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Dirichlet control problems in smooth and nonsmooth convex plain domains* †

by

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Abstract: In this paper we collect some results about boundary Dirichlet control problems governed by linear partial differential equations. Some differences are found between problems posed on polygonal domains or smooth domains. In polygonal domains some difficulties arise in the corners, where the optimal control is forced to take a value which is independent of the data of the problem. The use of some Sobolev norm of the control in the cost functional, as suggested in the specialized literature as an alternative to the L^2 norm, allows to avoid this strange behavior. Here, we propose a new method to avoid this undesirable behavior of the optimal control, consisting in considering a discrete perturbation of the cost functional by using a finite number of controls concentrated around the corners. In curved domains, the numerical approximation of the problem requires the approximation of the domain Ω typically by a polygonal domain Ω_h, this introduces some difficulties in comparing the continuous and the discrete controls because of their definition on different domains Γ and Γ_h, respectively. We complete the existing recent analysis of these problems by proving the error estimates for a full discretization of the control problem. Finally, some numerical results are provided to compare the different alternatives and to confirm the theoretical predictions.

Keywords: optimal control, boundary control, Dirichlet control.

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1. Introduction

In this paper we will study the control problem

$$
\text{(P)}\left\{\begin{array}{l}
\min J(u)=\dfrac{1}{2}\displaystyle\int_\Omega (y_u(x)-y_\Omega(x))^2+\dfrac{N}{2}\int_\Gamma u^2(x)\,d\sigma(x)\\
\text{subject to}\ \ (y_u,u)\in (L^\infty(\Omega)\cap H^{1/2}(\Omega))\times L^\infty(\Gamma),\\
u\in U_{ad}=\{u\in L^\infty(\Gamma):\ \alpha\le u(x)\le\beta\quad \text{for a.e.}\quad x\in\Gamma\},
\end{array}\right.
$$

where y_u is related to u by the equation

$$
\left\{\begin{array}{rcll}
-\Delta y &=& f & \text{in }\Omega,\\
y &=& u & \text{on }\Gamma,
\end{array}\right. \tag{1.1}
$$

the domain $\Omega\subset\mathbb{R}^2$ is bounded and convex, Γ is its boundary, $\alpha<\beta$ and $N>0$ are real constants and $f, y_\Omega:\Omega\to\mathbb{R}$ are measurable functions satisfying appropriate properties that will be described later.

In the last few years, several works have been devoted to the study of finite dimensional approximations of (P), the convergence of the solutions of the approximated problems to solutions of (P) and the proof of error estimates of these approximations in terms of some discretization parameter h.

The treatment of this topic is different when the boundary Γ is a polygonal line or a smooth curve.

The main results for polygonal domains were introduced in Casas and Raymond (2006) for problems governed by elliptic semilinear equations. A full discretization of (P) is performed: both the state and the control are discretized by using continuous piecewise linear functions on a quasiuniform family of triangulations of Ω. The state equation is replaced by its finite element approximation, and the boundary conditions are imposed on every boundary node x_j of the triangulation:

$$
y_h(x_j)=u_h(x_j)\ \forall\, x_j \text{ boundary node.} \tag{1.2}
$$

Theorem 7.1 in Casas and Raymond (2006) states that for local solutions $\bar u$ of (P) satisfying second order sufficient conditions, there is a sequence $\{\bar u_h\}$ of local solutions of the approximated problems (P_h) such that the following estimate holds:

$$
\|\bar u_h-\bar u\|_{L^2(\Gamma)}\le Ch^{1-1/p}. \tag{1.3}
$$

Here $p>2$ depends on ω, the biggest angle of Γ, and on the data of the problem (see Theorem 3.1). May, Rannacher and Vexler (2008) consider a linear quadratic problem without bound constraints. They provide error estimates for the control in the norm of $H^{-1/2}(\Gamma)$ and the state in $H^{-1}(\Omega)$ and are able to improve the error estimate for the state in the norm of $L^2(\Omega)$.

One of the characteristics of the polygonal case is that on the vertices $\{x_j\}$ of Γ, always

$$\bar{u}(x_j) = \mathrm{Proj}_{[\alpha,\beta]}(0), \tag{1.4}$$

independently of the rest of the data of (P). This well known property (see, for instance, Casas, Günther and Mateos (2011) for an explanation) causes a loss of efficiency of the optimal controls when we want to approximate a desired state y_Ω by taking the regularization parameter $N > 0$ small.

Some alternatives to the formulation of (P) are discussed in the literature (see, e.g., Gunzburger, Hou and Svobodny, 1991, or Kunisch and Vexler, 2007), which are mainly concerned with the change of the regularizing term $\|u\|^2_{L^2(\Gamma)}$ by some combination of this and $|u|^2_{H^{1/2}(\Gamma)}$ or $|u|^2_{H^1(\Gamma)}$. In Vexler (2007) the use of finite dimensional controls is proposed.

We propose an alternative formulation of (P) by adding some discrete term. This will avoid the use of gradient norms on the boundary and the undesirable property (1.4). This is done in Section 3.

The case of smooth curved domains has been studied in two papers. In both papers, the domain Ω is approximated by a sequence of domains Ω_h with a polygonal boundary Γ_h. To compare the controls defined on Γ_h with those defined on Γ a natural one-to-one mapping $g_h : \Gamma_h \to \Gamma$ is used. Casas and Sokolowski (2010) study the continuous problem posed in Ω_h. Let us denote it by (P^h). In Theorem 9.1, they establish that for local solutions $\bar{u}$ of (P) satisfying second order sufficient conditions, there is a sequence $\{\hat{u}^h\}$ of local solutions of the approximated problems (P^h) such that the following estimate holds:

$$\|\hat{u}^h \circ g_h^{-1} - \bar{u}\|_{L^2(\Gamma)} \leq Ch. \tag{1.5}$$

If we solve each of the continuous problems posed in Ω_h by a full discretization, as proposed in Casas and Raymond (2006), and we denote by $\bar{u}_h$ the solution of the discrete approximation, a direct application of (1.3) and (1.5) leads to an estimate of the form

$$\|\bar{u}_h \circ g_h^{-1} - \bar{u}\|_{L^2(\Gamma)} \leq Ch^{1/2} \tag{1.6}$$

since the exponent $p_h > 2$ depending on the biggest angle ω_h of Γ_h, tends to 2 as $h \to 0$ (see Section 3).

Deckelnick, Günther and Hinze (2009) considered the numerical approximation of (P). In this reference a semidiscretization of the problem is studied on each Ω_h, where the state is discretized by continuous piecewise linear functions on a triangulation $\mathcal{T}_h$ of Ω_h which is constructed in such a way that $\{\mathcal{T}_h\}_{h>0}$ can be seen as a quasiuniform family of triangulations of the original curved domain Ω. The control is not discretized and the state equation is solved using its finite element approximation imposing the boundary conditions in the variational way

$$(y_h, v_h)_{L^2(\Gamma)} = (u, v_h)_{L^2(\Gamma)} \ \forall v_h \in U_h, \tag{1.7}$$

where U_h is the space of continuous piecewise linear functions on the boundary of Ω_h (see Section 2). Although eventually the solution $\tilde{u}_h$ of the semidiscrete optimal control problem is a continuous piecewise linear function, in general it is not the trace of its related discrete state due to the presence of the bound constraints. The order of convergence obtained in a first step (Deckelnick, Günther and Hinze, 2009, Theorem 4.1) is $O(h\sqrt{|\log h|})$, valid even for three-dimensional domains. Nevertheless, it is shown in Deckelnick, Günther and Hinze (2009), Theorem 5.4, that if the triangulation is piecewise $O(h^2)$ irregular, then

$$\|\tilde{u}_h \circ g_h^{-1} - \bar{u}\|_{L^2(\Gamma)} \leq Ch^{3/2}. \tag{1.8}$$

This estimate may suggest that (1.5) is not optimal, since it seems rather paradoxical that the continuous approximation leads to a worse estimate than the semidiscrete one. Nevertheless, Casas, Günther and Mateos (2011) provide an example that shows that indeed (1.5) is optimal.

In Section 2 we prove the error estimates for the full discretization of the problem and obtain the same order of convergence as for the semidiscrete case.

We will develop all the results for convex two-dimensional domains. Let us comment about the results in some other domains. In the case of smooth domains, the estimate $O(h\sqrt{|\log h|})$ was proved in Deckelnick, Günther and Hinze (2009) for non-convex two- and three-dimensional domains. The estimate (1.8) was proved for convex and plane domains, we think that it can be extended for non-convex domains, but we do not know if it is still valid for three-dimensional domains. The results proved in Casas and Sokolowski (2010) can be extended to non-convex domains, but the three-dimensional case is open for the moment. The reader is referred to the paper by Apel, Pfefferer and Rösch (2011) for the study of a Neumann boundary control problem for non-convex polygonal or polyhedral domains. The authors use weighted Sobolev norms to obtain error estimates for approximations in graded meshes.

2. Smooth domains

Let Ω be a convex domain with a boundary Γ of class C^3. We will suppose that $y_\Omega \in W^{1,\bar{r}}(\Omega)$ for some $\bar{r} > 2$ and $f \in L^2(\Omega)$. For any control $u \in L^\infty(\Gamma)$, its related state $y_u \in H^{1/2}(\Omega) \cap L^\infty(\Omega)$ is the solution of (1.1) and its related adjoint state $\varphi_u \in H^2(\Omega) \cap H_0^1(\Omega)$ is the solution of

$$-\Delta\varphi_u = y_u - y_\Omega \text{ in } \Omega, \ \varphi_u = 0 \text{ on } \Gamma. \tag{2.1}$$

The derivative of the state with respect to the control in a direction $v \in L^\infty(\Gamma)$ is the function $z_v \in L^\infty(\Omega) \cap H^{1/2}(\Omega)$, solution of

$$-\Delta z_v = 0 \text{ in } \Omega, \ z_v = v \text{ on } \Gamma.$$

The derivative of the functional with respect to the control in a direction $v \in L^\infty(\Gamma)$ can be expressed as

$$J'(u)v = \int_\Omega (y_u - y_\Omega) z_v dx + N \int_\Gamma uv d\sigma(x) = \int_\Gamma (Nu - \partial_\nu \varphi_u) v d\sigma(x).$$

The following result is standard; see Deckelnick, Günther and Hinze (2009), and Casas and Raymond (2006) for details about regularity.

THEOREM 2.1 *Problem* (P) *has a unique solution* $\bar u \in C^{0,1}(\Gamma)$ *with related state* $\bar y \in H^{3/2}(\Omega)$ *and related adjoint state* $\bar\varphi = \varphi_{\bar u} \in W^{3,r}(\Omega)$ *for some* $2 < r \le \bar r$. *Moreover*

$$\int_\Omega (\bar y - y_\Omega)(y_u - \bar y) dx + N \int_\Gamma \bar u (u - \bar u) d\sigma(x) \ge 0 \quad \forall u \in U_{ad}, \tag{2.2}$$

$$\int_\Gamma (N\bar u - \partial_\nu \bar\varphi)(u - \bar u) d\sigma(x) \ge 0 \quad \forall u \in U_{ad}, \tag{2.3}$$

$$\bar u(x) = \mathrm{Proj}_{[\alpha,\beta]}\left(\frac{\partial_\nu \bar\varphi(x)}{N}\right) \quad \forall x \in \Gamma. \tag{2.4}$$

Let $\mathcal{T}_h$ be a quasi-uniform family of triangulations of Ω. For each $h > 0$ let $\Omega_h = \mathrm{int} \cup \{T_h : T_h \in \mathcal{T}_h\}$ and denote by Γ_h its boundary. Let n be the number of vertices of Γ_h that will be denoted by $\{x_j\}_{j=1}^n$, ordered counterclockwise, with $x_0 = x_n$ and $x_{n+1} = x_1$. As usual, we assume that $x_j \in \Gamma$ for all j. Consider

$$Y_h = \{y_h \in C(\bar\Omega_h) : \ y_{h|T} \in P_1(T) \ \forall T \in \mathcal{T}_h\},$$
$$U_h = \{u_h \in C(\Gamma_h) : u_{h|[x_j, x_{j+1}]} \in P_1([x_j, x_{j+1}]) \ \forall j = 1, \dots, n\}.$$

We also set $Y_{h0} = Y_h \cap H_0^1(\Omega_h)$. Let us note that U_h coincides with the space of the restrictions of functions in Y_h to Γ_h. For every $u \in L^\infty(\Gamma_h)$ we denote its projection onto U_h in the $L^2(\Gamma_h)$ sense by $\Pi_h u$, the unique element in U_h satisfying

$$(u, v_h)_{L^2(\Gamma_h)} = (\Pi_h u, v_h)_{L^2(\Gamma_h)} \ \forall v_h \in U_h$$

and its associated discrete state $y_h(u) \in Y_h$ as the unique solution of

$$\begin{cases} a_h(y_h(u), w_h) = \displaystyle\int_{\Omega_h} f(x) w_h(x)\, dx \quad \forall\, w_h \in Y_{h0}, \\ y_h(u) = \Pi_h u \text{ on } \Gamma_h, \end{cases}$$

where

$$a_h(y, w) = \int_{\Omega_h} \nabla y \cdot \nabla w \, dx \ \text{ for every } y, w \in H^1(\Omega_h).$$

Notice that for $u_h \in U_h$, $y_h(u_h) \equiv u_h$ on Γ_h.

For every $u \in C(\Gamma)$ we define its nodal interpolator $I_h u \in U_h$ as the unique element in U_h such that $I_h(u)(x_j) = u(x_j)$ for all $j = 1, \dots, n$.

We approximate (P) by the finite dimensional control problems

$$(\mathrm{P}_h)\begin{cases} \min J_h(u_h) = \dfrac{1}{2}\displaystyle\int_{\Omega_h} (y_h(u_h)(x) - y_\Omega(x))^2 dx + \dfrac{N}{2}\|u_h\|^2_{L^2(\Gamma_h)}, \\ u_h \in U_{h,ad} = \{u_h \in U_h :\ \alpha \le u_h(x) \le \beta\ \forall\, x \in \Gamma_h.\} \end{cases}$$

The derivative of the discrete state with respect to the control in a direction $v \in L^\infty(\Gamma_h)$, denoted by $z_h(v) \in Y_h$, is the unique solution of

$$\begin{cases} a_h(z_h(u), w_h) = 0\ \forall\, w_h \in Y_{h0}, \\ z_h(v) = \Pi_h v \text{ on } \Gamma_h. \end{cases}$$

The derivative of J_h with respect to the control in a direction $v \in L^\infty(\Gamma_h)$ is

$$J_h'(u)v = \int_{\Omega_h} (y_h(u) - y_\Omega) z_h(v) dx + N \int_{\Gamma_h} uv d\sigma_h(x).$$

The solution of (P_h) satisfies the following first order necessary optimality conditions.

THEOREM 2.2 *Let $\bar u_h \in U_h$ be the unique solution of* (P_h) *with related state $\bar y_h$, then*

$$\int_{\Omega_h} (\bar y_h - y_\Omega)(y_h(u_h) - \bar y_h) dx + N \int_{\Gamma_h} \bar u_h (u_h - \bar u_h) d\sigma_h(x) \ge 0\ \ \forall\, u_h \in U_{h,ad}. \quad (2.5)$$

To compare the solutions of the approximated problems to the solutions of (P), we will use $g_h : \Gamma_h \to \Gamma$, the natural one-to-one mapping that to every point $x \in \Gamma_h$ associates the one in Γ that intersects the line $\{x + \lambda\nu_h : \lambda \ge 0\}$, where ν_h is the unit exterior normal vector to Γ_h. At the vertices x_j of Γ we set $g_h(x_j) = x_j$.

The following relations will be useful for comparing integrals on Γ and on Γ_h (see Casas and Sokolowski, 2010, Eq. (4.3))

$$\begin{cases} \left|\displaystyle\int_\Gamma v(x)\, d\sigma(x) - \int_{\Gamma_h} v(g_h(x))\, d\sigma_h(x)\right| \le Ch^2\|v\|_{L^1(\Gamma)} \\ \displaystyle\int_{\Gamma_h} |v(g_h(x))|\, d\sigma_h(x) \le \int_\Gamma |v(x)|\, d\sigma(x) \end{cases} \forall\, v \in L^1(\Gamma). \quad (2.6)$$

For every $h > 0$ there exists a linear extension operator $E_h : H^1(\Omega_h) \to H^1(\mathbb{R}^2)$ such that $\|E_h y_h\|_{H^1(\mathbb{R}^2)} \le C\|y_h\|_{H^1(\Omega_h)}$ with C independent of h. The operator E_h can be built using Mikolskij's extension method (see Nečas, 1967, pp. 75–77) or Calderon's method (see Nečas, 1967, pp. 77–80). Abusing notation, we will usually write y_h instead of $E_h y_h$.

An important set on Γ is the set of "landing" or "kink" points of the optimal control on the bound constraints. It is the boundary, in the topology of Γ, of the set

$$\Gamma_S = \{x \in \Gamma : \bar{u}(x) = \alpha \text{ or } \bar{u}(x) = \beta\}.$$

This boundary will be denoted by $\partial\Gamma_S$. The estimates for $\bar{u} - \bar{u}_h \circ g_h^{-1}$ depend on the properties of the set $\partial\Gamma_S$ as well as on the properties of the triangulations $\mathcal{T}_h$. Let us indicate which are the good properties leading to better error estimates. We consider the assumptions

(A1) The number of points in $\partial\Gamma_S$, denoted by $\mathcal{N}$, is finite.

(A2) The mesh is $O(h^2)$ irregular.

For a more general assumption (A1) see Mateos and Rösch (2011), Remark 4.1.

The proof of the next theorem follows the lines of the proof given in Deckelnick, Günther and Hinze (2009) with some necessary modifications to deal with the full discretization of the control problem.

Theorem 2.3 *Let $\bar{u}$ be the solution of problem* (P) *and for all $h > 0$, let $\bar{u}_h$ be the solution of problem* (P_h),

$$\|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le Ch\sqrt{|\log h|}. \tag{2.7}$$

If, further, (A2) *is satisfied, then there exists a sequence $\{\varepsilon_h\}_{h>0}$ of positive numbers converging to zero such that*

$$\|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le \varepsilon_h h. \tag{2.8}$$

If both (A1) *and* (A2) *are satisfied, then there exists $C > 0$ such that*

$$\varepsilon_h \le Ch^{1/2}. \tag{2.9}$$

We will split the argument of the proof into several lemmas.

Lemma 2.1 *There exist a constant $C > 0$ and a sequence of positive numbers $\{\varepsilon_h\}_{h>0}$ converging to zero such that the following interpolation inequalities hold:*

$$\begin{aligned} \|\bar{u} - I_h\bar{u} \circ g_h^{-1}\|_{L^2(\Gamma)} &\le \varepsilon_h h & (2.10)\\ \|\bar{u} - I_h\bar{u} \circ g_h^{-1}\|_{L^\infty(\Gamma)} &\le Ch & (2.11)\\ \|\partial_\nu\bar{\varphi} - I_h\partial_\nu\bar{\varphi} \circ g_h^{-1}\|_{L^2(\Gamma)} &\le Ch^{3/2}. & (2.12) \end{aligned}$$

If we further suppose that (A1) *holds, then there exists $C > 0$ such that*

$$\varepsilon_h \le Ch^{1/2}. \tag{2.13}$$

Proof. Since $\bar{u} \in C^{0,1}(\Gamma)$ and $\partial_\nu \bar{\varphi} \in H^{3/2}(\Gamma)$, inequalities (2.10)–(2.12) are well known in the literature (see, e.g., Brenner and Scott, 1994). We only have to prove that (2.10) holds with ε_h satisfying (2.13). Let us denote

$$\begin{aligned} J_1 &= \{j = 1, \dots, n : \alpha < \bar{u}(x) < \beta \quad \forall x \in \widehat{x_j x_{j+1}}\}, \\ J_2 &= \{j = 1, \dots, n : \bar{u}(x) = \alpha \text{ or } \bar{u}(x) = \beta \quad \forall x \in \widehat{x_j x_{j+1}}\}, \\ J_3 &= \{1, \dots, n\} \setminus (J_1 \cup J_2), \end{aligned}$$

where $\widehat{x_j x_{j+1}}$ denotes the arch of Γ going from x_j to x_{j+1}.

If $j \in J_1$, (2.4) implies that $\bar{u}(x) = \frac{1}{N} \partial_\nu \bar{\varphi}(x)$ for all $x \in \widehat{x_j x_{j+1}}$.

If $j \in J_2$, then $\bar{u}$ is constant in $\widehat{x_j x_{j+1}}$ and therefore $\bar{u}(x) = I_h \bar{u}(x) \circ g_h^{-1}$ for all $x \in \widehat{x_j x_{j+1}}$.

From the above observations, the fact that the number of elements in J_3 is $\mathcal{N}$, $|x_{j+1} - x_j| \le h$ and using (2.11), we obtain

$$\begin{aligned} \int_\Gamma (I_h \bar{u} \circ g_h^{-1} - \bar{u})^2 d\sigma(x) &= \frac{1}{N^2} \sum_{j \in J_1} \int_{\widehat{x_j x_{j+1}}} (I_h \partial_\nu \bar{\varphi} \circ g_h^{-1} - \partial_\nu \bar{\varphi})^2 d\sigma(x) \\ &\quad + \sum_{j \in J_3} \int_{\widehat{x_j x_{j+1}}} (I_h \bar{u} \circ g_h^{-1} - \bar{u})^2 d\sigma(x) \\ &\le \frac{1}{N^2} \| I_h \partial_\nu \bar{\varphi} \circ g_h^{-1} - \partial_\nu \bar{\varphi} \|^2_{L^2(\Gamma)} + C \mathcal{N} h^3 \|\bar{u}\|^2_{C^{0,1}(\Gamma)} \\ &\le C h^3 \end{aligned}$$

∎

LEMMA 2.2 *Let $\bar{u}$ be the solution of problem* (P). *Then there exists a sequence of positive numbers $\{\varepsilon_h\}_{h>0}$ converging to zero such that*

$$\int_\Gamma (N\bar{u} - \partial_\nu \bar{\varphi})(I_h \bar{u} \circ g_h^{-1} - \bar{u}) d\sigma(x) \le \varepsilon_h h^2. \tag{2.14}$$

If we further suppose that (A1) *holds, then there exists $C > 0$ such that*

$$\varepsilon_h \le Ch. \tag{2.15}$$

Proof. Let us take J_1, J_2 and J_3 as in the proof of Lemma 2.1 and let us denote $\bar{d}(x) = N\bar{u}(x) - \partial_\nu \bar{\varphi}(x)$. Notice that $\bar{d} \in C^{0,1}(\Gamma)$. If $j \in J_3$, then there exists some $\xi_j \in \widehat{x_j x_{j+1}}$ such that $\alpha < \bar{u}(\xi_j) < \beta$ and using once again (2.4), we get $\bar{d}(\xi_j) = N\bar{u}(\xi_j) - \partial_\nu \bar{\varphi}(\xi_j) = 0$.

From the above observations, and using (2.10), we obtain (2.14) as follows:

$$\begin{aligned} &\int_\Gamma (N\bar{u} - \partial_\nu \bar{\varphi})(I_h \bar{u} \circ g_h^{-1} - \bar{u}) d\sigma(x) = \int_\Gamma \bar{d}(x)(I_h \bar{u} \circ g_h^{-1} - \bar{u}) d\sigma(x) \\ &= \sum_{j \in J_3} \int_{\widehat{x_j x_{j+1}}} (\bar{d}(x) - \bar{d}(\xi_j))(I_h \bar{u} \circ g_h^{-1} - \bar{u}) d\sigma(x) \\ &\le \sum_{j \in J_3} \int_{\widehat{x_j x_{j+1}}} |x - \xi_j| \|\bar{d}\|_{C^{0,1}(\Gamma)} (I_h \bar{u} \circ g_h^{-1} - \bar{u}) d\sigma(x) \end{aligned}$$

$$
\begin{aligned}
&\leq \quad \|\bar{d}\|_{C^{0,1}(\Gamma)} h \sum_{j\in J_3} \int_{\widehat{x_j x_{j+1}}} (I_h\bar{u}\circ g_h^{-1} - \bar{u}) d\sigma(x) \\
&\leq \quad Ch \int_\Gamma |I_h\bar{u}\circ g_h^{-1} - \bar{u}| d\sigma(x) \leq \varepsilon_h h^2.
\end{aligned}
$$

If we also use the fact that the number of elements in J_3 is $\mathcal{N}$ and (2.11), then we obtain (2.15) as follows:

$$
\begin{aligned}
&\int_\Gamma (N\bar{u} - \partial_\nu\bar{\varphi})(I_h\bar{u}\circ g_h^{-1} - \bar{u}) d\sigma(x) \\
\leq \quad & \sum_{j\in J_3} \int_{\widehat{x_j x_{j+1}}} |x-\xi_j| \|\bar{d}\|_{C^{0,1}(\Gamma)} \|I_h\bar{u}\circ g_h^{-1} - \bar{u}\|_{L^\infty(\Gamma)} d\sigma(x) \\
\leq \quad & C\mathcal{N} \|\bar{d}\|_{C^{0,1}(\Gamma)} \|\bar{u}\|_{C^{0,1}(\Gamma)} h^3 \leq Ch^3.
\end{aligned}
$$

■

Lemma 2.3 *Let $\bar{u}$ be the solution of Problem* (P) *and for all $h>0$ let $\bar{u}_h$ be the solution of problem* (P$_h$). *Then there exists $C>0$ such that*

$$
\|I_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma_h)} \leq Ch + \|\bar{u} - \bar{u}_h\circ g_h^{-1}\|_{L^2(\Gamma)} \tag{2.16}
$$

and

$$
\|I_h\bar{u}\circ g_h^{-1} - \bar{u}_h\circ g_h^{-1}\|_{L^2(\Gamma)} = Ch + \|\bar{u} - \bar{u}_h\circ g_h^{-1}\|_{L^2(\Gamma)}. \tag{2.17}
$$

Proof. Inequality (2.17) is an immediate consequence of (2.10) and (2.16) follows from (2.17) and the second inequality of (2.6). ■

Lemma 2.4 *Let $\bar{u}$ be the solution of Problem* (P) *and for all $h>0$ let $\bar{u}_h$ be the solution of problem* (P$_h$). *Then there exists a sequence of positive numbers $\{\varepsilon_h\}_{h>0}$ converging to zero such that*

$$
\begin{aligned}
\int_{\Gamma_h} \bar{u}_h(I_h\bar{u} - \bar{u}_h) d\sigma_h(x) \quad \leq \quad & \varepsilon_h^2 h^2 + \varepsilon_h h \|\bar{u}_h\circ g_h^{-1} - \bar{u}\|_{L^2(\Gamma)} \\
& + \int_\Gamma \bar{u}(I_h\bar{u}\circ g_h^{-1} - \bar{u}) d\sigma(x) \\
& + \int_\Gamma \bar{u}_h\circ g_h^{-1}(\bar{u} - \bar{u}_h\circ g_h^{-1}) d\sigma(x).
\end{aligned} \tag{2.18}
$$

If assumption (A1) *holds, then $\varepsilon_h \leq Ch^{1/2}$ for some constant $C>0$.*

Proof. Let us prove (2.18). First we write

$$\begin{aligned}\int_{\Gamma_h} \bar u_h(I_h\bar u - \bar u_h)d\sigma_h(x) &= \int_{\Gamma_h} \bar u_h(\bar u\circ g_h - \bar u_h)d\sigma_h(x)\\ &\quad+\int_{\Gamma_h} \bar u_h(I_h\bar u - \bar u\circ g_h)d\sigma_h(x)\\ &= \int_{\Gamma_h} \bar u_h(\bar u\circ g_h - \bar u_h)d\sigma_h(x)\\ &\quad+\int_{\Gamma_h} (\bar u_h - \bar u\circ g_h)(I_h\bar u - \bar u\circ g_h)d\sigma_h(x)\\ &\quad+\int_{\Gamma_h} \bar u\circ g_h(I_h\bar u - \bar u\circ g_h)d\sigma_h(x) = I+II+III.\end{aligned}$$

Using relation (2.6), Cauchy's inequality and the fact that $\{\|\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}\}_{h>0}$ is uniformly bounded, we deduce the existence of $C>0$ independent of h such that

$$\begin{aligned}I=\int_{\Gamma_h} \bar u_h(\bar u\circ g_h - \bar u_h)d\sigma_h(x) &\le \int_\Gamma \bar u_h\circ g_h^{-1}(\bar u - \bar u_h\circ g_h^{-1})d\sigma(x)\\ &\quad+Ch^2\int_\Gamma \left|\bar u_h\circ g_h^{-1}(\bar u - \bar u_h\circ g_h^{-1})\right| d\sigma(x)\\ &\le \int_\Gamma \bar u_h\circ g_h^{-1}(\bar u - \bar u_h\circ g_h^{-1})d\sigma(x)\\ &\quad+Ch^2\|\bar u - \bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}.\end{aligned}$$

Now, using the second inequality of (2.6) and (2.10) we obtain

$$\begin{aligned}II &= \int_{\Gamma_h} (\bar u_h - \bar u\circ g_h)(I_h\bar u - \bar u\circ g_h)d\sigma_h(x)\\ &\le \int_\Gamma |\bar u_h\circ g_h^{-1} - \bar u||I_h\bar u\circ g_h^{-1} - \bar u|\, d\sigma(x)\\ &\le \varepsilon_h h\|\bar u_h\circ g_h^{-1} - \bar u\|_{L^2(\Gamma)}.\end{aligned}$$

Finally, once again from (2.6) and (2.10) we get

$$\begin{aligned}III=\int_{\Gamma_h} \bar u\circ g_h(I_h\bar u - \bar u\circ g_h)d\sigma_h(x) &\le \int_\Gamma \bar u(I_h\bar u\circ g_h^{-1} - \bar u)d\sigma(x)\\ &\quad+Ch^2\int_\Gamma |\bar u(I_h\bar u\circ g_h^{-1} - \bar u)|d\sigma(x)\\ &\le \int_\Gamma \bar u(I_h\bar u\circ g_h^{-1} - \bar u)d\sigma(x) + \varepsilon_h^2 h^2\end{aligned}$$

and (2.18) follows from the above inequalities.

The last assertion follows directly from (2.13). ∎

LEMMA 2.5 *Let $\bar u$ be the solution of Problem* (P) *and for all $h > 0$ let $\bar u_h$ be the solution of problem* (P_h). *Then there exists $C > 0$*

$$\int_{\Omega_h}(\bar y_h - y_\Omega)(y_h(I_h\bar u) - \bar y_h)dx \le Ch^2\|\bar u - \bar u_h \circ g_h\|_{L^2(\Gamma)} + Ch^3 + \int_{\Omega}(\bar y_h - y_\Omega)(y_h(I_h\bar u) - \bar y_h)dx, \qquad (2.19)$$

where $y_h(I_h\bar u)$ denotes the discrete state associated to $I_h\bar u$.

Proof. Using Bramble and King (1994), (2.10), $|\Omega \setminus \Omega_h| \le Ch$ and (2.16), we get

$$\begin{aligned}\left|\int_{\Omega\setminus\Omega_h}(\bar y_h - y_\Omega)(y_h(I_h\bar u) - \bar y_h)dx\right| &\le \|\bar y_h - y_\Omega\|_{L^2(\Omega\setminus\Omega_h)}\|y_h(I_h\bar u) - \bar y_h\|_{L^2(\Omega\setminus\Omega_h)}\\ &\le C\|\bar y_h - y_\Omega\|_{L^\infty(\Omega\setminus\Omega_h)}|\Omega\setminus\Omega_h|^{1/2}\left(h\|y_h(I_h\bar u) - \bar y_h\|_{L^2(\Gamma_h)}\right.\\ &\qquad\left. + h^2\|\nabla y_h(I_h\bar u) - \nabla\bar y_h\|_{L^2(\Omega_h)}\right)\\ &\le C\left(h^2\|I_h\bar u - \bar u_h\|_{L^2(\Gamma_h)} + h^3\right) \le C\left(h^2\|\bar u - \bar u_h \circ {g_h}^{-1}\|_{L^2(\Gamma)} + h^3\right),\end{aligned} \qquad (2.20)$$

which implies (2.19). ∎

COROLLARY 2.1 *Let $\bar u$ be the solution of Problem* (P) *and for all $h > 0$ let $\bar u_h$ be the solution of problem* (P_h). *Then there exists a sequence of positive numbers $\{\varepsilon_h\}_{h>0}$ converging to zero such that*

$$\int_\Omega(\bar y_h - y_\Omega)(y_h(I_h\bar u) - \bar y_h)dx + N\int_\Gamma \bar u(I_h\bar u \circ g_h^{-1} - \bar u)d\sigma(x) + N\int_\Gamma \bar u_h \circ g_h^{-1}(\bar u - \bar u_h \circ g_h^{-1})d\sigma(x) + \varepsilon_h^2 h^2 + \varepsilon_h h\|\bar u_h \circ g_h^{-1} - \bar u\|_{L^2(\Gamma)} \ge 0. \qquad (2.21)$$

If (A1) *holds, then $\varepsilon_h \le Ch^{1/2}$ for some constant $C > 0$.*

Proof. Taking $u_h = I_h u$ in (2.5), we obtain

$$\int_{\Omega_h}(\bar y_h - y_\Omega)(y_h(I_h\bar u) - \bar y_h)dx + N\int_{\Gamma_h}\bar u_h(I_h\bar u - \bar u_h)d\sigma_h(x) \ge 0 \quad \forall u_h \in U_{h,ad}. \qquad (2.22)$$

So (2.21) follows directly from (2.22), (2.18) and (2.19). The last assertion follows directly from the corresponding one in Lemma 2.4. ∎

LEMMA 2.6 *Let $\bar u$ be the solution of Problem* (P) *and for all $h > 0$ let $\bar u_h$ be the solution of problem* (P_h). *Then there exists $C > 0$ such that*

$$\left|\int_\Gamma \partial_\nu\bar\varphi(I_h\bar u \circ g_h^{-1} - \bar u_h \circ g_h^{-1})d\sigma(x) - \int_{\Gamma_h}\partial_{\nu_h}\bar\varphi(I_h\bar u - \bar u_h)d\sigma_h(x)\right| \le C(h^3 + h^2\|\bar u - \bar u_h \circ g_h^{-1}\|_{L^2(\Gamma)}). \qquad (2.23)$$

Proof. Using (2.6) and (2.17) we obtain

$$\begin{aligned}
&\left|\int_\Gamma \partial_\nu\bar\varphi(I_h\bar u\circ g_h^{-1}-\bar u_h\circ g_h^{-1})d\sigma(x)-\int_{\Gamma_h}\partial_{\nu_h}\bar\varphi(I_h\bar u-\bar u_h)d\sigma_h(x)\right|\\
\leq\;&\left|\int_\Gamma(\partial_\nu\bar\varphi-\partial_{\nu_h}\bar\varphi\circ g_h^{-1})(I_h\bar u\circ g_h^{-1}-\bar u_h\circ g_h^{-1})d\sigma(x)\right|\\
&+C(h^3+h^2\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)})\\
\leq\;& C\|\partial_\nu\bar\varphi-\partial_{\nu_h}\bar\varphi\circ g_h^{-1}\|_{L^2(\Gamma)}(h+\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)})\\
&+C(h^3+h^2\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}).
\end{aligned}\tag{2.24}$$

Now we write

$$\begin{aligned}
\partial_\nu\bar\varphi(x)-\partial_{\nu_h}\bar\varphi(g_h^{-1}(x)) &= \nabla\bar\varphi(x)\cdot\nu(x)-\nabla\bar\varphi(g_h^{-1}(x))\cdot\nu_h(g_h^{-1}(x))\\
&= \left\{\nabla\bar\varphi(x)-\nabla\bar\varphi(g_h^{-1}(x))\right\}\cdot\nu_h(g_h^{-1}(x))\\
&\quad+\nabla\bar\varphi(x)\cdot\left\{\nu(x)-\nu_h(g_h^{-1}(x))\right\}.
\end{aligned}\tag{2.25}$$

On the one hand, using Bramble and King (1994), Eq. (2.12) and the regularity $\bar\varphi\in H^3(\Omega)$, we have that

$$\|\nabla\bar\varphi-\nabla\bar\varphi\circ g_h^{-1}\|_{L^2(\Gamma)}\leq Ch^2.\tag{2.26}$$

On the other hand, for every $x\in\Gamma$, $\{\nu(x),\tau(x)\}$, where $\tau(x)$ is the unit tangent vector to Γ at x, is an orthonormal basis of $\mathbb{R}^2$, and hence the vector $\nabla\bar\varphi(x)$ can be written as

$$\nabla\bar\varphi(x)=\alpha(x)\nu(x)+\beta(x)\tau(x).$$

Since $\bar\varphi\equiv 0$ on Γ, then the tangential component of the gradient is null ($\beta(x)=0$), and the normal component is precisely $\alpha(x)=\partial_\nu\bar\varphi(x)$. Using that both $|\nu_h(g_h^{-1}(x))|^2=|\nu(x)|^2=1$, we can write

$$\begin{aligned}
\left|\nabla\bar\varphi(x)\cdot\big(\nu(x)-\nu_h(g_h^{-1}(x))\big)\right| &= \left|\partial_\nu\bar\varphi(x)\nu(x)\cdot\big(\nu(x)-\nu_h(g_h^{-1}(x))\big)\right|\\
&= \left|-\frac{1}{2}\partial_\nu\bar\varphi(x)|\nu(x)-\nu_h(g_h^{-1}(x))|^2\right|\\
&\leq Ch^2,
\end{aligned}\tag{2.27}$$

where we have used the fact that $|\nu(x)-\nu_h(g_h^{-1}(x))|\leq Ch$ because Γ is of class C^2; see Casas and Sokolowski (2010), (4.1).

From (2.25)–(2.27) we have that

$$\|\partial_\nu\bar\varphi-\partial_{\nu_h}\bar\varphi\circ g_h^{-1}\|_{L^2(\Gamma)}\leq Ch^2$$

and (2.23) follows from this last inequality and (2.24). ∎

LEMMA 2.7 *Let $\bar{u}$ be the solution of Problem* (P) *and for all $h > 0$ let $\bar{u}_h$ be the solution of problem* (P_h). *Then for every $\mu > 0$ there exists $C_\mu > 0$ such that*

$$\int_{\Omega_h} \nabla\bar{\varphi}(\nabla y_h(I_h\bar{u}) - \nabla\bar{y}_h)dx \leq C_\mu h^2|\log h| + \mu\|\bar{u} - \bar{u}_h \circ g_h^{-1}\|^2_{L^2(\Gamma)}. \quad (2.28)$$

Suppose further that (A2) *holds. Then there exists $C > 0$ such that*

$$\int_{\Omega_h} \nabla\bar{\varphi}(\nabla y_h(I_h\bar{u}) - \nabla\bar{y}_h)dx \leq Ch^3 + Ch^{3/2}\|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^2(\Gamma)}. \quad (2.29)$$

Proof. We introduce the nodal interpolator in Ω, which we also denote $I_h : C(\bar{\Omega}) \to Y_h$. Using $I_h\bar{\varphi}$ as test function in the discrete state equation, Deckelnick, Günther and Hinze (2009), Eq. (4.10) and (2.16), we obtain

$$\begin{aligned} & \int_{\Omega_h} \nabla\bar{\varphi}(\nabla y_h(I_h\bar{u}) - \nabla\bar{y}_h)dx \\ = & \int_{\Omega_h} (\nabla\bar{\varphi} - \nabla I_h\bar{\varphi})(\nabla y_h(I_h\bar{u}) - \nabla\bar{y}_h)dx \\ \leq & \ C_\mu h^2|\log h| + \mu\|\bar{u} - \bar{u}_h \circ g_h^{-1}\|^2_{L^2(\Gamma)}. \end{aligned}$$

and (2.28) follows.

If we suppose (A2), using $I_h\bar{\varphi}$ as test function in the discrete state equation and the superconvergence property, Deckelnick, Günther and Hinze (2009), Lemma 5.2, we can write:

$$\begin{aligned} & \int_{\Omega_h} \nabla\bar{\varphi}(\nabla y_h(I_h\bar{u}) - \nabla\bar{y}_h)dx \\ = & \int_{\Omega_h} (\nabla\bar{\varphi} - \nabla I_h\bar{\varphi})(\nabla y_h(I_h\bar{u}) - \nabla\bar{y}_h)dx \\ \leq & \ C(h^2\|y_h(I_h\bar{u}) - \bar{y}_h\|_{H^1(\Omega_h)} + h^{3/2}\|I_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma_h)}.) \end{aligned} \quad (2.30)$$

Using the inequality

$$\|z_h\|_{H^1(\Omega_h)} \leq C\|z_h\|_{H^{1/2}(\Gamma_h)} \quad \forall h > 0,$$

for any $z_h \in Y_h$ such that $a(z_h, \psi_h) = 0$ for all $\psi_h \in Y_h$ and $z_h = \Pi_h u$ on Γ_h, see Bramble, Pasciak and Schatz (1986), Lemma 3.2, and the well known inverse inequality

$$\|u_h\|_{H^{1/2}(\Gamma_h)} \leq Ch^{-1/2}\|u_h\|_{L^2(\Gamma_h)} \ \forall\, u_h \in U_h,$$

we have that

$$\|y_h(I_h\bar{u}) - \bar{y}_h\|_{H^1(\Omega_h)} \leq Ch^{-1/2}\|I_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma_h)}.$$

From this inequality, (2.16), (2.17) and (2.30) we infer (2.29). ■

Proof of Theorem 2.3. Taking $u = \bar u_h \circ g_h^{-1}$ in (2.2) we get

$$\int_\Omega (\bar y - y_\Omega)(y_{\bar u_h \circ g_h^{-1}} - \bar y)dx + N\int_\Gamma \bar u(\bar u_h \circ g_h^{-1} - \bar u)d\sigma(x) \geq 0. \tag{2.31}$$

Adding up inequalities (2.31) and (2.21) we obtain

$$\begin{aligned} N\int_\Gamma (\bar u_h \circ g_h^{-1} - \bar u)^2 d\sigma(x) \quad \leq \quad & \int_\Omega (\bar y - y_\Omega)(y_{\bar u_h \circ g_h^{-1}} - \bar y)dx \\ & + \int_\Omega (\bar y_h - y_\Omega)(y_h(I_h\bar u) - \bar y_h)dx \\ & + N\int_\Gamma \bar u(I_h\bar u \circ g_h^{-1} - \bar u)d\sigma(x) \\ & + \varepsilon_h^2 h^2 + \varepsilon_h h\|\bar u_h \circ g_h^{-1} - \bar u\|_{L^2(\Gamma).} \end{aligned} \tag{2.32}$$

First we write

$$\begin{aligned} & \int_\Omega (\bar y - y_\Omega)(y_{\bar u_h \circ g_h^{-1}} - \bar y)dx + \int_\Omega (\bar y_h - y_\Omega)(y_h(I_h\bar u) - \bar y_h)dx \\ = \quad & -\int_\Omega (\bar y - \bar y_h)^2 dx + \int_\Omega (\bar y - \bar y_h)(\bar y - y_h(I_h\bar u))dx \\ & -\int_\Omega (\bar y - y_\Omega)\left\{(\bar y - y_{\bar u_h \circ g_h^{-1}}) - (y_h(I_h\bar u) - \bar y_h)\right\}dx. \\ & \leq -\frac{1}{2}\int_\Omega (\bar y - \bar y_h)^2 dx + \frac{1}{2}\int_\Omega (\bar y - y_h(I_h\bar u))^2 dx \\ & -\int_\Omega (\bar y - y_\Omega)\left\{(\bar y - y_{\bar u_h \circ g_h^{-1}}) - (y_h(I_h\bar u) - \bar y_h)\right\}dx. \end{aligned} \tag{2.33}$$

Let us bound the second addend:

$$\begin{aligned} \|\bar y - y_h(I_h\bar u)\|_{L^2(\Omega)} & \leq \|\bar y - y_{I_h\bar u \circ g_h^{-1}}\|_{L^2(\Omega)} + \|\bar y_{I_h\bar u \circ g_h^{-1}} - y_h(I_h\bar u)\|_{L^2(\Omega)} \\ & \leq C\left\{\|\bar u - I_h\bar u \circ g_h^{-1}\|_{L^2(\Gamma)} + h^{3/2}\right\} \leq \varepsilon_h h, \end{aligned} \tag{2.34}$$

where we have used the fact that $-\Delta(\bar y - y_{I_h\bar u \circ g_h^{-1}}) = 0$ in Ω, $\bar y - y_{I_h\bar u \circ g_h^{-1}} = \bar u - I_h\bar u \circ g_h^{-1}$ on Γ along with Casas and Raymond (2006), Lemma 2.1, the superconvergence property, Deckelnick, Günther and Hinze (2009), (5.10) and (2.10).

Inserting (2.33) into (2.32), and taking into account (2.34), we obtain that

$$\begin{aligned} & \frac{N}{2}\|\bar u_h \circ g_h^{-1} - \bar u\|^2_{L^2(\Gamma)} + \frac{1}{2}\|\bar y - \bar y_h\|^2_{L^2(\Omega)} \\ \leq \quad & -\int_\Omega (\bar y - y_\Omega)\left\{(\bar y - y_{\bar u_h \circ g_h^{-1}}) - (y_h(I_h\bar u) - \bar y_h)\right\}dx \\ & + N\int_\Gamma \bar u(I_h\bar u \circ g_h^{-1} - \bar u)d\sigma(x) + \varepsilon_h^2 h^2. \end{aligned} \tag{2.35}$$

Next we use $\bar\varphi$, the adjoint state related to $\bar u$. We apply Green's formula and obtain

$$
\begin{aligned}
& -\int_\Omega (\bar y - y_\Omega)\left\{(\bar y - y_{\bar u_h\circ g_h^{-1}}) - (y_h(I_h\bar u) - \bar y_h)\right\}dx \\
&= \int_\Omega \Delta\bar\varphi(\bar y - y_{\bar u_h\circ g_h^{-1}})dx - \int_{\Omega_h} \Delta\bar\varphi(y_h(I_h\bar u) - \bar y_h)dx \\
&\quad + \int_{\Omega\setminus\Omega_h} (\bar y - y_\Omega)(y_h(I_h\bar u) - \bar y_h)dx \\
&= -\int_\Omega \nabla\bar\varphi(\nabla\bar y - \nabla y_{\bar u_h\circ g_h^{-1}})dx + \int_\Gamma \partial_\nu\bar\varphi(\bar u - \bar u_h\circ g_h^{-1})d\sigma(x) \\
&\quad + \int_{\Omega_h} \nabla\bar\varphi(\nabla y_h(I_h\bar u) - \nabla\bar y_h)dx - \int_{\Gamma_h} \partial_{\nu_h}\bar\varphi(I_h\bar u - \bar u_h)d\sigma_h(x) \\
&\quad + \int_{\Omega\setminus\Omega_h} (\bar y - y_\Omega)(y_h(I_h\bar u) - \bar y_h)dx \\
&\le -\int_\Gamma \partial_\nu\bar\varphi(I_h\bar u\circ g_h^{-1} - \bar u)d\sigma(x) + \int_\Gamma \partial_\nu\bar\varphi(I_h\bar u\circ g_h^{-1} - \bar u_h\circ g_h^{-1})d\sigma(x) \\
&\quad + \int_{\Omega_h} \nabla\bar\varphi(\nabla y_h(I_h\bar u) - \nabla\bar y_h)dx - \int_{\Gamma_h} \partial_{\nu_h}\bar\varphi(I_h\bar u - \bar u_h)d\sigma_h(x) \\
&\quad + C\left(h^2\|\bar u - \bar u_h\circ {g_h}^{-1}\|_{L^2(\Gamma)} + h^3\right), \qquad (2.36)
\end{aligned}
$$

where we we have used again Green's formula and the equalities

$$
-\Delta(\bar y - y_{I_h\bar u\circ g_h^{-1}}) = 0
$$

in Ω and $\bar\varphi = 0$ on Γ to state that

$$
\begin{aligned}
\int_\Omega \nabla\bar\varphi(\nabla\bar y - \nabla y_{\bar u_h\circ g_h^{-1}})dx &= -\int_\Omega \bar\varphi\Delta(\bar y - y_{\bar u_h\circ g_h^{-1}})dx \\
&\quad + \int_\Gamma \bar\varphi\partial_\nu(\bar y - y_{\bar u_h\circ g_h^{-1}})d\sigma(x) = 0,
\end{aligned}
$$

and the same argument as in (2.20) to write

$$
\left|\int_{\Omega\setminus\Omega_h} (\bar y - y_\Omega)(y_h(I_h\bar u) - \bar y_h)dx\right| \le C\left(h^2\|\bar u - \bar u_h\circ g_h\|_{L^2(\Gamma)} + h^3\right).
$$

Inserting (2.36) in (2.35), we get

$$
\begin{aligned}
& \frac{N}{4}\|\bar u_h\circ g_h^{-1} - \bar u\|^2_{L^2(\Gamma)} + \frac{1}{2}\|\bar y - \bar y_h\|^2_{L^2(\Omega)} \\
&\le \int_\Gamma (N\bar u - \partial_\nu\bar\varphi)(I_h\bar u\circ g_h^{-1} - \bar u)d\sigma(x) + \int_\Gamma \partial_\nu\bar\varphi(I_h\bar u\circ g_h^{-1} - \bar u_h\circ g_h^{-1})d\sigma(x) \\
&\quad + \int_{\Omega_h} \nabla\bar\varphi(\nabla y_h(I_h\bar u) - \nabla\bar y_h)dx - \int_{\Gamma_h} \partial_{\nu_h}\bar\varphi(I_h\bar u - \bar u_h)d\sigma_h(x) + Ch^3. \qquad (2.37)
\end{aligned}
$$

Using now (2.14), (2.23) and (2.28) in (2.37), we have that

$$\frac{N}{4}\|\bar u_h\circ g_h^{-1}-\bar u\|^2_{L^2(\Gamma)}+\frac{1}{2}\|\bar y-\bar y_h\|^2_{L^2(\Omega)}\le C(h\varepsilon_h)^2+Ch\varepsilon_h\|\bar u_h\circ g_h^{-1}-\bar u\|_{L^2(\Gamma)}$$
$$+C_\mu h^2|\log h|+\mu\|\bar u_h\circ g_h^{-1}-\bar u\|^2_{L^2(\Gamma)}$$

and (2.7) follows by taking $\mu=N/8$.

If (A2) is satisfied, we use (2.14), (2.23) and (2.29) in (2.37), and we obtain

$$\frac{N}{4}\|\bar u_h\circ g_h^{-1}-\bar u\|^2_{L^2(\Gamma)}+\frac{1}{2}\|\bar y-\bar y_h\|^2_{L^2(\Omega)}\le \varepsilon_h^2h^2+\varepsilon_h h\|\bar u_h\circ g_h^{-1}-\bar u\|_{L^2(\Gamma)}$$

and (2.8) follows trivially from this.

Again, if we further suppose (A1), then Lemmas 2.1, 2.2 and 2.4 and Corollary 2.1 imply that $\varepsilon_h\le Ch^{1/2}$. ■

The following result is a direct consequence of Theorem 2.3:

COROLLARY 2.2 *Let $\bar u$ be the solution of problem* (P) *and for all $h>0$, let $\bar u_h$ be the solution of problem* (P$_h$). *Then there exists a constant $C>0$ such that*

$$\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^\infty(\Gamma)}\le Ch^{1/2}\sqrt{|\log h|}. \tag{2.38}$$

If, further, (A2) *is satisfied then there exists a sequence of positive numbers $\{\varepsilon_h\}_{h>0}$ converging to zero such that $\lim_{h\to 0}\varepsilon_h=0$ and*

$$\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^\infty(\Gamma)}\le \varepsilon_h h^{1/2}. \tag{2.39}$$

If both (A1) *and* (A2) *are satisfied then there exists $C>0$ such that*

$$\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^\infty(\Gamma)}\le Ch. \tag{2.40}$$

Proof. We will use the following inverse inequality (see, e.g., Brenner and Scott, 1994, Theorem 4.5.11): There exists a constant $C>0$ independent of h such that for every $u_h\in U_h$

$$\|u_h\|_{L^\infty(\Gamma_h)}\le Ch^{-1/2}\|u_h\|_{L^2(\Gamma_h)}.$$

Using the second inequality in (2.6) we have that

$$\|\bar u_h\circ g_h^{-1}\|_{L^\infty(\Gamma)}=\|\bar u_h\|_{L^\infty(\Gamma_h)}\le Ch^{-1/2}\|\bar u_h\|_{L^2(\Gamma_h)}\le Ch^{-1/2}\|\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}. \tag{2.41}$$

With the triangular inequality, (2.11), (2.41) and (2.12) we get

$$\begin{aligned}\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^\infty(\Gamma)} &\le \|\bar u-I_h\bar u\circ g_h^{-1}\|_{L^\infty(\Gamma)}\\ &\quad+\|I_h\bar u\circ g_h^{-1}-\bar u_h\circ g_h^{-1}\|_{L^\infty(\Gamma)}\\ &\le Ch+Ch^{-1/2}\|I_h\bar u\circ g_h^{-1}-\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}\\ &\le Ch+Ch^{-1/2}\|I_h\bar u\circ g_h^{-1}-\bar u\|_{L^2(\Gamma)}\\ &\quad+Ch^{-1/2}\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}\\ &\le Ch+Ch^{1/2}\varepsilon_h+Ch^{-1/2}\|\bar u-\bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}.\end{aligned}$$

So, (2.38) follows from (2.7). If we suppose (A2), (2.8) implies (2.39). If we suppose (A1) and (A2), then (2.13) and (2.9) imply (2.40). ∎

Numerical experiment. In Example 1 of Deckelnick, Günther and Hinze (2009), the following problem is solved. The domain is $\Omega = B_1(0,0) \subset \mathbb{R}^2$, $\alpha = 0$, $\beta = 1$ and $N = 1$. Let r, ϕ be the usual polar coordinates. The functions involved in the problem are defined by

$$\begin{aligned} y_\Omega &= (7r^2\cos^2\phi + 6r^2 - 6r)\cos\phi + r^3\max(0,\cos^3\phi), \\ f &= -6r\max(0,\cos\phi) \end{aligned}$$

and the solution is

$$\begin{aligned} \bar{u} &= \max\{0,\cos^3\phi\}, \\ \bar{y} &= r^3\max(0,\cos^3\phi), \\ \bar{\varphi} &= r^3(r-1)\cos^3\phi. \end{aligned}$$

We have that $\partial\Gamma_s = \{(0,1),(1,0),(0,-1)\}$. The family of triangulations is made in Deckelnick, Günther and Hinze (2009) by congruent refinement of an initial grid formed by 8 triangles of vertices

$$\begin{aligned} &(0,0),\ (\cos(2\pi j/8),\sin(2\pi j/8)), \\ &(\cos(2\pi(j+1)/8),\sin(2\pi(j+1)/8)),\ \text{for } j = 0:7 \end{aligned}$$

and the "landing" points are nodes of all the meshes, so the semidiscrete and the full discretization approach coincide.

We have solved the same problem rotating the mesh by 0.5 radians so that the points in $\partial\Gamma_s$ are no longer mesh points. Both assumptions (A1) and (A2) are satisfied.

We summarize our results in Table 2. Mesh data are shown in Table 1. For the reader's convenience, we have followed the same conventions of notation in the tables as in Deckelnick, Günther and Hinze (2009): i is the number of refinements from the initial mesh, n is the number of sides of the polygon approximating the boundary of the domain (and also the dimension of U_h), $h = |x_j - x_{j-1}|$, nt is the number of elements of $\mathcal{T}_h$, np is the number of nodes of the triangulation (and also the dimension of Y_h), $E^0_u(h) = \|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^2(\Gamma)}$, $E^\infty_u(h) = \|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^\infty(\Gamma)}$ and $E^0_y(h) = \|\bar{y} - \bar{y}_h\|_{L^2(\Omega_h)}$. As usual, EOC denotes the experimental order of convergence

$$EOC(h_1,h_2) = \frac{\log(E(h_1)) - \log(E(h_2))}{\log(h_1) - \log(h_2)}.$$

Both theoretical predictions (2.9) and (2.40) about the errors in the approximation of the control are confirmed in our experiment. The order of convergence of the error in the approximation of the state is, nevertheless, bigger than what could be expected. The same unexplained phenomenon appears in the semidiscrete approach in Deckelnick, Günther and Hinze (2009), Table 4.

Table 1. Mesh parameters for a sequence of $O(h^2)$ irregular meshes

i	n	h	nt	np
1	8	0.765367	8	9
2	16	0.390181	32	25
3	32	0.196034	128	81
4	64	0.098135	512	289
5	128	0.049082	2048	1089
6	256	0.024543	8192	4225
7	512	0.012272	32768	16641
8	1024	0.006136	131072	66049
9	2048	0.003068	524288	263169

Table 2. Order of convergence for a perturbation of Example 1 in Deckelnick, Günther and Hinze (2009)

i	$E^0_u(h_i)$	EOC^0_u	$E^\infty_u(h_i)$	EOC^∞_u	$E^0_y(h_i)$	EOC^0_y
1	0.2874809		0.324129		0.316139	
2	0.1349552	1.122	0.204850	0.681	0.111547	1.546
3	0.0889099	0.606	0.135616	0.599	0.039778	1.498
4	0.0454429	0.970	0.134135	0.016	0.012942	1.623
5	0.0194054	1.228	0.086700	0.630	0.003740	1.792
6	0.0075206	1.368	0.048723	0.832	0.001020	1.875
7	0.0027959	1.428	0.025933	0.910	0.000271	1.914
8	0.0010164	1.460	0.013411	0.951	0.000071	1.928
9	0.0003649	1.478	0.006828	0.974	0.000019	1.940

3. Polygonal domains

Let Ω be a polygonal domain of boundary Γ. Let us denote by n the number of its sides and $\{x_j\}_{j=1}^n$ the vertexes of Γ, ordered counterclockwise. Denote also $x_0 = x_n$ and $x_{n+1} = x_1$. Through this section we will suppose that $y_\Omega \in L^r(\Omega)$ for some $r > 2$.

Let $0 < \omega < \pi$ be the biggest interior angle of Ω. The Sobolev exponent giving the maximum regularity is (see Grisvard, 1985)

$$q_\Omega = \frac{2\omega}{2\omega - \pi} \text{ if } \omega > \pi/2. \tag{3.1}$$

If $0 < \omega \leq \pi/2$, then we can choose any $q_\Omega < +\infty$. In any case, notice that $q_\Omega > 2$. This roughly means that the solutions of the Poisson equation belong to any space $W^{2,r}(\Omega)$ with $r < q_\Omega$.

Let us formulate a modification of the control problem (P). We choose $\{e_j\}_{j=1}^n$, functions in $C^{0,1}(\Gamma)$ such that $e_i(x_j) = \delta_{ij}$ for $1 \leq i,j \leq n$ and $0 \leq e_j(x) \leq 1$ for $1 \leq j \leq n$. Typically, the support of each function e_j is small and they are disjoint. For any $\mathbf{v} = (v_j)_{j=1}^n \in \mathbb{R}^n$, we will denote $v = \sum_{j=1}^n v_j e_j$. We also choose two regularization parameters $0 < N$ and $0 < M$. Then we formulate the following problem

$$(\tilde{\mathrm{P}}) \begin{cases} \min \tilde{J}(u,\mathbf{v}) = \dfrac{1}{2}\displaystyle\int_\Omega (y_{u,\mathbf{v}}(x) - y_\Omega(x))^2 dx + \dfrac{N}{2}\|u\|^2_{L^2(\Gamma)} + \dfrac{M}{2}|\mathbf{v}|^2 \\ \text{subject to } -\Delta y_{u,\mathbf{v}} = f \text{ in } \Omega,\ y_{u,\mathbf{v}} = u + v \text{ on } \Gamma, \\ \alpha \leq u(x) \leq \beta \text{ for a.e. } x \in \Gamma,\ \alpha \leq v_i \leq \beta \text{ for all } i = 1,\ldots,n, \end{cases}$$

where $|\mathbf{v}|^2 = \sum_{j=1}^n v_j^2$. We will denote

$$W_{ad} = \{(u,\mathbf{v}) \in L^\infty(\Gamma)\times\mathbb{R}^n :\ \alpha \leq u(x) \leq \beta,\ \alpha \leq v_i \leq \beta \text{ for all } i = 1,\ldots,n\}.$$

For any control pair $(u,\mathbf{v})$, we define its related adjoint state $\varphi_{u,\mathbf{v}}$ as the solution of the homogeneous Dirichlet problem

$$-\Delta\varphi = y_{u,\mathbf{v}} - y_\Omega \text{ in } \Omega,\ \varphi = 0 \text{ on } \Gamma.$$

The derivative of the functional at a point $(\bar{u},\bar{\mathbf{v}})$ in a direction $(u,\mathbf{v}) \in L^2(\Gamma) \times \mathbb{R}^n$ is

$$\tilde{J}'(\bar{u},\bar{\mathbf{v}})(u,\mathbf{v}) = \int_\Gamma (N\bar{u} - \partial_\nu\bar{\varphi})u d\sigma(x) + \sum_{j=1}^n \big(M\bar{v}_j - \int_\Gamma \partial_\nu\bar{\varphi}e_j d\sigma(x)\big)v_j.$$

where $\bar{\varphi} = \varphi_{\bar{u},\bar{\mathbf{v}}}$. The derivative of the state at a point $(\bar{u},\bar{\mathbf{v}})$ in a direction $(u,\mathbf{v}) \in L^\infty(\Gamma) \times \mathbb{R}^n$ is given by $\bar{z}_{u,\mathbf{v}}$, solution of

$$-\Delta\bar{z}_{u,\mathbf{v}} = 0 \text{ in } \Omega,\ \bar{z}_{u,\mathbf{v}} = u + v \text{ on } \Gamma.$$

THEOREM 3.1 *Problem* $(\tilde{\mathrm{P}})$ *has a unique solution* $(\bar{u},\bar{\mathbf{v}}) \in L^\infty(\Gamma) \times \mathbb{R}^n$ *with related state* $\bar{y} = y_{\bar{u},\bar{\mathbf{v}}} \in H^{1/2}(\Omega) \cap L^\infty(\Omega)$ *and related adjoint state* $\bar{\varphi} = \varphi_{\bar{u},\bar{\mathbf{v}}}$. $\bar{u} \in W^{1-1/p,p}(\Gamma)$, $\bar{y} \in W^{1,p}(\Omega)$ *and* $\bar{\varphi} \in W^{2,p}(\Omega)$ *for* $p = r$ *if* $r < q_\Omega$ *and all* $2 < p < q_\Omega$ *if* $r \geq q_\Omega$. *Moreover, the optimal pair satisfies the variational inequality*

$$\int_\Gamma (N\bar{u} - \partial_\nu\bar{\varphi})(u - \bar{u})d\sigma(x) + \sum_{j=1}^n \big(M\bar{v}_j - \int_\Gamma \partial_\nu\bar{\varphi}e_j d\sigma(x)\big)(v_j - \bar{v}_j) \geq 0 \quad \forall\,(u,\mathbf{v}) \in W_{ad} \tag{3.2}$$

and the projection formulae:

$$\bar{u}(x) = \mathrm{Proj}_{[\alpha,\beta]}\left(\frac{\partial_\nu\bar{\varphi}(x)}{N}\right) \text{ for a.e. } x \in \Gamma,$$

$$\bar{v}_i = \mathrm{Proj}_{[\alpha,\beta]}\left(\frac{1}{M}\int_\Gamma \partial_\nu\bar{\varphi}(x)e_i(x)d\sigma(x)\right)\ \forall i = 1\ldots,n.$$

Proof. A direct computation of the derivatives of $\tilde{J}$ and first order necessary optimality conditions for $(\tilde{\mathrm{P}})$ lead to the projection formulae in a standard way. Next, the regularity of the state, the control and the adjoint state can be deduced as in Casas and Raymond (2006), Theorems 2.2 and 3.4. ∎

REMARK 3.1 *If $\alpha < 0 < \beta$, we have still that $\bar{u}(x_j) = 0$, but now $v(x_j)$ need not be zero and hence the optimal state is not forced to take a prescribed value at the corners independent of the data of the problem. Notice also that the original bound constraints can be overshot by $\bar{u} + \bar{v}$. Nevertheless, since $\bar{u}(x_j) = 0$, this effect should be small in general.*

For a fixed pair $(\bar{u}, \bar{\mathbf{v}})$, we will define $(\bar{d}, \bar{\mathbf{d}}) \in L^2(\Gamma) \times \mathbb{R}^n$ as

$$\bar{d}(x) = N\bar{u}(x) - \partial_\nu\bar{\varphi}(x),\ x \in \Gamma, \quad \text{and} \quad \bar{d}_j = M\bar{v}_j - \int_\Gamma \partial_\nu\bar{\varphi}e_j d\sigma(x),\ j = 1, \ldots, n.$$

The discretization is made formally as in Section 2. Notice that now $\Omega_h = \Omega$ and $\Gamma_h = \Gamma$ for all $h > 0$. Of course, g_h is the identity mapping and disappears in this section. We take

$$W_h = U_h \times \mathbb{R}^n \text{ and } W_{h,ad} = W_{ad} \cap W_h.$$

For every $(u, \mathbf{v}) \in L^\infty(\Gamma) \times \mathbb{R}^n$ we define $y_h(u, \mathbf{v}) \in Y_h$, the unique solution of

$$\begin{cases} \displaystyle\int_\Omega \nabla y_h(u, \mathbf{v})(x) \cdot \nabla w_h(x) dx) = \int_\Omega f(x) w_h(x) dx \ \forall\, w_h \in Y_{h0} \\ y_h(u, \mathbf{v}) = \Pi_h(u + v) \text{ on } \Gamma. \end{cases}$$

Notice that for $u_h \in U_h$, the discrete state on the boundary satisfies $y_h = u_h + \sum_{j=1}^n v_j \Pi_h e_j$. We approximate $(\tilde{\mathrm{P}})$ by the finite dimensional problem

$$(\tilde{\mathrm{P}}_\mathrm{h}) \begin{cases} \min \tilde{J}_h(u_h, \mathbf{v}) = \dfrac{1}{2}\displaystyle\int_\Omega (y_h(u, \mathbf{v}) - y_\Omega)^2 dx + \frac{N}{2}\|u_h\|^2_{L^2(\Gamma)} + \frac{M}{2}|\mathbf{v}|^2 \\ (u_h, \mathbf{v}) \in W_{h,ad} \end{cases}.$$

The discrete adjoint state associated to a control $(u, \mathbf{v})$ is the unique solution $\varphi_h(u, \mathbf{v}) \in Y_{h0}$ of

$$\int_\Omega \nabla w_h(x) \cdot \nabla \varphi_h(u, \mathbf{v})(x) dx = \int_\Omega (y_h(u, \mathbf{v}) - y_\Omega(x)) w_h dx \quad \forall\, w_h \in Y_{h0}.$$

The variational normal derivative $\partial_\nu^h \varphi_h(u, \mathbf{v}) \in U_h$ is defined through the variational equality

$$\begin{aligned} \int_\Gamma \partial_\nu^h \varphi_h(u, \mathbf{v}) w_h d\sigma(x) = &\int_\Omega \nabla w_h(x) \cdot \nabla\varphi_h(u, \mathbf{v})(x) dx \\ &- \int_\Omega (y_h(u, \mathbf{v}) - y_\Omega(x)) w_h dx \quad \forall\, w_h \in Y_h. \end{aligned}$$

Theorem 3.2 *Problem* $(\tilde{\mathrm{P}}_\mathrm{h})$ *has a unique solution* $(\bar{u}_h, \bar{\mathbf{v}}_h)$ *with associated discrete state and adjoint state* $\bar{y}_h$ *and* $\bar{\varphi}_h$*, respectively. The following variational inequality is satisfied:*

$$\int_\Gamma (N\bar{u}_h - \partial_\nu^h \bar{\varphi}_h)(u_h - \bar{u}_h) d\sigma(x) + \sum_{j=1}^{n} \big(M\bar{v}_{hj} - \int_\Gamma \partial_\nu^h \bar{\varphi}_h e_j d\sigma(x)\big)(v_j - \bar{v}_{hj}) \geq 0 \tag{3.3}$$

for all $(u_h, \mathbf{v}_h) \in W_{ad,h}$.

We define $(\bar{d}_h, \bar{\mathbf{d}}_h) \in W_h$ by

$$\bar{d}_h(x) = N\bar{u}_h(x) - \partial_\nu^h \bar{\varphi}_h(x) \quad \text{and} \quad \bar{d}_{hj} = M\bar{v}_{hj} - \int_\Gamma \partial_\nu^h \bar{\varphi}_h e_j d\sigma(x).$$

First order necessary condition (3.3) can be written as

$$\int_\Gamma \bar{d}_h(x)(u_h(x) - \bar{u}_h(x)) d\sigma(x) \geq 0 \quad \forall\, u_h \in U_{h,ad}, \tag{3.4}$$

$$\bar{d}_{hj}(t - \bar{v}_{hj}) \geq 0 \quad \forall\, t \in [\alpha, \beta]. \tag{3.5}$$

We will obtain now estimates for $\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}$ and $|\bar{\mathbf{v}} - \bar{\mathbf{v}}_h|$ in terms of the discretization parameter h. We obtain the same order of convergence as in Casas and Raymond (2006).

Theorem 3.3 *Let* $(\bar{u}, \bar{\mathbf{v}})$ *be the solution of* $(\tilde{\mathrm{P}})$ *and for* $h > 0$ *let* $(\bar{u}_h, \bar{\mathbf{v}}_h)$ *be the solution of* $(\tilde{\mathrm{P}}_\mathrm{h})$*. Then there exist* $h_0 > 0$ *and* $C > 0$ *such that for all* $0 < h < h_0$

$$\sqrt{\|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)} + |\bar{\mathbf{v}} - \bar{\mathbf{v}}_h|^2} \leq Ch^{1-1/p}$$

where p *is given in Theorem 3.1.*

Proof. We have the following error estimate (see Casas and Raymond, 2006, Theorem 5.7): for $u \in H^{1/2}(\Gamma)$ and $\mathbf{v} \in \mathbb{R}^n$ with $(u, \mathbf{v}) \in W_{ad}$

$$\|\partial_\nu \varphi(u, \mathbf{v}) - \partial_\nu^h \varphi_h(u, \mathbf{v})\|_{L^2(\Gamma)} \leq C(\|u + v\|_{H^{1/2}(\Gamma)} + 1)h^{1-1/p}. \tag{3.6}$$

According to Casas and Raymond (2006), Eq. (7.9) and Lemma 7.5, there exists a control $u_h^* \in U_{ad,h}$ such that

$$\tilde{J}'(\bar{u}, \bar{\mathbf{v}})(u_h^*, \bar{\mathbf{v}}) = \tilde{J}'(\bar{u}, \bar{\mathbf{v}})(\bar{u}, \bar{\mathbf{v}}) \tag{3.7}$$

and

$$\|\bar{u} - u_h^*\|_{L^2(\Gamma)} \leq Ch^{1-1/p}. \tag{3.8}$$

Since the problem is linear-quadratic we have that

$$N\|\bar u-\bar u_h\|^2_{L^2(\Gamma)}+M|\bar{\mathbf{v}}-\bar{\mathbf{v}}_h|^2\le(\tilde J'(\bar u,\bar{\mathbf{v}})-\tilde J'(\bar u_h,\bar{\mathbf{v}}_h))(\bar u-\bar u_h,\bar{\mathbf{v}}-\bar{\mathbf{v}}_h). \quad (3.9)$$

Next we take $(u,\mathbf{v})=(\bar u_h,\bar{\mathbf{v}}_h)$ in (3.2) and $(u_h,\mathbf{v}_h)=(u_h^*,\bar{\mathbf{v}})$ in (3.3) and we get

$$\tilde J'(\bar u,\bar{\mathbf{v}})(\bar u-\bar u_h,\bar{\mathbf{v}}-\bar{\mathbf{v}}_{\mathbf{h}})\le 0\le\tilde J_h'(\bar u_h,\bar{\mathbf{v}}_h)(u_h^*-\bar u_h,\bar{\mathbf{v}}-\bar{\mathbf{v}}_h).$$

We add at the right hand side $\pm\tilde J_h'(\bar u_h,\bar{\mathbf{v}}_h)(\bar u,\mathbf{0})$ to get

$$\tilde J'(\bar u,\bar{\mathbf{v}})(\bar u-\bar u_h,\bar{\mathbf{v}}-\bar{\mathbf{v}}_{\mathbf{h}})\le\tilde J_h'(\bar u_h,\bar{\mathbf{v}}_h)(u_h^*-\bar u,\mathbf{0})+\tilde J_h'(\bar u_h,\bar{\mathbf{v}}_h)(\bar u-\bar u_h,\bar{\mathbf{v}}-\bar{\mathbf{v}}_h).$$

We subtract at both sides $\tilde J'(\bar u_h,\bar{\mathbf{v}}_h)(\bar u-\bar u_h,\bar{\mathbf{v}}-\bar{\mathbf{v}}_h)$ and add at the right hand side $-\tilde J'(\bar u,\bar{\mathbf{v}})(u_h^*-\bar u,\mathbf{0})$ which is equal to 0 due to (3.7),

$$\begin{aligned}&(\tilde J'(\bar u,\bar{\mathbf{v}})-J'(\bar u_h,\bar{\mathbf{v}}_h))(\bar u-\bar u_h,\bar{\mathbf{v}}-\bar{\mathbf{v}}_h)\\ \le\;&(\tilde J_h'(\bar u_h,\bar{\mathbf{v}}_h)-\tilde J'(\bar u,\bar{\mathbf{v}}))(u_h^*-\bar u,\mathbf{0})\\ &+(\tilde J_h'(\bar u_h,\bar{\mathbf{v}}_h)-\tilde J'(\bar u_h,\bar{\mathbf{v}}_h))(\bar u-\bar u_h,\bar{\mathbf{v}}-\bar{\mathbf{v}}_h).\end{aligned}\quad (3.10)$$

To estimate the first summand we write directly the expressions for the derivatives and use (3.6) and (3.8) to get

$$(\tilde J_h'(\bar u_h,\bar{\mathbf{v}}_h)-\tilde J'(\bar u,\bar{\mathbf{v}}))(u_h^*-\bar u,\mathbf{0})\le Ch^{2(1-1/p)}. \quad (3.11)$$

To estimate the second term we use (3.6) to obtain

$$\begin{aligned}&(\tilde J_h'(\bar u_h,\bar{\mathbf{v}}_h)-\tilde J'(\bar u_h,\bar{\mathbf{v}}_h))(\bar u-\bar u_h,\bar{\mathbf{v}}-\bar{\mathbf{v}}_h)\\ =\;&\int_\Gamma(\partial_\nu\varphi_{\bar u_h,\bar{\mathbf{v}}_h}-\partial_\nu^h\bar\varphi_h)(\bar u-\bar u_h)d\sigma(x)\\ &+\sum_{j=1}^n\int_\Gamma(\partial_\nu\varphi_{\bar u_h,\bar{\mathbf{v}}_h}-\partial_\nu^h\bar\varphi_h)e_j d\sigma(x)(\bar v_j-\bar v_{hj})\\ &\le Ch^{1-1/p}\sqrt{\|\bar u-\bar u_h\|^2_{L^2(\Gamma)}+|\bar{\mathbf{v}}-\bar{\mathbf{v}}_h|^2}.\end{aligned}\quad (3.12)$$

And the result follows from relations (3.9)–(3.12). ∎

Numerical experiments. Let Ω be a polygonal domain in $\mathbb{R}^2$, $y_\Omega\in L^r(\Omega)$ for some $r>2$, $f(x)=0$, $A=-\Delta$, $\alpha=-2$ and $\beta=2$. We want to check if our perturbation technique leads to a better approximation of y_Ω. In the numerical tests we fix $N=1$ and the parameter M decreases from 1 to 0.001. We have chosen the functions e_j such that

$$\operatorname{supp}e_j=\{x_j+t(x_{j-1}-x_j):0\le t\le 1/8\}\cup\{x_j+t(x_{j+1}-x_j):0\le t\le 1/8\}$$

and $e_j(x_j)=1$, and e_j is linear in the segments $\{x_j+t(x_{j-1}-x_j):0\le t\le 1/8\}$ and $\{x_j+t(x_{j+1}-x_j):0\le t\le 1/8\}$.

We have tested three examples:

1. Set $\Omega = (0,1)^2$ and $y_\Omega = 1/|x|^{2/3}$. This problem was studied in Casas and Raymond (2006).
2. Ω is taken as the regular octagon inscribed in the unit circle with $x_1 = (1,0)$. Here we take $y_\Omega = 1$.
3. Ω is the same regular octagon, but $y_\Omega = 3$ (in this case $y_\Omega > \beta$).

The results of the tests are summarized in Tables 3, 4 and 5. We have included the quantity of interest $\|\bar y - y_\Omega\|_{L^2(\Omega)}$ and the cost $\frac{1}{2}N\|\bar u\|^2_{L^2(\Gamma)} + \frac{1}{2}M|\bar{\mathbf{v}}|^2$. It can be observed that our perturbation technique increases the quality of the approximation of the desired state as M tends to 0.

Table 3. Results for $y_\Omega = 1/|x|^{2/3}$ in $\Omega = (0,1)^2$.

Test 1	$J(u)$	$\tilde J(u,\mathbf{v})$			
		$M=1$	$M=0.1$	$M=0.01$	$M=0.001$
$\|\bar y - y_\Omega\|_{L^2(\Omega)}$	1.3195	1.3181	1.3057	1.2495	1.2353
Cost	0.1733	0.1741	0.1813	0.1916	0.1650
$\bar u_h(x_1)$	4.22×10^{-2}	4.22×10^{-2}	4.22×10^{-2}	4.25×10^{-2}	4.25×10^{-2}
CG *iterations*	10	15	13	23	21

Table 4. Results for $y_\Omega = 1$ in an octagon.

Test 2	$J(u)$	$\tilde J(u,\mathbf{v})$			
		$M=1$	$M=0.1$	$M=0.01$	$M=0.001$
$\|\bar y - y_\Omega\|_{L^2(\Omega)}$	1.1375	1.1357	1.1206	1.0141	0.9756
Cost	0.3093	0.3099	0.3144	0.3313	0.2401
$\bar u_h(x_1)$	6.73×10^{-3}	6.69×10^{-3}	6.35×10^{-3}	3.73×10^{-3}	2.63×10^{-3}
CG *iterations*	10	16	15	10	20

Table 5. Results for $y_\Omega = 3$ in an octagon.

Test 3	$J(u)$	$\tilde J(u,\mathbf{v})$			
		$M=1$	$M=0.1$	$M=0.01$	$M=0.001$
$\|\bar y - y_\Omega\|_{L^2(\Omega)}$	3.4125	3.4072	3.3618	3.2341	3.2341
Cost	2.7834	2.7885	2.8295	2.6740	2.5300
$\bar u_h(x_1)$	2.02×10^{-2}	2.01×10^{-2}	1.90×10^{-2}	1.61×10^{-2}	1.61×10^{-2}
CG iterations	10	16	15	21	22

We have solved the problems numerically with a regular mesh having 256 boundary edges on each side (this means 2.6×10^5 and 5.1×10^5 triangles, respectively, on the square and the octagon). We have used a primal-dual active set strategy (as described in Casas, Mateos and Tröltzsch, 2005, and based on the algorithm developed by Bergounioux, Ito and Kunisch, 1999) to solve the constrained optimization problem. To deal with the unconstrained optimization problems that come up in the procedure, we have used the conjugate gradient method. As an indicator of the conditioning of the problem, we have included in a row of the table the total number of conjugate gradient iterations. It can be seen that this number increases very slowly as M decreases.

One of the difficulties in the resolution of the numerical problems is that the convergence of $u_h(x_j) \to 0$ may be very slow. Applying the same technique of Corollary 2.2 we can expect that $|\bar{u}_h(x_j)| \leq Ch^{1/2-1/p}$. In the first case $y_\Omega \in L^r(\Omega)$ for all $r < 3$. Since $\omega = \pi/2$, we have that $p < 3$. In the other cases, $r = +\infty$, but for an octagon, $q_\Omega = 3$. So again $p < 3$. Therefore, we have that the convergence rate at the corners is very slow:

$$|\bar{u}_h(x_j)| \leq Ch^\theta, \quad \text{with } \theta < 1/6.$$

For a polygon with 16 sides we have that $q_\Omega = 7/3$ and hence $1/2 - 1/p < 1/14$ and in the numerical solution the results at the corners are not reliable at all. That is the reason we have not included polygons with more sides in our tests. We have included the data on $\bar{u}_h(x_1)$ in the tables to stress this fact.

References

Apel, T., Pfefferer, J. and Rösch, A. (2011) Finite element error estimates for Neumann boundary control problems on graded meshes. *Comput. Optim. Appl. Online First*, DOI 10.1007/510589-011-9427x.

Bergounioux, M., Ito, K. and Kunisch, K. (1999) Primal-dual strategy for constrained optimal control problems. *SIAM J. Control and Optim.*, **37**, 1176–1194.

Bramble, J.H. and King, J.T. (1994) A robust finite element method for nonhomogeneous Dirichlet problems in domains with curved boundaries. *Math. Comp.* **63** (207), 1–17.

Bramble, J.H., Pasciak, J.E. and Schatz, A.H. (1986) The construction of preconditioners for elliptic problems by substructuring. I. *Math. Comp.* **47** (175), 103–134.

Brenner, S.C. and Scott, L.R. (1994) The Mathematical Theory of Finite Element Methods. *Texts in Applied Mathematics*, **15**. Springer-Verlag, New York.

Casas, E., Günther, A. and Mateos, M. (2011) A Paradox in the Approximation of Dirichlet Control Problems in Curved Domains. *SIAM J. Control Optim.*, **49**(5), 1998–2007.

Casas, E. and Mateos, M. (2002) Second order optimality conditions for semilinear elliptic control problems with finitely many state constraints. *SIAM J. Control Optim.* **40** (5), 1431–1454 (electronic).

Casas, E., Mateos, M. and Raymond, J.P. (2007) Error estimates for the numerical approximation of a distributed control problem for the steady-state Navier-Stokes equations. *SIAM J. Control Optim.*, **46**, 952–982

Casas, E., Mateos, M. and Tröltzsch, F. (2005) Error estimates for the numerical approximation of boundary semilinear elliptic control problems. *Comput. Optim. Appl.* **31**, 193–219.

Casas, E. and Raymond, J.P. (2006) Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations. *SIAM J. Control Optim.*, **45**, 1586–1611.

Casas, E. and Sokolowski, J. (2010) Approximation of boundary control problems on curved domains. *SIAM J. Control Optim.*, **48**, 3746–3780.

Deckelnick, K., Günther, A. and Hinze, M. (2009) Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains. *SIAM J. Control Optim.*, **48**, 2798–2819.

Grisvard, P. (1985) *Elliptic Problems in Nonsmooth Domains.* Pitman, Boston.

Gunzburger, M., Hou, L. and Svobodny, T. (1991) Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with Dirichlet controls. *RAIRO Modél. Math. Anal. Numér.*, **25**, 711–748.

Kunisch, K. and Vexler, B. (2007) Constrained Dirichlet boundary control in L^2 for a class of evolution equations. *SIAM J. Control Optim.*, **46**, 1726–1753.

Mateos, M. and Rösch, A. (2009) On saturation effects in the Neumann boundary control of elliptic optimal control problems. *Comput. Optim. Appl.* **49**, 359–378.

May, S., Rannacher, R. and Vexler, B. (2008) Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems. Preprint 05/2008, Lehrstuhl für Numerische Mathematik, Universtät Heidelberg, Germany, `http://ganymed.iwr.uni-heidelberg.de/Paper/may-rannacher-vexler-2008.pdf`

Nečas, J. (1967) *Les Méthodes Directes en Théorie des Equations Elliptiques.* Editeurs Academia, Prague.

Vexler, B. (2007) Finite element approximation of elliptic Dirichlet optimal control problems. *Numer. Funct. Anal. Optim.* **28**, 957–973.