



## **THE THEORY OF ISOTOPE SEPARATION IN CASCADES: PROBLEMS AND SOLUTIONS**

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### **ABSTRACT**

The status of the theory of isotope mixture separation in cascades is presented. The problems that limit the concept of an ideal cascade in the case of separation of binary mixtures and arbitrary separation factors at cascade stages are discussed. A new type of separating cascades which have a total flow less than the flow of the corresponding ideal cascade is presented. Furthermore, these "optimum" cascades may permit mixing concentrations and still produce higher cascade separative work. In the paper, a criterion to define efficiency for the separation of multi-isotope mixtures is analyzed. The analysis is based on the concept of the match-abundance ratio cascade (MARC). The approach has demonstrated that in order to obtain the optimum parameters of a single stage in a cascade it is necessary to minimize the linear combination of the inverse values of "partial separative powers" for all mixture components.

*Keywords:* Isotope separation; Cascade; Binary and multicomponent mixtures

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### **INTRODUCTION**

In the theory of isotope mixture separation the problem of searching for the optimum parameters the separation process plays an extremely important role. Until now, the theory of the best multistage installations (cascades) was developed in a comprehensive way only for the so-called

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"fine" separation of binary mixtures, i.e. in the case when the separation factor for each separating stage of a cascade is very close to unity. The theory is based on the concept of the non-mixing condition for the concentrations at each point in the cascade where streams merge (such a cascade is called "an ideal cascade"). For ideal cascades, the concepts of "a separative power", "a separation potential", "a value function" and some others have been introduced. These concepts have proven to be very useful for calculations and optimizing various cascade installations. The numerous attempts to generalize this theory for arbitrary separation factors at a single stage or for the case of multi-isotope mixtures lead to difficulties that do not allow these results to be applied for the practical production of isotopes.

The paper examines the reasons for these difficulties and analyzes possible ways to find optimum conditions of separation in the case of large enrichment at each separation stage of a cascade with no limit in the number of separated components of an isotope mixture.

## SEPARATION OF BINARY MIXTURES

### Small separation factors at each cascade stage

To specify the problem under investigation, we shall review the basic propositions in the theory of ideal cascades for the case of the "fine" separation of binary isotope mixtures.

The cost of work for separation of isotope mixtures is the biggest component in the price of the enriched material. For the series of isotope separation methods these costs can be estimated as proportional to the total number of separating units (elements), each of which has usually one input stream (feed) and two withdrawal streams (product and waste). If the total flow in a cascade is known and all the units are working under the same conditions, the whole number of separating elements  $n$  can be calculated as

$$n = \frac{\sum L}{G} \quad (1)$$

where  $\sum L$  is the total flow in the cascade and  $G$  is the flow entering each element of a cascade. Eq. (1) can be derived easily by using the concept of the "ideal cascade" introduced by Cohen [1] to get the most efficient cascade. In such cascade, the interstage flows that merge at each confluent point have identical compositions, i. e. no mixing of streams with different concentrations takes place:

$$C_S = C'_{S-1} = C''_{S+1} \quad (2)$$

or

$$R_S = R'_{S-1} = R''_{S+1} \quad (3)$$

where  $C_S$  is the concentration (mole fraction) of the desired component contained in the  $S$  th stage feed flow. Primes ( ' and " ) denote the enriched and depleted flows, respectively, and the symbols without primes pertain to the feed flow.  $R$  is the abundance ratio

$$R = C/(1 - C) \quad (4)$$

As it has been shown in [1-3], the non-mixing conditions (2-3) for a desired isotope result in a most efficient cascade in which the total flow, and hence, the total number of separating elements, is minimized. In other words, in the particular case of "fine separation" of binary isotope mixtures, an ideal cascade is the standard with which all other cascades must be compared.

To calculate the total flow for the ideal cascade, the following expression for the total number of separating units in a cascade is applied:

$$n_{id} \equiv n_{min} = \frac{\Delta U}{\delta U} \quad (5)$$

where  $n_{id}$  - is the total number of separating units in the ideal cascade,

$$\Delta U = PV(C^P) + WV(C^W) - FV(C^F) \quad (6)$$

$$V(C) = (2C - 1) \ln \frac{C}{1 - C} \quad (7)$$

$P, W, F$  - are the product, waste, and feed flows, respectively,  
 $C^P, C^W, C^F$  - are the concentrations of a desired isotope in the product, waste, and feed flows, respectively,

$$\delta U = G \frac{(q - 1)^2}{8} \quad (8)$$

$q = R' / R''$  - is the overall separation factor of a single separating unit of the cascade.

Note that Eq. (5) has been derived with the assumption that the separation factor  $q$  does not depend on concentration, i.e. it is the same for all separating units of the cascade. Thus, the numerator in Eq. (5) depends on neither the nature of the separation process being used nor the scheme followed in the stage connection, but it merely depends on the external parameters of the cascade  $P, W, F, C^P, C^W, C^F$ . This value is called the "separation performance" of a cascade, and one may consider  $\Delta U$  as an external load.

The denominator in the same formula, defined by Eq. (8), is called the "separative power" of a unit and depends on the separation properties of a single element. Thus, for a "fine" separation of binary isotope mixtures,

the process of optimization is reduced to the definition of the operation parameters providing maximum value of separative power  $\delta U$ .

If in Eq. (6) the flows  $W$  and  $F$  are expressed by  $P$ , using the balance equation for a cascade as a whole, it can be rewritten as follows:

$$\Delta U = P\Phi(C^P, C^F, C^W) \quad (9)$$

where

$$\begin{aligned} \Phi(C^P, C^F, C^W) = & (2C^P - 1) \ln \frac{C^P}{1 - C^P} + \frac{C^P - C^F}{C^F - C^W} (2C^W - 1) \ln \frac{C^W}{1 - C^W} \\ & - \frac{C^P - C^W}{C^F - C^W} (2C^F - 1) \ln \frac{C^F}{1 - C^F} \end{aligned} \quad (10)$$

Since the value of  $\Delta U$  characterizes the useful separation work of a cascade per time unit, the function  $\Phi(C^P, C^F, C^W)$  can be interpreted as the specific separation work under the condition of defined concentration of desired isotope in the ingoing and outgoing flows. In the theory of isotope separation  $\Phi(C^P, C^F, C^W)$  is called the "value function". The metric for measuring the separation of binary mixture of uranium isotopes is the "separation work unit" (SWU) [4]. A SWU is the work that was spent to produce of 1kg of uranium with the enrichment of the uranium-235 isotope that is equal to a quantitative change of unity in the value function in the separation process. For example, the production of 1kg of uranium-235 with the enrichment of 30% from natural uranium containing uranium-238 of 99.289% concentration and uranium-235 (0.711 %) requires 4.3 SWU on the condition that the concentration of uranium-235 in the waste flow is 0.2 %. The price of SWU is determined on the basis of the cost of isotope separation production and plays a crucial role in the evaluation of the economic efficiency of a separation process.

Relations (6)-(8) may be obtained by another, more formal method which uses the axiomatic approach suggested by Dirac-Peierls [1]. This approach exploits three main conditions. The first condition defines the value of an isotope mixture as production of an extensive value (the number of moles of a separating mixture) to an intensive one (a "separation potential"  $V(C)$ , depending only on isotope composition). According to the second condition, it possible to attain the non-mixing condition at each point in the cascade where streams merge. The last condition defines a kind of a separation potential  $V(C)$ . Using the above conditions, suppose that stage number  $S$  of a cascade is separating  $L_S$  moles with a concentration of the desired isotope  $C_S$  into  $\theta_S L_S$  moles with a concentration  $C'_S$  and  $(1 - \theta_S)L_S$  moles with a concentration  $C''_S$ . Here  $\theta_S$  is the cut of the stage. The separative power of the stage can now be written as:

$$dU_s = L_s[\theta_s V(C_s') + (1 - \theta_s)V(C_s'') - V(C_s)] \quad (11)$$

After summing Eq. (11) over all stages of the cascade and taking into account that in the ideal cascade the separative powers of the stages are fully used, one can get  $\sum_S dU_s$  from the left side of the equation. Summing up the right side of equation (11), it is necessary to assume that the flows in the ideal cascade are connected without mixing concentrations. This means that during the summing up, all members describing the interstage flows disappear except for three external ones with their own concentrations. As the result, one will get

$$\sum_S dU_s = PV(C^P) + WV(C^W) - FV(C^F) \quad (12)$$

Thus, on the left side of the Eq. (12), in the absence of mixing losses, the sum of stage separative powers obtained; and, on the right side, the separative power considering an ideal cascade as a single unit is formulated. This allows us to write

$$\Delta U = PV(C^P) + WV(C^W) - FV(C^F) \quad (13)$$

Hence, the only problem to solve is a separation potential  $V(C)$ . Formally, as it follows from Eqs. (12) and (13), the function  $V(C)$  may be introduced in an arbitrary form, but, from a practical point of view, the choice of a function  $V(C)$  should produce a form of the stage (element) separative power that is independent of the mixture concentrations. If all the separating units (elements) in the cascade have identical parameters, then its separative power may be presented as

$$dU_s = n_s \delta U \quad (14)$$

where  $n_s$  is the number of separating elements connected in parallel;  $\delta U$  is the separative power of an individual element not depending on concentration.

So, for the cascade as a whole, one can write

$$\sum_S dU_s = n_{id} \delta U \quad (15)$$

In the Dirac-Peierls approach, the question whether the value of  $n_{id}$  is minimal or not, remains open.

The expressions for the stage separative power and a separation potential are usually found by solving the functional equations obtained from Eq. (15)

$$dU = L[\theta V(C') + (1 - \theta)V(C'') - V(C)] = \text{const.} \quad (16)$$

For the symmetrical case ( $\theta = 0.5$ ), the separative power is identical to expression (8) as well as a separation potential defined by Eq. (7), if this function has a zero value for  $C = 0.5$ . Similar approaches exist for the definition of separation potential and separative power for the ideal cascade designed from asymmetric elements ( $\theta \neq 0.5$ ).

### Large separation factors at cascade stages

The theory of ideal cascades for the case when the degree of the desired isotope enrichment per stage unit is not small and the heads and tails separation factors are equal to each other

$$\alpha = \beta = \sqrt{q} \quad (17)$$

has been developed in [1].

Here  $\alpha$ ,  $\beta$  are the heads and tails separation factors ( $\alpha = R' / R$ ,  $\beta = R'' / R$ ). Equation (17) means that each separating unit in such a cascade must be operated symmetrically.

In theory, it was also assumed that the overall separation factor per stage  $q = \alpha\beta$  is independent of the mixture concentrations and cut  $\theta$ .

The relation (5) for such a cascade is still valid, the function  $V(C)$  is analogous to (7), and the element separative power is written as

$$\delta U = G \frac{\sum \sqrt{q} - 1}{\sum \sqrt{q} + 1} \ln \sqrt{q} \quad (18)$$

It is easy to make sure that the analogous result can be obtained by solving the functional equation (16) for arbitrary values of an overall separation factor  $q$  in the cascade designed from the symmetrical elements satisfying condition (17). In [1], the problem of the optimal parameters in this cascade is not addressed.

Later, the principal possibility to construct ideal cascades designed from asymmetrical elements with arbitrary enrichment at each stage was demonstrated in [5-7]. However, the theory developed does not yield a relation such as (5) to evaluate the number of separating elements  $n_{id}$ . The problem of the optimal cascade parameters (in the sense of the minimum total flow) in this cascade still remains unsolved.

The solution of the functional equation (16) derived in [7] for the case of  $\alpha \neq \beta$  was written as follows:

$$V(C, \alpha, \beta) = f(\alpha, \beta) \left[ \frac{C}{b} - \frac{1-C}{a} \right] \ln \frac{C}{1-C} \quad (19)$$

$$dU = L f(\alpha, \beta) \quad (20)$$

where

$$a = \frac{\beta(\alpha - 1) \ln \beta - (\beta - 1) \ln \alpha}{\alpha\beta - 1} \quad (21)$$

$$b = \frac{\alpha(\beta - 1) \ln \alpha - (\alpha - 1) \ln \beta}{\alpha\beta - 1} \quad (22)$$

$f(\alpha, \beta)$  is an arbitrary function.

From Eqs. (19, 20), one can see that separation potential depends on either the mixture composition or the stage heads and tails factors,  $\alpha$  and  $\beta$ , and on any choice of the arbitrary function  $f(\alpha, \beta)$  that contradicts the first condition of the Dirac-Peierls axiomatic approach. This contradiction makes use the expressions obtained for the separative power and separation potential unacceptable for practical calculations [7,8]. First, such separation potential will depend on the separation characteristics of a single stage in the cascade. In particular it means that two ideal cascades, designed from elements with non-identical separation factors  $\alpha$  and  $\beta$  and provided with the same feed flow rates and isotope compositions, have to possess at the output the product and waste flows with similar flow rates and separation compositions. In other words, two ideal cascades executing the same separation program have to possess the same separation power. However, calculations with Eqs. (19)-(22) lead to different values of the separative power for these two cascades [7]. Second, a use of Eq. (19) for calculating the separative power of the ideal cascade constructed from asymmetrical elements with variable separation factors will be impossible. It is due to impossibility to get an expression for evaluation the number of separating elements in a cascade for solving this or that practical problem.

In [7,8], attempts to remove the above contradictions were suggested. One suggestion was to allow the separative power of a single stage in the cascade to be dependent on the composition of the isotopic mixture. At the same time, it was suggested that the separation potential be allowed to have the classical form as given in equation (7). With these assumptions, it can easily be shown [8] that the expression for the separative power will have the following form:

$$dU = L[a(1 - C) + bC] \quad (23)$$

where the coefficients  $a$  and  $b$  are defined by the expressions (21, 22).

As in the case of the "fine" separation and the symmetrical operation mode of separating stages in the cascade, Eq. (23) gives the formula (18). Thus, for uranium isotope enrichment ( $C \ll 1$ ), the formula (23) will be written in the following form [8]

$$dU = L\{\ln[1 + \theta(q - 1)] + \theta \ln q\} \quad (24)$$

where  $\theta = \frac{\beta - 1}{q - 1}$ .

As one can see in this case, the separative power does not depend on concentration. The analysis of the expression (24) has shown that the specific stage separation power ( $dU/L$ ) for the fixed value of  $q$  has its maximum value with respect to  $\theta$  (or  $\beta$ ). The optimum values of  $\theta$  and  $\beta$  will be defined as

$$\begin{aligned} \theta_{opt} &= \frac{1}{\ln q} - \frac{1}{q - 1} \\ \beta_{opt} &= \frac{q - 1}{\ln q} \end{aligned} \quad (25)$$

As a result, the maximum value of the specific stage separative power will be calculated from the formula

$$(dU / L) = \ln \left[ \frac{q - 1}{\ln q} \right] + \frac{\ln q}{q - 1} - 1 \quad (26)$$

It is important to note that the value of the cut corresponding to the symmetric regime of a stage (an element)

$$\theta_{sym} = \frac{1}{\sqrt{q} + 1} \quad (27)$$

differs from  $\theta_{opt}$  calculated from Eq. (25). Fig. 1 demonstrates the growth of this disagreement with  $q$ .

In a general case, the condition  $\alpha_S = \beta_{S+1}$  takes place in the ideal cascade. It means that the values of  $\theta$  and  $dU/L$  vary over the cascade stages. As a result, it is impossible to design the ideal cascade with the stages operating in the optimum regime.



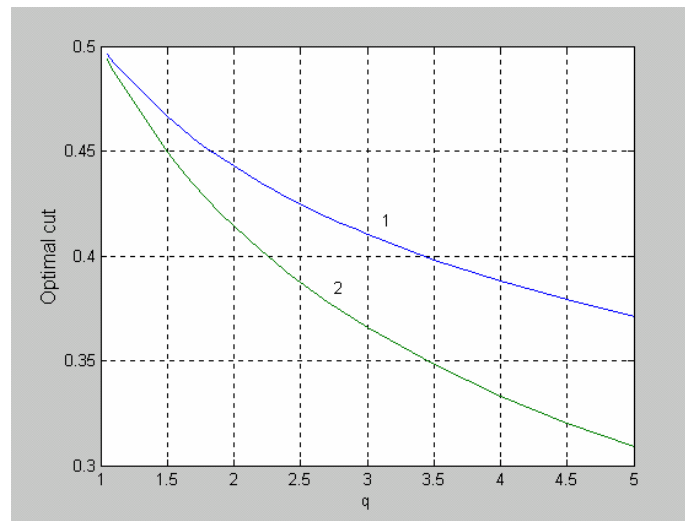


Fig.1. The cut versus a value of an overall separation factor in  
1 - optimum and 2 - symmetric cases.

Besides, the formula (23) is of little help when solving the major problem of isotope separation production - that is for the evaluation of the number of separation elements in the ideal cascade for production of the necessary amount of the final product at a given concentration of the key component. Note that in the case of large separation factors at the cascade stages, the question whether the ideal cascade is the best one in the sense of the total flow (as it was in the case of a small separation factors) has no definite answer.

### Optimum and ideal cascades

For the first time, the conclusion that the concepts of "the ideal cascade" (nomixing concentrations in merge flows) and "the optimum cascade" (i.e. the cascade with the minimum number of separation elements) in a general case may not coincide was made in [10]. The special calculation research has shown that the ideal and optimum cascades coincide by a total flow at any enrichments at a separation stage (that is equivalent to the arbitrary values of an overall separation factor  $q$ ), when condition (17) is valid for each separation element, and the value of  $q$  does not vary over the length of the cascade [10-12].

A quite different picture takes place if the condition of the symmetric work of separation elements in a cascade is not fulfilled. In this case, at the same value of a separation factor at each stage, the total flow in the ideal cascade can exceed that in the optimum cascade, in some cases for more than 50%. In other words, in any cascade designed from asymmetrical elements, the no-mixing condition expressed by the relation  $\alpha_S = \beta_{S+1}$ ,

$S = 1, 2, \dots, N-1$ , where  $S$  is the stage number, and  $N$  is the total number of elements in a cascade, does not coincide with the condition of the minimum total flow.

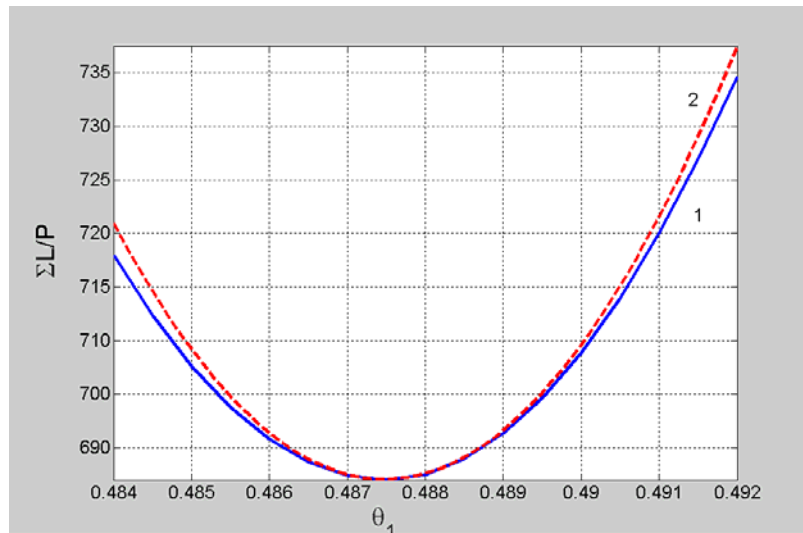


Fig.2. The total flow in a cascade versus a cut number at its first stage for 1 – an ideal and 2 – optimum cascades.

In Fig. 2, two curves representing the total flow versus the cut number at the first stage of the ideal and optimal cascades for the fixed overall separation factor, product flow rate, and the concentration of a key component in the product flow ( $q = 1.59$ ,  $C^F = 0.711\%$ ,  $C^P = 4.4\%$ ,  $C^W = 0.45\%$ ,  $P = 1$  g/s) are shown. In the ideal cascade, the total flow has its minimum value for the cut number corresponding to the symmetrical separation elements and can be calculated by expression (27). It is explained by the fact that in any asymmetrical case ( $\beta_1 \neq \alpha_1$ ) a cut number is changing over cascade stages. So, if the maximum value of the specific stage separation power is provided by the choice of the cut at the first stage, the value of separative power at the next stage with changing cut in contrast to the previous one will be lower. Finally, the average value of the specific stage separation power will occur lower than that of the one calculated from the formula (26). In other words, only the ideal cascade designed of symmetric separation stages is able to provide full use of separation performance of all its elements and the total flow in such a cascade will be minimum one.

The computational experiments have demonstrated that the total flow in a cascade made of asymmetric elements can be minimized by the special choice of the  $\alpha(S)$  and  $L(S)$  distributions that will smoothen the serrated

distribution of the specific separation power over the cascade stages and bring it closer to the value calculated by Eq. (26) [12].

After the numerous computational experiments, we came to the conclusion that even in the case of arbitrary separation factors at the cascade stages the separation performance of a cascade as a whole may be evaluated by the classic formula of separation potential  $V(C)$ . Despite this convenience, the theory does not allow to optimize the parameters of each separation stage of the cascade. However, the process of the total flow minimization will entail a search that will lead to a set of optimal stage parameters.

### SEPARATION OF MULTICOMPONENT MIXTURES

The practical requirements for evaluating the efficiency of multicomponent isotope mixtures separation requires introducing of the efficiency criteria analogous to (7) and (8). In this case, the difficulties with defining for the separative performance of a separation unit are connected with the problem that the no-mixing condition in a cascade is not applicable because it is not usually possible to match more than one concentration in a stream. The first attempts to introduce the expressions for  $dU$  and  $V(C_1, C_2, \dots, C_m)$  in the case of multi-isotope mixtures (see e.g. [13-16]), have been made to solve the functional Eq. (16) or to extend the theory of binary mixture separation in the multi-isotope separation case. Because the results obtained in all the papers quoted are practically the same, we will analyze only one of them [14].

Let us consider the separation stage with one input and two withdrawal streams intended for multicomponent isotope mixture separation (Fig.3) where  $m$  is the number of components.

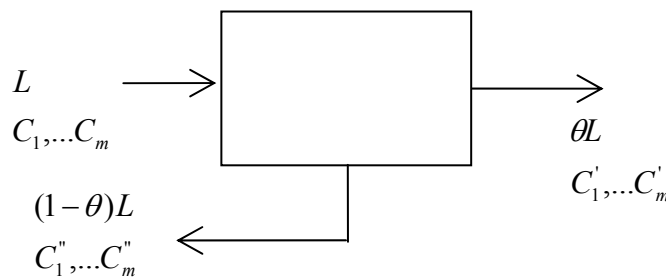


Fig.3. The separation stage for multi-isotope mixture separation.

If the “value” of a mass unit for a multicomponent isotope mixture is defined as  $V(C_1, C_2, \dots, C_m)$ , the useful separation work for a time unit analogous with the case of a binary mixture can be written as follows

$$dU = \theta LV(C'_1, \dots, C'_m) + (1 - \theta) LV(C''_1, \dots, C''_m) - LV(C_1, \dots, C_m) \quad (28)$$

### The case of low enrichment at cascade stages

Assuming that enrichment at each cascade stage is low (or in other words a separation factor for a pair of components of a separating mixture  $q_{ij} = \frac{C'_i}{C'_j} / \frac{C''_i}{C''_j}$  is close to unity), one can expect that the change in concentrations at a separating element of a cascade will be small [17]

$$C'_i - C_i = (1 - \theta)C_i \sum_{j=1}^m \varepsilon_{ij} C_j \quad (29)$$

$$C_i - C''_i = \theta C_i \sum_{j=1}^m \varepsilon_{ij} C_j \quad (30)$$

where  $\varepsilon_{ij} = \ln q_{ij}$ .

Expanding the Eq. (28) into the Taylor's series near the point  $(C_1, \dots, C_m)$ , in  $m$  dimensional space, retaining only the second order terms, and taking into account the balance equation for each component of the separating mixture

$$LC_i = \theta LC'_i + (1 - \theta)LC''_i \quad (31)$$

as well as using Eqs. (29) and (30), one will get the following equation:

$$dU = \frac{\theta(1-\theta)L}{2} \left[ \sum_{i=1}^m C_i \left( \sum_{j=1}^m \varepsilon_{ij} C_j \right) \frac{\partial}{\partial C_i} \right]^2 V \quad (32)$$

The expression in the square brackets is the differential operator acting on the function  $V(C_1, C_2, \dots, C_m)$ . For many separation processes (including gas centrifugation), one can present the values of relative enrichment coefficients can be presented in the form

$$\varepsilon_{ij} = \varepsilon_0 (M_j - M_i) \quad (33)$$

where  $M_i, M_j$  are masses of  $i$  th and  $j$  th components, respectively;  $\varepsilon_0$  is the enrichment coefficient for a unit mass difference.

Assuming that  $\varepsilon_0$  depends on neither the mixture composition, nor cut  $\theta$ , Eq. (32) can be rewritten as

$$dU = \frac{\theta(1-\theta)L\varepsilon_0^2}{2} \left[ \sum_{i=1}^m C_i \left( \sum_{j=1}^m k_{ij} C_j \right) \frac{\partial}{\partial C_i} \right]^2 V(C_1, \dots, C_m) \quad (34)$$

where  $k_{ij} = M_i - M_j$ .

From the physical point of view, the separative power of the separating element is its own characteristic that is defined by its constructive and physical peculiarities, and consequently, it should not depend on the mass structure and concentrations of mixture components. This means that the second factor in Eq. (34) should be a constant, which without loss of generality can be taken as unity. Therefore, Eq. (34) is split into two equations:

$$dU = \frac{\theta(1-\theta)L\varepsilon_0^2}{2} \tag{35}$$

$$\left[ \sum_{i=1}^m C_i \left( \sum_{j=1}^m k_{ij} C_j \right) \frac{\partial}{\partial C_i} \right]^2 V(C_1, \dots, C_m) = 1 \tag{36}$$

For the following boundary conditions

$$\begin{aligned} V(1,0,\dots,0) &= \infty \\ \dots & \\ V(0,0,\dots,1) &= \infty \end{aligned} \tag{37}$$

which means that the production of a pure isotope (with the concentration of 100 %) requires infinite separation work, the solution of equation (37) is given by

$$\begin{aligned} V(C_1, \dots, C_m) &= \frac{2}{m(m-1)} \sum_{i,j} \frac{1}{k_{ij}^2} V_{ij} \\ V_{ij} &= (C_i - C_j) \ln \frac{C_i}{C_j}, (i < j, 1 \leq i, j \leq m, i \neq j) \end{aligned} \tag{38}$$

The solution of (38) for any  $m$  is proved by mathematical induction.

Thus, the separation potential (38) is a linear combination of the terms  $(C_i - C_j) \ln \frac{C_i}{C_j}$ , each of which, from the thermodynamic point of view, characterizes the level of the “order” in a concentration distribution between the  $i$  th and  $j$  th mixture components ( $i = 1, \dots, m; j = 1, \dots, m$ ) [18].

The question now is whether the results obtained have real practical meaning. One can easily derive that, for any ordinary cascade used for separating of a multicomponent isotope mixture, the following inequality will hold

$$\sum \frac{\theta(1-\theta)L\varepsilon_0^2}{2} > PV(C_1^P, \dots, C_m^P) + WV(C_1^W, \dots, C_m^W) - FV(C_1^F, \dots, C_m^F) \quad (39)$$

Here  $C_i^P, C_i^W, C_i^F$  are the concentration of the  $i$ th component in the product, waste, and feed flows, and the functional  $V(\vec{C})$  is defined by the expression (38).

The right side of the inequality (39) characterizes the separative power of some abstract cascade, which operates in the regime of no-mixing of concentrations in the points of merging flows in the cascade. However, because in any multicomponent separation cascade, concentrations are mixed, the total separative capacity of all the cascade stages will always be more than the separative power of such a cascade. In other words, the separation potential obtained from Eq. (36), contrary to the case of a binary mixture, does not allow to evaluate the total number of elements in the cascade. To do this, it is necessary to solve a specific separation problem and, hence, it is not applicable as a cost characteristic of a manufactured isotope product. Nevertheless, it is believed that the potential (38) can be useful to assess the separation work at the consecutive steps of the multi-phase separation campaigns.

#### Arbitrary separation factors at cascade stages

A more basic approach to the evaluation of the multicomponent separation performance results from the theory of a multicomponent separation cascade which is analogous to the two-component "ideal cascade". The theory of such a cascade was established by De La Garza et al. [19,20] and then developed in detail in a number of papers [21-32]. A multicomponent analogy of an ideal cascade for binary isotope mixture separation is a cascade in which the abundance ratio between the  $n$ th and  $(n+k)$ th components  $R_{n,n+k} = C_n/C_{n+k}$  ( $n$  and  $n+k$  are designated as the "key" components) is matched whenever two streams come together; i.e. at all the inter-stage connections and feed points. Such a cascade is referred to as a matched abundance ratio cascade (MARC). Hence, for MARC one can write:

$$R'_{n,n+k}(S-1) = R_{n,n+k}(S) = R''_{n,n+k}(S+1) \quad (40)$$

where  $S$  is the current number of a stage in the cascade.

The properties of MARC allow it to be used for solving the problem of introduction of "a separative power" and a separation potential for the multi-isotope mixtures. For the first time, the expressions for the "separative power" and the "separation potential" for the common case of arbitrary overall separation factors  $q_{ij}$  and without limitation on the number of components have been obtained in [33]. In that paper a counterflow

symmetric (for a chosen pair of components) MARC consisting of  $N$  steps and intended for separation of a  $m$ -component isotope mixture was examined in detail.

For many separation processes, a stage separation factor for a pair of components can be expressed as a function of a difference between their molecular masses  $M_k - M_i$ . For example, in the case of gas centrifugation, it is appropriate to use the following approximation [28]

$$q_{ik} = q_0^{M_k - M_i} \tag{41}$$

where  $M_k, M_i$  are molecular masses of components with numbers  $k$  and  $i$ ,  $q_0$  is the overall separation factor per unit mass difference.

Thus, in the case of the isotope separation by gas centrifuges

$$\gamma_j = \frac{q_{j,n+1}}{\sqrt{q_{n,n+1}}}, j = 1, 2, \dots, m \text{ can be transformed into the expression [33]}$$

$$\gamma_j = \alpha_0^{M^* - M_j}, (j \neq n) \tag{42}$$

where  $q_{j,n+1}$  is the overall stage separation factor for a pair of components with the numbers  $j$  and  $n+1$  and  $M^* = \frac{M_n + M_{n+1}}{2}$ .

As a result, the total flow in MARC  $\sum_{S=1}^N L_S$  was obtained as

$$\sum_{S=1}^N L_S = \sum_j^m \frac{PC_i^P \ln R_{n,n+1}^P + WC_i^W \ln R_{n,n+1}^W - FC_i^F \ln R_{n,n+1}^F}{\frac{\gamma_j - 1}{\gamma_j + 1} \ln \beta_n} \tag{43}$$

If productivity ( $G$ ) value is identical for all the elements in the cascade, the expression for the total number of separating units in the cascade per unit of the product flow may be written as

$$\frac{n}{P} = \sum_{j=1}^m \frac{V_j(C_j^P, R_{n,n+1}^P) + \frac{W}{P} V_j(C_j^W, R_{n,n+1}^W) + \frac{F}{P} V_j(C_j^F, R_{n,n+1}^F)}{\delta U_j} \tag{44}$$

where

$$V_j(C_j, R_{n,n+1}) = C_j \ln R_{n,n+1} \tag{45}$$

$$\delta U_j = G \frac{\gamma_j - 1}{\gamma_j + 1} \ln \gamma_n \quad (46)$$

It is quite obvious that  $\delta U_j$  can be interpreted as the separation performance for the  $j$  th component of the separated mixture and  $V_j$  may be identified as the partial separation potential referring to the component with the number  $j$ .

The final expression for  $\delta U_j$  will be as follows

$$\delta U_j = G \frac{(q_0)^{M^* - M_j} - 1}{(q_0)^{M^* - M_j} + 1} \ln(q_0)^{M_{n+1} - M_n} \quad (47)$$

The numerator in (44) does not depend on the characteristics of the single separating element. However, it does depend on the values of  $R_{n,n+1}^P, R_{n,n+1}^W, R_{n,n+1}^F$  and the mass differences of the components  $M^* - M_i$  and  $M_{n+1} - M_n$ .

Thus, in the case of a multi-isotope separation, to obtain the optimal parameters of a single stage in the cascade, it is necessary to minimize the linear combination of the value  $1/\delta U_j$  for all the components. For the case of a “fine” or “difficult” separation ( $q_0 \sim 1$ ), the expressions (46) may be transformed as

$$n = \frac{P\Phi(R_{n,n+1}^P, R_{n,n+1}^W, R_{n,n+1}^F)}{G\varepsilon_0^2/8} \quad (47)$$

where the function

$$\Phi(R_{n,n+1}^P, R_{n,n+1}^W, R_{n,n+1}^F) = \sum_{j=1}^m \frac{C_i^P \ln R_{n,n+1}^P + \frac{W}{P} \ln R_{n,n+1}^W - \frac{F}{P} \ln R_{n,n+1}^F}{(M_{n+1} - M_n)^2 \left( \frac{M_{n+1} + M_n}{2} - M_j \right)} \quad (48)$$

can be considered as the value function for a multicomponent isotope mixture and the value of  $G\varepsilon_0^2/8$  can be interpreted as the separative power of a single element.

The examples of the successful application of the developed efficiency criterion for optimization of the cascades and single gas centrifuges for separation of multicomponent isotope mixtures, have been presented in [33,34].



### CONCLUSIONS

1. Until now, the theory of isotope separation in cascades has not reached its final form. Important questions arising from practical problems still remain unanswered. The main reason is the difficulty with defining the best (standard) cascade in the general case of arbitrary enrichment at cascade stages.
2. An important step in defining such a standard cascade was fixing the fact that an ideal (non-mixing) cascade and an optimal cascade (with minimum number of separating elements) are coincident in the sense of total flow for any enrichments at cascade stages only in the case of the symmetric regime of their exploitation.
3. The formulas to calculate “the separative power” and “the separation potential” for a multi-isotope separation in a cascade with arbitrary separation factors of its units are proposed. In contrast with numerous, previously applied approaches, the theory developed here allows to evaluate the number of separating units in a cascade.

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