

Stabilisation of LC ladder network with the help of
delayed output feedback*

by

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Abstract: The paper presents a method of stabilisation of an LC ladder network with a delayed output feedback. A discussion of certain properties of tridiagonal matrices and formulation of the considered system in state space equations is included. Formal stability of the arising infinite dimensional system is described and stability conditions are formulated based on properties of characteristic quasi-polynomial. The method of D-partitions is used to determine the stability regions in the controller parameter space. Paper includes examples for ladders of dimensions 1, 2 and 3 and a comparison with Padé approximation.

Keywords: stabilisation, delay differential equation, stabilisation, LC ladder network, D-partitions.

1. Introduction

In this paper, we consider stabilisation of LC ladder networks such as given in Fig. 1. This problem is practically important, and was considered in earlier works. The system is an undamped oscillatory system, for which stabilisation by an output feedback is difficult. Systems of this kind along with different types of feedback were considered in the earlier works - distributed feedback (active systems) Mitkowski (1973), linear feedback, Mitkowski, (1978b), boundary feedback Mitkowski, (1987a), and dynamical feedback, Mitkowski (2003). For a survey of different methods of stabilisation, see Mitkowski (2004b). Non-linear dynamical feedback was considered in Skruch (2005, p. 30). It was shown that such system can be stabilised by dynamical linear and nonlinear output feedbacks, but a static output feedback is unable to stabilise this system. Here we present an approach to stabilisation of the system with an application of linear delayed output feedback. It is a continuation of authors' earlier works (see, for example, Baranowski, Mitkowski and Skruch, 2009).

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2. Preliminaries

At first we will present certain information regarding properties of tridiagonal matrices (also called Jacobi matrices). This is a very important matrix class with many interesting applications. Some results, regarding them, are collected in Ilin and Kuznyetsow (1985), and in earlier author's works (see for example Mitkowski, 1991a, 1996). It should be noted that applications can be found in biology, circuit theory, signal processing and other areas. It should be also noted that research in this area is still active (see, for example, Cheng and Berger, 2009).

We will present a lemma on the characteristic polynomial of a particular tridiagonal matrix.

LEMMA 1 (see, for example, Mitkowski, 1996). *Characteristic polynomial $J_n(s)$ of matrix \mathbf{E} , defined as*

$$J_n(s) = \det[s\mathbf{I} - \mathbf{E}] \quad (1)$$

where

$$\mathbf{E} = \begin{bmatrix} b & c & 0 & \dots & 0 & 0 \\ a & b & c & \dots & 0 & 0 \\ 0 & a & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & c \\ 0 & 0 & 0 & \dots & a & b \end{bmatrix}_{n \times n} \quad (2)$$

is given by the following recurrence formula

$$J_k(s) = (s - b)J_{k-1}(s) - acJ_{k-2}(s)$$

$$J_0(s) = 1$$

$$J_1(s) = s - b$$

$$k = 2, 3, 4, \dots, n.$$

To complement this lemma, we will prove the following corollary, regarding the constant value of this polynomial at 0.

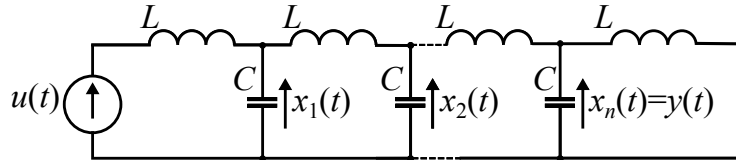


Figure 1. LC ladder network

COROLLARY 2 $J_n(0)$ is given by:

$$J_n(0) = \begin{cases} \text{If } b^2 \neq 4ac: \\ \quad (-1)^n \left(\frac{r_2 - b}{r_2 - r_1} r_1^n + \frac{b - r_1}{r_2 - r_1} r_2^n \right) \\ \quad \text{where} \\ \quad r_1 = \frac{1}{2} \left(b + \sqrt{b^2 - 4ac} \right) \\ \quad r_2 = \frac{1}{2} \left(b - \sqrt{b^2 - 4ac} \right) \\ \text{If } b^2 = 4ac: \\ \quad (-1)^n (1+n) \left(\frac{b}{2} \right)^n. \end{cases} \quad (3)$$

Proof. $J_n(0)$ is equal to the constant term of $J_n(s)$. It is known (see Turowicz, 2005, p. 117), that characteristic polynomial of any square, $n \times n$ matrix \mathbf{F} is given by

$$\det[s\mathbf{I} - \mathbf{F}] = s^n - S_1 s^{n-1} + S_2 s^{n-2} + \dots + (-1)^n S_n \quad (4)$$

where S_k is the sum of all principal minors of degree k of matrix \mathbf{F} . In particular, $S_1 = \text{tr}\mathbf{F}$ and $S_n = \det\mathbf{F}$. That is why in order to find the constant value of $J_n(s)$ we only need to compute the determinant of matrix \mathbf{E} . The formula for this determinant for $b^2 \neq 4ac$ is given in Turowicz (2005, p. 55). For $b^2 = 4ac$ we have to consider a difference equation

$$\zeta_k - b\zeta_{k-1} + ac\zeta_{k-2} = 0 \quad (5)$$

$$\zeta_0 = 1 \quad (6)$$

$$\zeta_1 = b. \quad (7)$$

It should be noted that $\det\mathbf{E} = \zeta_n$. Characteristic equation of (5) has one double root $\zeta = b/2$. General solution is then

$$\zeta_n = K_1 \left(\frac{b}{2} \right)^n + n K_2 \left(\frac{b}{2} \right)^n. \quad (8)$$

For initial conditions (6) and (7) we get $K_1 = K_2 = 1$, so

$$\det\mathbf{E} = (1+n) \left(\frac{b}{2} \right)^n. \quad (9)$$

3. Problem formulation

Mathematical model of a LC ladder network with $y(t) = x_n(t)$, depicted in Fig. 1 is given by the following second order equation of dimension n

$$\ddot{x}(t) + \mathbf{A}x(t) = \mathbf{B}u(t) \quad (10)$$

$$y(t) = \mathbf{C}x(t) \quad (11)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and

$$\mathbf{A} = \frac{1}{LC} \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}_{n \times n} \quad (12)$$

$$\mathbf{B} = \frac{1}{LC} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \quad \mathbf{C} = [0 \ \dots \ 0 \ 0 \ 1]_{1 \times n}.$$

System (10) is oscillatory and undamped – its eigenvalues are imaginary and equal to $\pm j\omega_i$, $j = \sqrt{-1}$, $i = 1, 2, \dots, n$ where

$$\omega_i = \sqrt{\lambda_i(\mathbf{A})} \quad (13)$$

$$\lambda_i(\mathbf{A}) = \frac{2}{LC}(1 - \cos \varphi_i) \quad (14)$$

$$\varphi_i = \frac{i\pi}{n+1}, \quad i = 1, 2, \dots, n \quad (15)$$

(see, for example, Mitkowski, 2003, 2004b). General results on stabilisation of such second order systems are provided by Skruch (2005) and Mitkowski (2004a). Main conclusion of these and other works is that oscillatory system such as (10), (11) with matrices given by (12) can be stabilised by an appropriate static state feedback or dynamical output feedback.

In this paper we will consider a feedback in the form of proportional, time delayed controller

$$u(t) = Ky(t-h) \quad (16)$$

where $K \in \mathbb{R}$ is the gain and $h > 0$ is the time delay. Usually, in control application focus is on elimination of the influence of delay (which is usually negative),

what leads to difficult control problems. On the other hand, introducing or increasing a delay to the system is very simple - it can be implemented with appropriate buffers. That is why such controller can be easily applied. This type of control structure is presented in Fig. 2.

Equation of the closed loop system is given by

$$\ddot{x}(t) + \mathbf{A}x(t) - \mathbf{B}K\mathbf{C}x(t-h) = 0 \quad (17)$$

or equivalently

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & \mathbf{I}_{n \times n} \\ -\mathbf{A} & 0_{n \times n} \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} 0_{n \times 1} \\ \mathbf{B} \end{bmatrix} K [\mathbf{C} \quad 0_{1 \times n}] \begin{bmatrix} x(t-h) \\ \dot{x}(t-h) \end{bmatrix}. \quad (18)$$

In order to analyse the behaviour of the closed loop system, we need to discuss the stability of time delay systems.

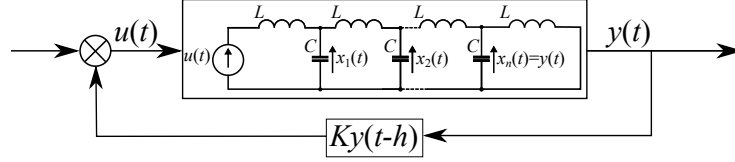


Figure 2. The control structure considered

4. Stability of time delay systems

Let us consider a class of dynamical systems generated by the following differential equation (see, for example, Klamka, 1990, p. 166):

$$\dot{x}(t) = \mathbf{A}_0 x(t) + \mathbf{A}_1 x(t-h) + \mathbf{B}u(t) \quad (19)$$

with $t \geq 0$, $h > 0$, and initial condition

$$x_0 \in M^2([-h, 0]; \mathbb{R}^n) = \mathbb{R}^n \times L^2([-h, 0]; \mathbb{R}^n) \quad (20)$$

with $\mathbf{A}_0, \mathbf{A}_1 \in \mathbb{R}^{n \times n}$, $u \in L^2_{loc}([0, \infty), \mathbb{R}^m)$, $B \in \mathbb{R}^{n \times m}$.

Let us denote the solution of (19) at $t \geq 0$ with initial condition x_0 and control u by $x(t; x_0, u)$. This solution exists and is unique. It can be obtained, for example, with a step method. This solution can also be also interpreted as a function of time, for which values are elements of $M^2([-h, 0]; \mathbb{R}^n)$. M^2 is a Hilbert space with a scalar product

$$(\phi, \psi)_{M^2} = \phi(0)^\top \psi(0) + \int_{-h}^0 \phi(s)^\top \psi(s) ds. \quad (21)$$

Also

$$\begin{aligned}\tilde{x}(t) &= \begin{bmatrix} x(t; x_0, u) \\ x_t(\bullet; x_0, u) \end{bmatrix} \in M^2([-h, 0]; \mathbb{R}^n) \\ x_t(\bullet; x_0, u) &= x(t + \tau; x_0, u) \\ \tau &\in [-h, 0].\end{aligned}\tag{22}$$

The state of the dynamical system generated by equation (19) is a solution of an abstract state equation in Hilbert space $M^2([-h, 0]; \mathbb{R}^n)$ given by (see Klamka, 1990)

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \quad t \geq 0\tag{23}$$

where

$$\tilde{A} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} \mathbf{A}_0 v(0) + \mathbf{A}_1 v(-h) \\ \dot{v}(\bullet) \end{bmatrix}\tag{24}$$

$$\tilde{B} = \begin{bmatrix} \mathbf{B}u \\ 0 \end{bmatrix}\tag{25}$$

$$\tilde{B} \in L(\mathbb{R}^m; M^2([-h, 0]; \mathbb{R}^n))\tag{26}$$

$$\begin{bmatrix} w \\ v \end{bmatrix} \in D(\tilde{A})\tag{27}$$

$$D(\tilde{A}) = \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \in M^2([-h, 0]; \mathbb{R}^n) : v, \dot{v} \in L^2([-h, 0]; \mathbb{R}^n), v(0) = w \right\}\tag{28}$$

$$\overline{D(\tilde{A})} = M^2([-h, 0]; \mathbb{R}^n).\tag{29}$$

The operator \tilde{A} given by (24) is an infinitesimal generator of C^0 semigroup \tilde{S} in the $M^2([-h, 0]; \mathbb{R}^n)$ space (see Delfour, 1980, and also Mitkowski, 1991b, pp. 178, 182, 243 and 247). The solution of (23) takes the form

$$\tilde{x}(t) = \tilde{S}(t)\tilde{x}(0) + \int_0^t \tilde{S}(t - \tau)\tilde{B}u(\tau)d\tau.\tag{30}$$

Generator \tilde{A} has only a discrete, numerable (sometimes finite) spectrum (see Delfour and Manitius, 1980a,b; Manitius, 1980), consisting of the roots s_i of the equation

$$\begin{aligned}W(s) &= 0 \\ W(s) &= \det[s\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_1 e^{-sh}].\end{aligned}\tag{31}$$

Equation (31) is called characteristic equation, and function $\det[s\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_1 e^{-sh}]$ is called characteristic quasi-polynomial. If roots s_i of characteristic equation fulfil

$$\operatorname{Re} s_i < 0, \quad \forall i\tag{32}$$

then system (23) is exponentially stable. If so, the solutions of the equation (19) are also exponentially stable (see Elsgolts and Norkin, 1973, pp. 119 and 123, and also Pandolfi, 1975; Triggiani, 1975). For other important properties see Górecki (1971), Górecki, Fuksa, Grabowski and Korytowski (1989).

This analysis is also valid for the system (10) with feedback (16), because it is equivalent to the system (19) with matrices

$$\mathbf{A}_0 = \begin{bmatrix} 0_{n \times n} & \mathbf{I}_{n \times n} \\ -\mathbf{A} & 0_{n \times n} \end{bmatrix} \quad \mathbf{A}_1 = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} \\ \mathbf{BKC} & 0_{n \times n} \end{bmatrix} \quad (33)$$

with \mathbf{A} , \mathbf{B} , \mathbf{C} given by (12).

5. Application to the considered system

For the remainder of this paper we will assume that $a = c = 1$ and $b = -2$ (see (2)). This will cause $\mathbf{E} = -LCA$. With the help of Lemma 1 we will prove the following

THEOREM 3 *Characteristic quasi-polynomial $W(s)$ of the closed loop system*

$$\ddot{x}(t) + \mathbf{A}x(t) = \mathbf{B}u(t) \quad (34)$$

$$y(t) = \mathbf{C}x(t) \quad (35)$$

$$u(t) = Ky(t-h) \quad (36)$$

with matrices \mathbf{A} , \mathbf{B} and \mathbf{C} given by (12) takes the form

$$W(s) = J_n(LCs^2) - Ke^{-sh}. \quad (37)$$

Proof. We will prove this theorem through the analysis of the transfer function of system (10). This system is a single output single input system, and so its transfer function has the form

$$G(s) = \frac{l(s)}{m(s)}. \quad (38)$$

Let us assume that initial conditions for system (10) are equal to zero, then application of Laplace transform to its equations results in

$$s^2X(s) + \mathbf{A}X(s) = \mathbf{B}U(s) \quad (39)$$

$$Y(s) = \mathbf{C}X(s). \quad (40)$$

In consequence

$$X(s) = [s^2\mathbf{I} + \mathbf{A}]^{-1}\mathbf{B}U(s) \quad (41)$$

$$Y(s) = \mathbf{C}[s^2\mathbf{I} + \mathbf{A}]^{-1}\mathbf{B}U(s) = G(s)U(s). \quad (42)$$

Let us denote e_i as the i -th unit vector (a column vector, of which all elements are equal to zero except for the i -th one). One can easily see that

$$\mathbf{C} = e_n^\top, \quad \mathbf{B} = \frac{1}{LC} e_1. \quad (43)$$

We can now see that

$$G(s) = \frac{1}{\det[s^2\mathbf{I} + \mathbf{A}]} \cdot \frac{1}{LC} \cdot e_n^\top \text{Adj}[s^2\mathbf{I} + \mathbf{A}] e_1. \quad (44)$$

Let us analyse the expression $\det[s^2\mathbf{I} + \mathbf{A}]$. If we introduce an auxiliary variable $z = LCs^2$, we get

$$\begin{aligned} \det[s^2\mathbf{I} + \mathbf{A}] &= \det \left[\frac{z^2}{LC} \mathbf{I} + \mathbf{A} \right] = \det \left[\frac{1}{LC} (z\mathbf{I} + LC\mathbf{A}) \right] = \\ &= \det \left[\frac{1}{LC} (z\mathbf{I} - \mathbf{E}) \right] = \left(\frac{1}{LC} \right)^n \det[z\mathbf{I} - \mathbf{E}] = \\ &= \left(\frac{1}{LC} \right)^n J_n(z) = \left(\frac{1}{LC} \right)^n J_n(LCs^2). \end{aligned}$$

It should be noted that for matrix $\mathbf{F} = [f_{ij}]_{n \times n}$ we have

$$e_n^\top \mathbf{F} e_1 = f_{1n}. \quad (45)$$

In that case and because \mathbf{A} is symmetric

$$e_n^\top \text{Adj}[s^2\mathbf{I} + \mathbf{A}] e_1 = \mathcal{A}(1, n) \quad (46)$$

where $\mathcal{A}(1, n)$ denotes the $(1, n)$ cofactor of matrix $[s^2\mathbf{I} + \mathbf{A}]$. From the definition of cofactor we get

$$\begin{aligned} \mathcal{A}(1, n) &= (-1)^{1+n} \det \begin{bmatrix} -\frac{1}{LC} & \frac{s^2+2}{LC} & -\frac{1}{LC} & \cdots & 0 \\ 0 & -\frac{1}{LC} & \frac{s^2+2}{LC} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -\frac{1}{LC} & \frac{s^2+2}{LC} \\ 0 & \cdots & 0 & 0 & -\frac{1}{LC} \end{bmatrix} = \\ &= (-1)^{1+n} \left(-\frac{1}{LC} \right)^{n-1} = \left(\frac{1}{LC} \right)^{n-1}. \end{aligned} \quad (47)$$

The transfer function (44), with the above results, takes the form

$$G(s) = \frac{\frac{1}{LC} \left(\frac{1}{LC} \right)^{n-1}}{\left(\frac{1}{LC} \right)^n J_n(LCs^2)} = \frac{1}{J_n(LCs^2)}. \quad (48)$$

It is well known from rules of block schematic transformations, that for control systems, such as shown in Fig. 2, closed loop transfer function takes the form

$$G_z(s) = \frac{G(s)}{1 - G(s)R(s)} \quad (49)$$

where $R(s)$ is the transfer function of the controller. In the considered case

$$R(s) = Ke^{-sh}. \quad (50)$$

Closed loop transfer function becomes

$$G_z(s) = \frac{1}{\frac{J_n(LCs^2)}{Ke^{-sh}} - \frac{1}{J_n(LCs^2)}} = \frac{1}{J_n(LCs^2) - Ke^{-sh}} \quad (51)$$

and denominator of (51) is the desired characteristic quasi-polynomial. ■

6. The method of D-partitions

In order to verify the stability of system (10) with feedback (16) we have applied the method of D-partitions (see Esgolts and Norkin, 1973, p. 132), which showed to be very useful in the considered problem. D-partitions is a method of determining regions, in one or two parameter space, for which a certain number of roots of polynomial (or in our case quasi-polynomial) have positive real part. In stability analysis the goal is to find the areas, where quasi-polynomial has zeros whose real parts are negative (we will call them regions of stability).

The main idea of the method is that on the boundary of regions of stability quasi-polynomial roots have to “go through” the imaginary axis. In that case, finding all the curves in parameter space, for which quasi-polynomial has at least one zero on imaginary axis gives us boundaries of all stability regions (and also many other regions). To find the region of stability one has to check stability for any point inside the given area determined by the curves. If this point corresponds to a stable closed loop system, then this area is a region of stability.

We will now apply the method of D-partitions to characteristic quasi-polynomial of the closed loop system (10), (11), (16) obtained from Theorem 3.

Analysis takes two steps, first we check for which parameters (K, h) the system has the root $s = 0$

$$W(s)\Big|_{s=0} = 0, \quad W(s) = J_n(LCs^2) - Ke^{-sh}. \quad (52)$$

Using formula (3), from Corollary 2 we get the equation of the first curve

$$K = 1 + n. \quad (53)$$

To verify other zeros on the imaginary axis we have to analyse the following equation

$$W(s)\Big|_{s=j\beta} = 0, \quad j = \sqrt{-1}, \quad \beta > 0. \quad (54)$$

We limit ourselves to only positive β because complex roots of quasi-polynomial with real coefficients are present only as conjugate pairs. After substitution we get

$$0 = J_n(LC(j\beta)^2) - Ke^{-j\beta h} \quad (55)$$

$$0 = J_n(-LC\beta^2) - K(\cos \beta h - j \sin \beta h). \quad (56)$$

Equation (56) corresponds to the following system of equations

$$0 = K \sin \beta h \quad (57)$$

$$0 = J_n(-LC\beta^2) - K \cos \beta h. \quad (58)$$

From (57) another curve can be easily determined

$$K = 0. \quad (59)$$

It is especially interesting, because it corresponds to a situation, when delay vanishes from the system, and roots of quasi-polynomial become the roots of polynomial of the open loop system (10). From (57) we can also obtain the dependence of β on h , precisely

$$\beta = \frac{p\pi}{h}, \quad p = 1, 2, 3, 4, \dots \quad (60)$$

Substitution into (58) results in a numerable set of curves given by

$$K = (-1)^p J_n \left(-\frac{LCp^2\pi^2}{h^2} \right) \quad (61)$$

The set of curves given by (53), (59) and (61) gives us complete information regarding division of the parameter space $K \times h$ into the potential regions of stability. What is interesting and at the same time practically important, is that in order to analyse the bounded subset of parameter space (i.e. $\{K \in [K_{min}, K_{max}], h \in [h_{min}, h_{max}]\}$) only a finite number of curves is required. Also, it is clearly visible that analysis of gain K as a function of h in determination of stability regions is fully justified, as it comes naturally from curve equations (53), (59) and (61).

7. Examples

In this section we will analyse the stability regions of the closed loop system (10), (11), (16) for $n = 1, 2, 3$. It corresponds to the ladder network of Fig. 1 with one, two or three loops. First, in Figs. 3, 4, 5 we present a set of curves for $h \in (0, 15]$. We have plotted curves (53), (59) and curves (61) for $p = 1, 2, \dots, 9$. There is no loss of information caused by this limitation, because curves for $p > 9$ do not cross this area. This restriction was made because this area contains a “manageable” number of curve intersections, for manual determination of candidate regions. For greater delays the number of intersections rises, which is already visible for $n = 3$ and $h > 10$.

Curves on the plots were not marked, but they are easily distinguishable using appropriate formulas. It can be easily seen that curves (53) and (59) are actually parallel lines and curves (61) start from the left side of the axis, alternating signs - for even n the initial curve is negative for h close to zero and it is reversed for odd n .

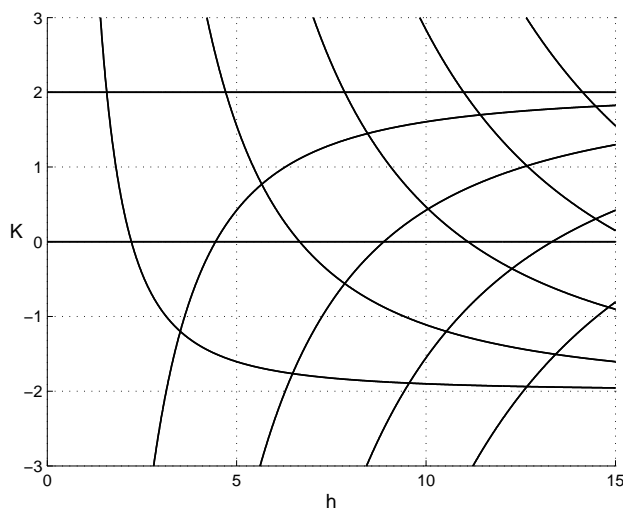
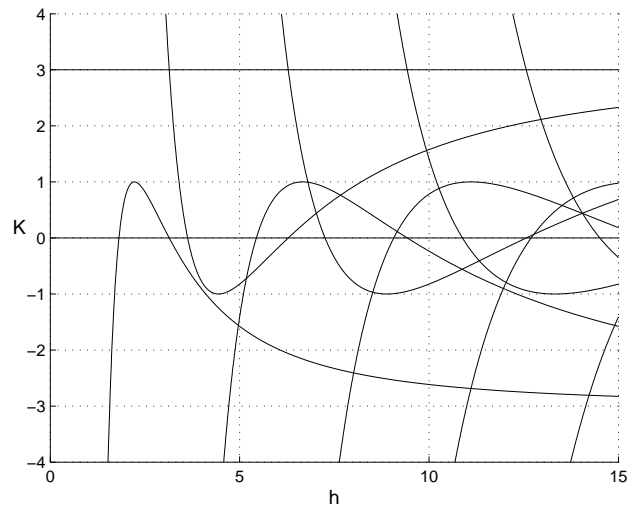
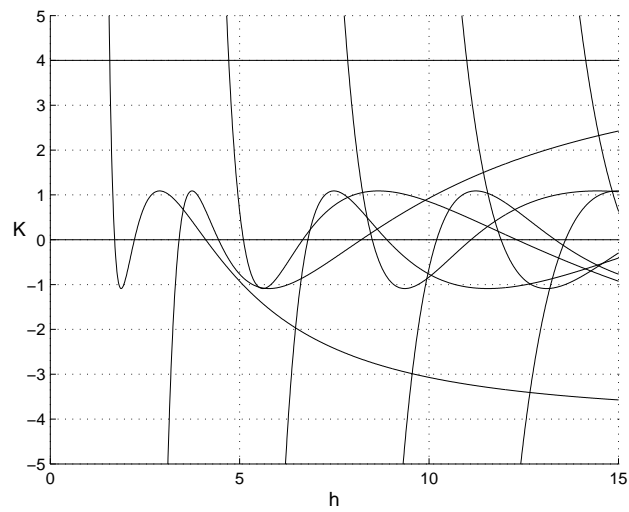
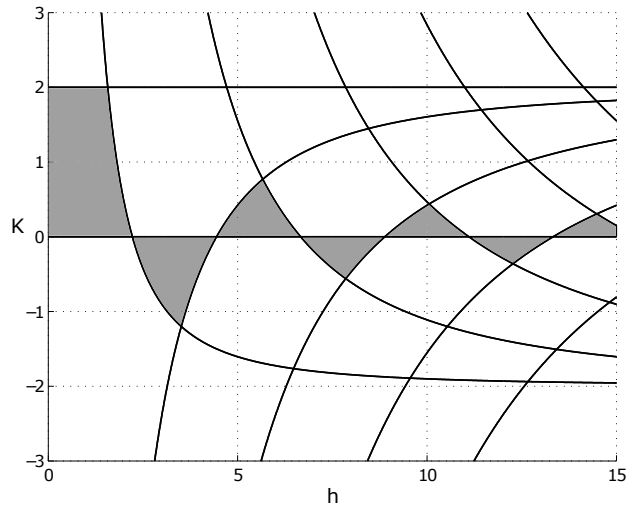
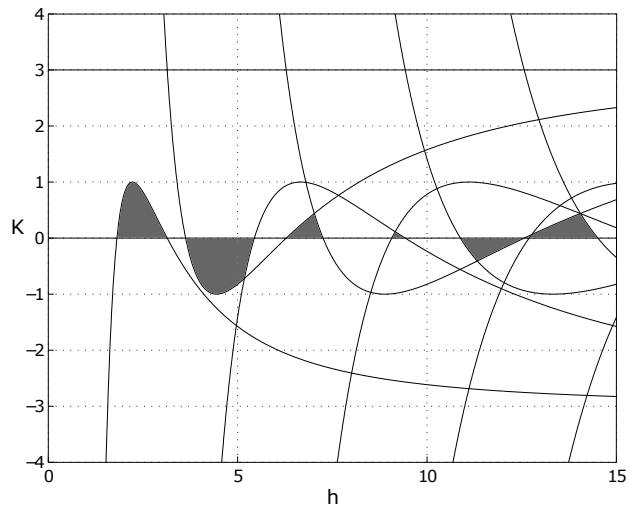


Figure 3. Curves of D-partitions for $n = 1$

In Figs. 6, 7 and 9 we have marked the stability regions on the parameter surface divided with D-partitions curves. As it can be seen, these areas are irregular, and especially for $n = 1$ the sign of feedback alternates. Stability regions for $n = 2$ are more irregular (and this trend continues with rising n , see Baranowski et al., 2009) and interesting structural properties can be observed. In Fig. 8 we can see a close-up of a stability region for $n = 2$ for $h \in [12.5, 14.5]$.

As it can be seen, one of the D-partitions curves goes through the stability

Figure 4. Curves of D-partitions for $n = 2$ Figure 5. Curves of D-partitions for $n = 3$

Figure 6. Stability regions for $n = 1$ Figure 7. Stability regions for $n = 2$

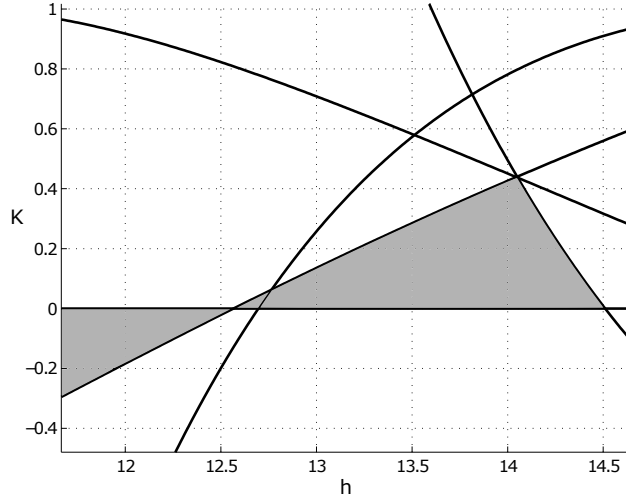


Figure 8. Close up of a stability region for $n = 2$ – the D-partitions curve splits the stability region into two

region. It means that in this region roots of quasi-polynomial have negative parts, except for points located on the splitting curve. Those points correspond to at least one conjugate pair of imaginary roots.

What is especially interesting, is that these imaginary roots are strongly structurally unstable, even numerical precision up to 6 digits does not allow observing undamped oscillations.

This increases for $n = 3$ as regions are more irregular and “split” regions also occur. This can be observed in Fig. 10. It should be also noted that for $n = 3$ we have limited h to the interval $[0, 10]$. It was caused by the desire to keep the figures legible as no particularly interesting phenomena occurred for larger h . Moreover, as it can be observed, the minimal delay required for stabilisation increases.

Probably interesting irregularities and “splitting” of regions are more common for larger n and h , but this is only a speculation.

8. Additional remarks

In previous works regarding applications of (16) to oscillatory systems two approaches were dominant. The first one, based on matrix pencils was considered in Abdallah, Dorato, Benitez-Read and Byrne (1993), Niculescu and Abdallah (2000), where some general results were derived. The other approach used Nyquist stability criterion and its results were shown in Mitkowski and Skruch

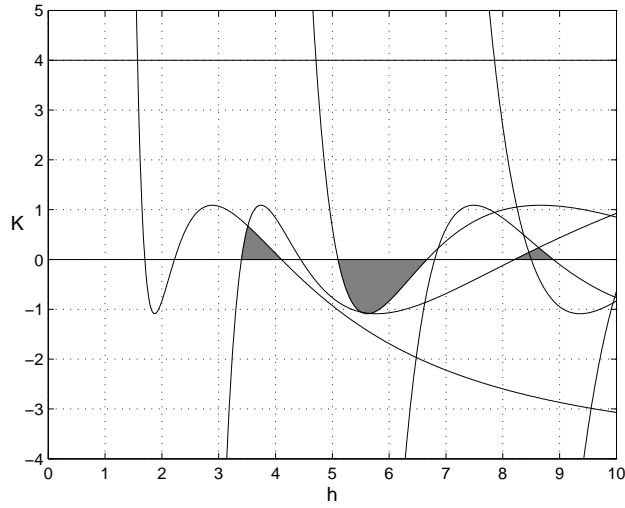


Figure 9. Stability regions for $n = 3$

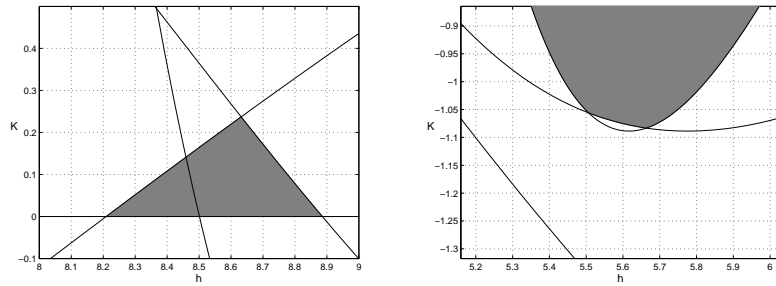


Figure 10. Close up of stability region for $n = 3$ – left - D-partitions curve splits stability region into two, right - strong irregularity of stability region

(2009). As it was shown in cited works, for simple cases an analytical solution could be obtained. Both of these directions were characterized by two aspects:

- only positive feedbacks were considered ($K > 0$),
- the focus was on finding such h for which given K will cause the closed loop system to be asymptotically stable.

Authors' previous works (Baranowski and Mitkowski, 2009; Baranowski et al., 2009) used a different approach when addressing these two weaknesses. A finite dimensional Padé approximation of controller (16) was used. The transfer function

$$R(s) = Ke^{-sh} \quad (62)$$

was approximated with

$$R(s) \approx K \frac{Q_q(-sh)}{Q_q(sh)} \quad (63)$$

where Q_q is a polynomial of arbitrarily chosen q -th order given by

$$Q_q(sh) = \int_0^\infty t^q (t + sh)^q e^{-t} dt = \sum_{j=0}^q \frac{(2q-j)!q!}{j!(q-j)!} (sh)^j = \sum_{j=0}^q \frac{(q+j)!q!}{j!(q-j)!} (sh)^j.$$

also recurrence formulas for these polynomials are possible. Stability of the finite dimensional closed loop system was verified with the application of optimisation algorithm based on constraints automatically generated from Hurwitz criterion (for details see Baranowski et al., 2009).

Padé approximation has many beneficial properties, most notable of which are:

- $Q_q(\lambda)$ has roots in left open complex half plane for any q ,
- frequency characteristics of system in the form (62) can be approximated with arbitrary precision in the sense of $L^\infty(\mathbb{R})$ norm.

Extensive analysis of the approach has, however, shown two substantial flaws of Padé approximation:

- Polynomials Q_q become ill-conditioned for all $q > 10$, and become numerically unusable for $q > 40$ (these bounds do not come from a precise analysis, but are presented in order to give the reader a general concept)
- Spectrum of time delayed system (roots of quasi-polynomial) for large h (for $LC = 1$ it was $h > 10$) greatly differs from the roots of Padé approximation, which makes stability analysis unreliable.

This motivated us to find a method of determining the stability regions free of all the shortcomings of previous methods. It was also desirable to find a

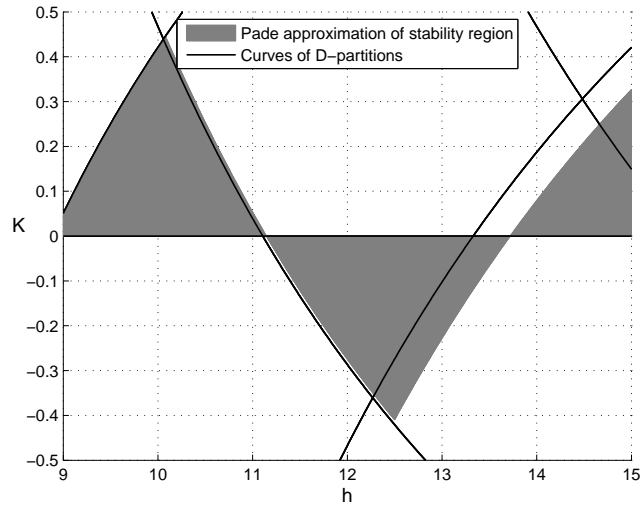


Figure 11. Padé approximation vs. D-partitions curves - $n = 1$

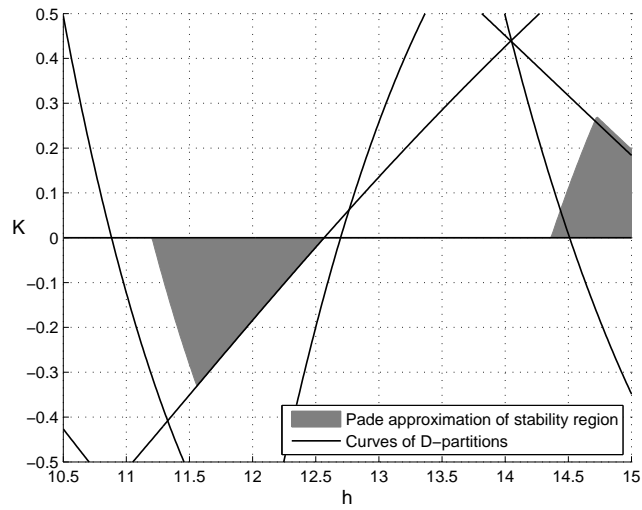


Figure 12. Padé approximation vs. D-partitions curves - $n = 2$

method based on analytical computations in order to remove the possibilities of numerical errors. The method of D-partitions described above is such a method.

In Figs. 11 and 12 we compare results from D-partitions with those obtained in earlier works via Padé approximation. Our comparison was limited only to the part of stability regions for large h . As mentioned before, Padé approximation gives poor results for large delays and this can be observed here. The approximate regions of stability obtained for lower h are not presented, because they were very close to the exact ones, so we focused on the part where problems were present.

For $n = 1$, where the structure of stability regions is relatively simple, we observe only a slight shift in the ending of the region (see Fig. 11). This shift extends for higher h . This error is a minor one because the character of the regions was preserved.

For $n = 2$ (Fig. 12) the differences between approximated and exact regions are more substantial. The beginning of stability region for $h \in [10.8, 12.6]$ was detected incorrectly, and information regarding possibly bigger, stable gains was lost. Much more significant failure is visible for $h \in [12.6, 14.5]$. This is the location of “split” stability region illustrated in Fig. 8. As it can be seen, the first part of the region was completely ignored, and from the second part only a little part for $h \in [14.4, 14.5]$ was found stable, but also not correctly. Moreover, an additional false stability region was generated, for parameters of which the system is unstable.

For $n = 3$ the approximation based approach fails completely and was not presented in the paper.

9. Conclusions and future work

In this paper we have presented a method for stability analysis of time delay feedback control of LC ladder network. An analytical formula for characteristic quasi-polynomial of closed loop control system was determined for any given n . The method of D-partitions was used to determine the stability regions in parameter space. Curves separating the region candidates were determined analytically also for any n . Obtained results were compared with those known from literature and those of authors' earlier works. Main benefits of the method are direct analytical results, allowing for avoiding numerical problems and no restrictions caused by higher order LC ladder.

It should be also noted that analysis of similar LC ladder systems is important, because of possible applications in approximation of infinite-dimensional systems (see, for example, Mitkowski, 1976). This class is especially interesting when using spatial discretisations of hyperbolic partial differential equations.

For example a lossless transmission cable is described by the following equations

$$\begin{aligned} LC \frac{\partial^2 x(t, z)}{\partial t^2} &= \frac{\partial^2 x(t, z)}{\partial z^2} \\ x(t, 0) &= u(t) \\ x(t, l) &= 0 \\ t &\geq 0, \quad 0 \leq z \leq l. \end{aligned}$$

After applying appropriate difference approximation

$$\frac{\partial^2 x(t, z)}{\partial z^2} \approx \frac{1}{\Delta} \left(\frac{x(t, z + \Delta) - 2x(t, z) + x(t, z - \Delta)}{\Delta} \right) \quad (64)$$

where $\Delta = l/n$ and $z = (2k - 1)\Delta/2$ for $k = 1, 2, \dots, n$ we get system (10) with the following matrices

$$\begin{aligned} \mathbf{A} &= \frac{n^2}{LC} \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}_{n \times n} \\ \mathbf{B} &= \frac{n^2}{LC} [1 \ 0 \ 0 \ \dots \ 0]_{1 \times n}^T. \end{aligned} \quad (65)$$

It should be noted, though, that approximations of this kind should be performed carefully, because in hyperbolic systems every mode carries the same amount of energy, so if system is undamped, approximation might be inadequate.

There are also areas of further work regarding the stability analysis. An open problem is automatising the algorithm for determining the stability regions. Currently region candidates are chosen manually and verified by the user. A promising approach is the application of graph theory. Intersections of D-partitions curves can be interpreted as vertexes, and appropriate sections of curves as edges. Intersections cannot be determined analytically, however computing them is equivalent to finding roots of certain polynomials. Those roots can be obtained by many numerically reliable procedures, available in such packages as MapleTM and MATLABTM. The problem of finding stability region candidates could be then solved as a problem of finding cycles in a graph. Having the stability region candidates allows for checking stability of a region through stability of one point in the region. It can be performed in many ways,

for example via Nyquist criterion (see, for example, Mitkowski, 1991b, p. 99) or even through direct simulation.

Another interesting aspect of the problem are the asymptotic properties of stability regions with $n \rightarrow \infty$. When analysing spatial discretisation of hyperbolic system it is very interesting, whether the results can be used for stabilisation of an actual infinite-dimensional system.

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