitive definition, while still remaining true to the intuitive aspects. Both the crisp case and the broad boundary case are incorporated in the model, and the results are normalized for a normalized input. Even spatial discontinuities will not occur for the core of the region, and can only occur in quite artificial examples for other membership grades. The difference between the boundaries $\Delta\tilde{C_1}$ and $\Delta\tilde{D_1}$ is also a bit undesirable: the region $\tilde{D_1}$ could be seen as a discrete approximation of $\tilde{C_1}$, yet their boundaries look quite different. In the continuous case, the region differs from the intuitive definition because the membership grades of the boundary do not decrease towards the core of the region.

Apart from looking at the boundary, it is worth considering the interior and exterior. For the region $\tilde{C_1}$, the interior will consist of the interior of the core of the region (and these points will have membership grade 1), and then the points for which $\mu_{\tilde{C_1}}(p) < 0.5$, but these points will get very low values: no longer is there a gradual transition as before, and the local minimum that occurs in the boundary now becomes a local maximum. There is a similar difference with the interior of $\tilde{D_1}$ as was observed for the boundaries. The exterior has not changed because of the change of boundary definition.

6. α-level based

In this section, the property that the boundary of every α-level equals the same α-level of the fuzzy boundary is put forward:

$$\partial\tilde{R}_\alpha = (\Delta\tilde{R})_\alpha. \tag{6}$$

From this property, a number of possible definitions for boundary and other concepts are derived.

6.1. Boundary

The above property implies that the definition of the boundary is of the form:

Definition 11

$$\begin{array}{rcl} \mu_{\Delta\tilde{R}} : U & \to & [0,1] \\ p & \mapsto & \begin{cases} x & \textit{if} \nexists \alpha : p \in \partial\tilde{R}_\alpha \\ \sup\{\alpha : p \in \partial\tilde{R}_\alpha\} & \textit{elsewhere} \end{cases} \end{array}$$

with x still to be determined.

Independent of the values of x, this definition already matches the first three desired properties. The most obvious candidates for x are 0, $\mu_{\tilde{R}}(p)$ and some value in between (for the sake of argumentation, we will use $\mu_{\tilde{R}}(p)^2$). These three options translate to: remaining points belong to the same extent to the boundary as they do to the region (x), do not belong to the boundary (0), or belong to a lesser extent to the boundary than to the region. If $x > \mu_{\tilde{R}}(p)$, the property we put forward (6) would no longer hold. Note that if the membership grades of the region are continuously and strictly decreasing, for all points p with membership grade in $]0,1[$ there $\exists\alpha : p \in \partial\tilde{R}_\alpha$, implying all points with membership grade less than 1 will be in the boundary with that same membership grade; this can also occur for only a portion of the boundary.

The case where $x = 0$ only considers boundary points to be those points that belong to the boundary of some α level, and assigns them this α as membership grade.

Definition 12

$$\begin{aligned} \mu_{\Delta\tilde{R}} : U &\rightarrow [0,1] \\ p &\mapsto \begin{cases} 0 & \textit{if } \nexists\alpha : p \in \partial\tilde{R}_\alpha \\ \sup\{\alpha : p \in \partial\tilde{R}_\alpha\} & \textit{elsewhere.} \end{cases} \end{aligned}$$

If $x = \mu_{\tilde{R}}(p)$, then all points that are not part of the boundary of any α level are assigned the same membership grade as in the original region. The full equality in (6) no longer holds, but is weakened to $\partial\tilde{R}_\alpha \subseteq (\Delta\tilde{R})_\alpha$ (yet both will have the same outline). Points, for which $\mu_{\tilde{R}}(p) = 1$, would all be assigned a membership grade 1, making the whole region part of the boundary. As this is not desired, it is necessary to add an exception for this case; yielding:

Definition 13

$$\begin{aligned} \mu_{\Delta\tilde{R}} : U &\rightarrow [0,1] \\ p &\mapsto \begin{cases} \mu_{\tilde{R}}(p) & \textit{if } \nexists\alpha : p \in \partial\tilde{R}_\alpha \\ 0 & \textit{if } p \in \tilde{R}_1^\circ \\ \sup\{\alpha : p \in \partial\tilde{R}_\alpha\} & \textit{elsewhere.} \end{cases} \end{aligned}$$

The case where $x \in]0, \mu_{\tilde{R}}(p)[$ (e.g. $\mu_{\tilde{R}}^2(p)$) is similar, only now the points that are not part of the boundary in any α-level are assigned a smaller value.

Definition 14

$$\begin{aligned} \mu_{\Delta\tilde{R}} : U &\rightarrow [0,1] \\ p &\mapsto \begin{cases} (\mu_{\tilde{R}}(p))^2 & \textit{if } \nexists\alpha : p \in \partial\tilde{R}_\alpha \\ 0 & \textit{if } p \in \tilde{R}_1^\circ \\ \sup\{\alpha : p \in \partial\tilde{R}_\alpha\} & \textit{elsewhere.} \end{cases} \end{aligned}$$

The compatibility with the broad boundary regions is not possible for $x = 0$, but is possible for $x \in]0, \mu_{\tilde{R}}(p)]$, as it suffices to assign the points in the broad boundary a membership grade in $]0,1[$ to make them part of the boundary.

6.2. Interior and exterior

The use of the α-levels allows for a straightforward definition of the interior if we - in analogy to the boundary - impose that the property

$$(\tilde{R}_\alpha)^\circ = (\tilde{R}^\circ)_\alpha \tag{7}$$

must be satisfied,

$$\begin{aligned} \mu_{\tilde{R}^\circ} : U &\rightarrow [0,1] \\ p &\mapsto \begin{cases} \sup\{\alpha : p \in \tilde{R}^\circ_\alpha\} & \text{if } \nexists \alpha : p \in \partial \tilde{R}_\alpha \\ x & \text{elsewhere.} \end{cases} \end{aligned}$$

Similarly as with the boundary Definition 11, there are different possibilities for the parameter x. As for the boundary, it is impossible that $x > \mu_{\tilde{R}}(p)$; however, it is now also impossible for $x = \mu_{\tilde{R}}(p)$, as this would violate (7). The only valid values for x are therefore $x \in [0, \mu_{\tilde{R}}(p)[$.

The exterior can be considered quite analogue to the interior, except that now the property changes to

$$(\tilde{R}_\alpha)^- = (\tilde{R}^-)_\alpha \tag{8}$$

$$\begin{aligned} \mu_{\tilde{R}^-} : U &\rightarrow [0,1] \\ p &\mapsto \begin{cases} \sup\{\alpha : p \in \tilde{R}^-_\alpha\} & \text{if } \nexists \alpha : p \in \partial \tilde{R}_\alpha \\ x & \text{elsewhere.} \end{cases} \end{aligned}$$

Again, several options remain for the parameter x, but they are similar as for the interior: any $x \geq \mu_{\tilde{R}}(p)$ would violate (8), leaving $x \in [0, \mu_{\tilde{R}}(p)[$ as valid values.

6.3. Digression

6.3.1. Examples

We will now look into the performance of these definitions with the examples from Section 4.2.

Crisp region A, represented as $\tilde{A}$ Regardless of the value of x, the definition reverts back to the crisp definition: the only points in the boundary will be the points that are in $\partial \tilde{A}_\alpha$, for $\alpha = 1$. The interior and exterior are also exactly the same as in the crisp definition.

Fuzzy region $\tilde{C}_1$, with continuously strictly decreasing membership grades from the core outward As the membership grades are strictly decreasing, every point p for which $\mu_{\tilde{C}_1}(p) \in]0,1[$ will belong to the boundary to the same extent as to the region. The points in $\partial \tilde{A}_\alpha$, for $\alpha = 1$ will also belong to it to the full extent.

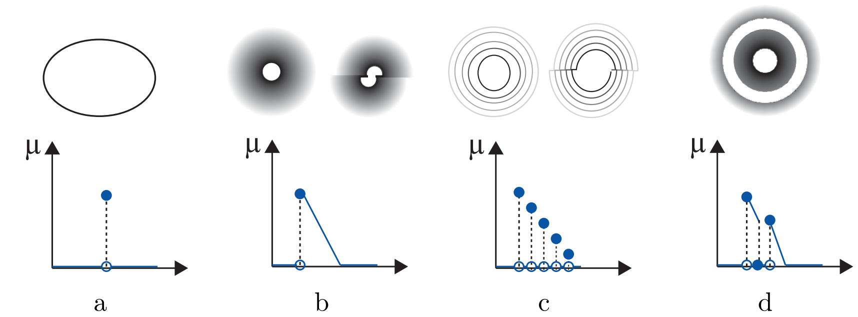


Figure 7. Graphical illustration of the boundaries of the example regions, using Definition 12. The top row shows a drawing of the region with grey scales to indicate membership grades, the bottom row a cross section from the core outward. (a) crisp region, (b) continuous decreasing membership grades (with and without spatial discontinuity), (c) discontinuous membership grades (with and without spatial discontinuity), (d) non-decreasing membership grade

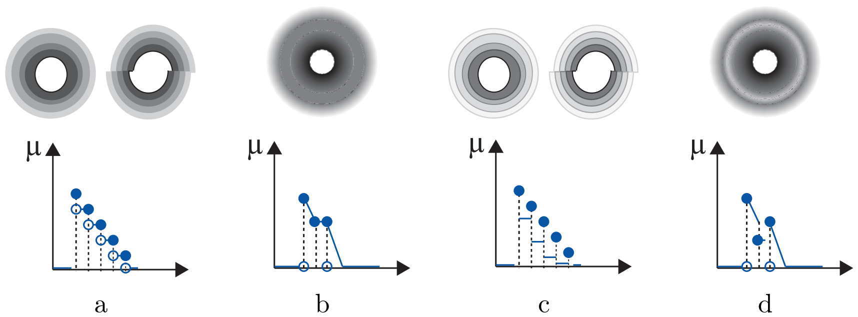


Figure 8. Graphical illustration of the boundaries of the example regions that have different boundaries when changing the value of x: using Definition 13 (a), (b) and Definition 14 (c), (d). The top row shows a drawing of the region with grey scales to indicate membership grades, the bottom row a cross section of from the core outward. (a),(c) continuous decreasing membership grades (with and without spatial discontinuity), (b),(d) discontinuous membership grades (with and without spatial discontinuity)

Fuzzy region $\tilde{C}_2$, with continuously strictly decreasing membership grades from the core outward, but no closed line The boundary looks similar to the above boundary, however the disconnected parts are closed.

Fuzzy region $\tilde{D}_1$, with decreasing membership grades $\in \{0.2, 0.4, 0.6, 0.8, 1\}$. Points for which $\mu_{\tilde{D}_1}(p) \in \partial D_{1\alpha}$, for some α are given that α as membership grade in the boundary. Other points are given x, which results for x respectively 0, $\mu_{\tilde{D}_1}(p)$ and $\mu_{\tilde{D}_1}(p)^2$ in the boundaries in Figs. 7d, 8a and 8c.

Fuzzy region $\tilde{D}_2$, with decreasing membership grades $\in \{0.2, 0.4, 0.6, 0.8, 1\}$. The boundary will look similar to the boundary of $\tilde{D}_1$, but the lines will be closed.

Fuzzy region $\tilde{E}$ Points for which $\mu_{\tilde{E}}(p) \in \partial E_\alpha$, for some α are given that α as membership grade in the boundary. Other points are given the value x, for x respectively 0, $\mu_{\tilde{E}}(p)$ and $\mu_{\tilde{E}}(p)^2$ the boundaries are shown in Figs. 7d, 8b and 8d.

Broad boundary region B, represented as $\tilde{B}$ The definition is immediately compatible with the crisp case, but a trick is required to make it compatible with the broad boundary region. First, it should be noted that regardless of the value x, it is not possible to have all points in the boundary get the same value: the boundary of the core will have membership grade 1, whereas other points of the boundary will get a lower membership grade. Following the definition, all points in the broad boundary should belong to the boundary of some α-level of the region. The only way to achieve this is to assign membership grades that are strictly decreasing from the kernel outward, in which case the boundary will be similar to that of $\tilde{C}_1$, Fig. 7b. When $x > 0$ in the definition, other constructions are also possible, in which case the boundary resembles that of $\tilde{D}_1$, Fig. 8a and Fig. 8c, albeit with fewer discontinuities in the membership grade distribution.

6.3.2. Properties

Required properties The definitions will never yield an empty boundary for a non-empty region: as soon as there are points, there will be points that are part of the boundary of some α level. The same reasoning can be applied for points with membership grade 1, so as soon as there are points with membership grade 1 in the region, there will be such points in the boundary. As the boundary always contains the points of the boundary of core of the fuzzy region, the core of the fuzzy boundary will form a closed line around the core of the region. All the mandatory properties are satisfied.

Desired properties The first desired property concerns a logical connection between neighbouring points in the boundary. For the case where $x = 0$, points just next to the core of the boundary can get a membership grade 0, so this is quite a crisp distinction. However, for the other values of x, there is a softer transition. By construction, we have equality between $(\Delta\tilde{R})_\alpha$ and $\partial\tilde{R}_\alpha$ (or rather equality between the outer outline of $(\Delta\tilde{R})_\alpha$). As shown in the examples, both crisp regions and broad boundary regions can be represented in a way that is compatible with the concepts for boundary, interior and exterior.

6.3.3. Overview

This approach improves a lot on the previous definitions; even if it moves further away from the initial intuitive definition, it does stay quite close to the crisp concepts. The presence of the parameter x allows for a number of distinct definitions, each with slightly different properties. The differences between the example regions $\tilde{C}_1$ and $\tilde{D}_1$ are much smaller than before, and especially for x equal to the membership grade in the original region, $\Delta\tilde{D}_1$ resembles a discrete version of $\Delta\tilde{C}_1$.

The membership grades in the interior and exterior are - unlike in the previous definition - independent from the boundary. A similar parameter here allows for a number of distinct definitions as well. Some combinations of the parameters in the boundary and interior lead to interesting results. If both parameters are set to 0, then no point belongs to both the interior and the boundary at the same time. Increasing the parameter in the boundary definition will increase the membership grade of the points in the boundary, for which it is applicable. The highest possible value will make those points belong as much to the boundary as to the interior. Increasing the parameter in the interior, on the other hand, impacts on the points that belong to a boundary at some α level. It should be noted that it is possible to assign values that may yield (7) failing. The impact of the parameter in the definition of the exterior is similar.

7. Determining topology

7.1. Intersection matrices

7.1.1. Definition

Once the concepts of interior, boundary and exterior are known, the topology can be investigated using an extension of the 9-intersection matrix as shown in (4). the 9-intersection matrix for fuzzy regions is:

$$\begin{pmatrix} h(\tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ) & h(\tilde{A}^\circ \tilde{\cap} \Delta\tilde{B}) & h(\tilde{A}^\circ \tilde{\cap} \tilde{B}^-) \\ h(\Delta\tilde{A} \tilde{\cap} \tilde{B}^\circ) & h(\Delta\tilde{A} \tilde{\cap} \Delta\tilde{B}) & h(\Delta\tilde{A} \tilde{\cap} \tilde{B}^-) \\ h(\tilde{A}^- \tilde{\cap} \tilde{B}^\circ) & h(\tilde{A}^- \tilde{\cap} \Delta\tilde{B}) & h(\tilde{A}^- \tilde{\cap} \tilde{B}^-) \end{pmatrix}. \tag{9}$$

Here $h(X)$ is a shorthand notation for the $height(X)$ of a fuzzy set X, Definition 4. The intersection is the fuzzy intersection, by means of a t-norm (e.g. minimum). Note that matrix elements are no longer limited to $\{0, 1\}$, but can have any value in the range $[0, 1]$. This, in turn, will influence how the intersection matrices will be interpreted. A full case study of the fuzzy 9-intersection matrix has been made using the intuitive definitions for the topological concepts. This yields a large number of cases, but there are similarities between some cases, and there are different ways of grouping these similar cases together. Even though the choice of the definitions may have some impact on the cases, the methodology is completely independent of the definitions used, even though the results may differ.

7.1.2. Methodology

Using the definitions, it is possible to perform a case study as illustrated in Verstraete (2010b), where the initial intuitive definitions were used. The matrix elements are in the range $[0, 1]$, implying that an infinite number of cases is possible. It can, though, be greatly reduced by considering a limited number of values and value ranges for each intersection element. In Verstraete (2010b), we considered those are equal to 0 or to 1, or have a value in $[0, 1[$ or $]0, 1]$. The interpretations are that the intersection is either empty, is an intersection to the full extent, can be empty or can be to the full extent. While this leaves 4^9 possible options, not all of them can occur, just as not all of the 2^9 possible cases can occur in the traditional 9-intersection matrix. It is also possible for cases to be grouped together, which resulted in a manageable 44 cases for the case study with the intuitive definition. It is important to realize that the different cases are not mutually exclusive.

A full case study is in progress, but methods are applied to find similarities in order to speed up generating the cases for the different families of definitions.

As in the crisp case (and the broad boundary model), it is also possible to create a *conceptual neighbourhood graph*. This is a graph in which all cases are present as nodes, and an edge connects two nodes that have a minimal number of differences between them. For example if two intersection cases have the same elements, but one (in one matrix this element is 0 whereas in the other it is $[0, 1[$), then these matrices will be connected with an edge.

In the approach, it is interesting to also group the different intersection cases; for each matrix element this will yield at most four groups (in the case study with the intuitive definition, many elements only yielded two or three groups).

7.2. Usage

7.2.1. Identifying matching cases

For two given fuzzy regions, it is fairly easy to construct the intersection matrix, which is a matrix with nine elements in the range $[0, 1]$. Using the groups that were defined above, it is possible to determine which cases in the case study match the given situation. In general, more than one case will match: for any value in the range $]0,1[$, both cases $[0,1[$ and $]0,1]$ will match. To be able to compare the different matching cases, the concept of *match values* was introduced.

7.3. Match values

As there can be multiple cases that match a given value, it is necessary to provide some qualitative measure to compare them.

Definition 15 (Match value m_x^i for a case i and a matrix element x)

$$m_x^i = \begin{cases} x & \text{if range of case } i =]0,1] \\ 1-x & \text{if range of case } i = [0,1[. \end{cases}$$

For a given intersection matrix, we will now have different matching cases, and match values that indicate for each element how well it matches a given case. By aggregating the match values of all the matrix elements of a single case, a single number is obtained. This number is considered representative for how well this particular case matches the given matrix.

8. Conclusion

In this contribution, some options for the topology of fuzzy regions were considered. A family of definitions for boundary, interior and exterior was derived and tested against a number of representative cases. These concepts are a first and important step towards a full topological case study. The methodology of the case study has been mentioned (and referenced) and it has been performed using the previous definitions of the concepts. The full case study has been performed with the methodology, and will be performed again using the definitions presented here. From that study we know the topological concepts are not simple unique cases, but that any random example may match multiple cases. This is also described in the article. The methodology we described provides for an algorithm to reason with all the possible cases, and a new case study considering all topological cases (using the described methodology) is ongoing.

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