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Boussinesq-type Equations for Long Waves in Water of Variable Depth

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Abstract

The paper deals with the problem of the transformation of long gravitational waves propagating in water of variable depth. The main attention of the paper is focused on the derivation of equations describing this phenomenon. These equations are derived under the assumption that the non-viscous fluid is incompressible and rotation free, and that the fluid velocity components may be expressed in the form of the power series expansions with respect to the water depth. This procedure makes it possible to transform the original two-dimensional problem into a one-dimensional one, in which all unknown variables depend on time and a horizontal coordinate. The partial differential equations derived correspond to the conservation of mass and momentum. The solution of these equations is constructed by the finite difference method and an approximate discrete integration in the time domain. In order to estimate the accuracy of this formulation, theoretical results obtained for a specific problem were compared with experimental measurements carried out in a laboratory flume. The comparison shows that the proposed theoretical formulation is an accurate description of long waves propagating in water of variable depth.

Key words: long waves, wave propagation, variable water depth

1. Introduction

An important part of the theory of water waves is concerned with the analysis of the propagation of waves in a fluid of small, non-uniform depth. Since wave lengths are large compared to the water depth, we can speak of long waves propagating in shallow water. Usually, these waves undergo transformations depending on changes in the water depth. In a theoretical description of this phenomenon, we introduce certain approximations which simplify the analysis of the problem considered. Generally, these approximations enable us to eliminate one of the spatial dimensions of the description. For instance, a three-dimensional flow problem may be reduced to a two-dimensional one. In many applications it can also be assumed that the wave amplitude is a small quantity. In order to develop a more accurate theory, however, finite amplitudes of these waves should be taken into account. In the literature on the subject, there are two main approaches to describing the long wave phenomenon (Ursell 1953). The first is based on the assumption that the pressure at any point in the fluid is equal to the hydrostatic head of water above that point. This assumption of the hydrostatic pressure leads to the so-called Airy's shallow water theory of long waves. For a two-dimensional problem, considered in this paper, this theory provides a system of two non-linear partial differential equations:

$$\begin{gathered} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(gh + \frac{1}{2}u^2\right) = 0, \\ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0, \end{gathered} \tag{A}$$

where $u(x, t)$ is the horizontal velocity component independent of the depth coordinate, $h(x, t)$ is the water depth, g is the gravitational acceleration, x is the horizontal coordinate, and t is the time.

In the second approach to the long wave phenomenon, the description of the fluid pressure takes into account the fluid acceleration. With this dynamic description of the pressure, we arrive at the so-called Boussinesq theory of long waves in shallow water (Whitham 1974). For the latter case, the representative system of differential equations for the two-dimensional problem reads

$$\begin{gathered} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(gh + \frac{1}{2}u^2 + \frac{1}{3}h_0\frac{\partial^2 h}{\partial t^2}\right) = 0, \\ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0, \end{gathered} \tag{B}$$

where h_0 is the constant, still water depth.

A formulation similar to Boussinesq's one was developed by Korteveg and de Vries, who derived a single non-linear equation for the free surface elevation known as the KdV equation (Whitham 1974). With regard to equations (A), Stoker (1948) derived a complete solution to these equations by the method of characteristics. In particular, he presented closed solutions to the problem of shallow water waves propagating along a sloping shore with a constant slope. Since this work by Stoker, a number of works have appeared in which equations (A) are a starting point in analyses of specific problems considered. Among other contributions, an important one has been made by Carrier and Greenspan (1958). On the basis of shallow water approximations, these authors discovered a hodograph transformation which enabled them to transform the original non-linear differential equations, defined in a physical space, into a single linear equation for a potential function defined in a transformation space. This linear equation enabled them to calculate the run-up of a wave of small amplitude on a sloping beach. It was shown that the wave can climb the beach without breaking. In order to obtain a better description of steep waves propagating in water of variable depth, Peregrine (1967) derived Boussinesq equations in which depth-averaged velocity was used as a dependent variable. The standard Boussinesq equations are based on the assumption of weak dispersion and weak non-linearity, and thus, they are restricted to shallow water. In order to improve linear dispersion characteristics in deeper water, Madsen et al (1991) derived a new form of Boussinesq equations by adding third order terms to the momentum equations, written for a constant water depth. These terms, derived from the long wave equations, are insignificant in shallow water, and thus they do not affect the accuracy of the description. Another set of Boussinesq equations was derived by Nwogu (1993), who used the velocity at a certain distance from the still water level as the velocity variable instead of the commonly used depth-averaged velocity. In this way the linear dispersion properties of Boussinesq equations have been improved.

Boussinesq-type equations for surface gravity waves are also discussed in Madsen and Schäffer (1999), where a number of formulations of the problem, known from the literature, are reviewed. In particular, the authors discuss the velocity potential formulations in terms of an infinite power series expansion. In most of the formulations, the final result is a set of partial differential equations dependent on selected approximations in the description of the phenomenon.

A variety of problems related to the description of shallow water waves are described in Dingemans's monograph (1997). In particular, a number of Boussinesq-type models for uneven bottoms are discussed in detail. This book provides techniques for analysing problems of wave propagation in water of non-uniform depth. In most of the derivations of Boussineq-like equations, the starting point is a power series solution to the Laplace equation for the velocity potential. It may be important to add here that all Boussinesq-like models are asymptotically equivalent but may differ in practical applicability (Dingemans 1997).

In this paper, an alternative derivation of Boussinesq-type equations for water of variable depth is presented. The starting point of our analysis are formulae describing the velocity field in a fluid of variable depth. Velocity components are expressed in the form of power series expansions with respect to the water depth. The derivation of equations describing the fluid motion is similar to that given in Van Groesen's and de Jager's monograph (1999) for the derivation of the Korteweg and de Vries equation for shallow water waves propagating in a fluid of constant depth. The procedure developed in this paper provides a hierarchy of non-linear partial differential equations describing the fluid motion. In order to estimate the accuracy of the formulation, selected results of theoretical solutions were compared with experimental data obtained in a laboratory flume.

2. Non-linear Theory of Waves in Water of Variable Depth

In what follows we confine our attention to a two-dimensional potential motion of an incompressible non-viscous fluid, as shown schematically in Fig. 1. A water wave arrives from the left and propagates to the right, in the direction of positive values of the x coordinate. Our main goal is to construct a solution that would be valid for waves of finite heights. The starting point of the analysis is the derivation of differential equations of the problem considered. Thus, with respect to the Cartesian system of coordinate axes shown in the figure, let us assume that the two components of the velocity field, i.e. $u(x, z, t)$ and $v(x, z, t)$, are expressed in the following form:

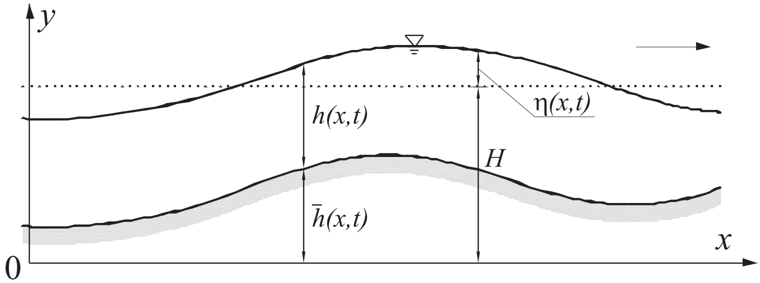

Fig. 1. A long wave propagating in fluid of variable depth

$$\begin{aligned} u(x, y, t) &= \sum_{n=0}^{\infty} (z + h_b)^n f_n(x, t), \\ v(x, y, t) &= \sum_{n=0}^{\infty} (z + h_b)^n \varphi_n(x, t), \end{aligned} \tag{1}$$

where $h_b(x)$ describes the bottom elevation (the still water depth), and $f_0, f_1, f_2, ...$ and $\varphi_0, \varphi_1, \varphi_2, ...$ are unknown functions of the description.

The fluid is assumed to be incompressible, and thus

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0. \tag{2}$$

From the substitution of equations (1) into the above condition one obtains

$$\sum_{n=0}^{\infty} (z + h_b)^n \left[f_n' + (n + 1)(m f_{n+1} + \varphi_{n+1}) \right] = 0, \tag{3}$$

where hereinafter the primes denote differentiation with respect to x.

At the fluid bottom $z = -h_b(x)$, the normal component of the velocity vector should be equal to zero, and thus we have

$$v\left[x, -h_b(x), t\right] = \frac{dh_b}{dx} u\left[x, -h_b(x), t\right], \quad \rightarrow \quad \varphi(x, t) = -m f_0(x, t), \tag{4}$$

where $dh_b/dx = m(x)$.

By collecting terms with the same power in $(z + h_b)$ in equation (3), we arrive at the system of equations

$$f'_{n-1} + mnf_n + n\varphi_n = 0, \quad n = 1, 2, \ldots \tag{5}$$

From relations (4) and (5) the following formulae result:

$$\begin{aligned} \varphi_0 &= -mf_0, \\ \varphi_n &= -\left(mf_n + \frac{1}{n}f'_{n-1}\right), \quad n = 1, 2, \ldots \end{aligned} \tag{6}$$

The incompressibility condition and the boundary condition at the fluid bottom enable us to express all functions $\varphi_0, \varphi_1, \varphi_2, \ldots$ in terms of the functions $f_0, f_1, f_2, \ldots$. Moreover, for the assumed rotation-free fluid motion, the following condition holds:

$$\frac{\partial u}{\partial z} - \frac{\partial v}{\partial x} = 0, \tag{7}$$

which gives

$$\sum_{n=1}^{\infty} (z + h_b)^{n-1} \left[nf_n - \varphi'_{n-1} - mn\varphi_n\right] = 0. \tag{8}$$

As in the previous case, the collection of terms with the same power in $(z + h_b)$, leads to the formula

$$f_n = \frac{1}{n}\varphi'_{n-1} + m\varphi_n, \quad n = 1, 2, \ldots \tag{9}$$

From equations (6) and (9) the following recurrence formulae may be derived:

$$\begin{aligned} f_n &= \frac{1}{n}\frac{1}{1+m^2}\left(\varphi'_{n-1} - mf'_{n-1}\right), \\ \varphi_n &= -\frac{1}{n}\frac{1}{1+m^2}\left(f'_{n-1} + m\varphi'_{n-1}\right), \quad n = 1, 2, \ldots \end{aligned} \tag{10}$$

On account of equations (6), the vertical velocity component may be written in the form

$$v = -\sum_{n=1}^{\infty} \left[m(z + h_b)^{n-1} f_{n-1} + \frac{1}{n}(z + h_b)^n f'_{n-1}\right] \tag{11}$$

For points at the free surface of the fluid, i.e. for $z = \eta(x, t) = h(x, t) - h_b(x)$, where $\eta(x, t)$ is the free surface elevation, the fluid pressure equals the atmospheric pressure, which is taken as a constant. Therefore, the differentiation of the pressure with respect to the arc length s of $\eta(x, t)$ gives

$$\frac{\partial p}{\partial x} + \left(\frac{\partial h}{\partial x} - m\right) \frac{\partial p}{\partial z}\bigg|_{z=h-h_b} = 0, \tag{12}$$

where $h(x,t)$ is the water depth.

The derivatives of the pressure in the above equation result from Euler's equations of fluid motion

$$
\begin{aligned}
-\frac{1}{\rho}\frac{\partial p}{\partial x} &= \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2\right),\\
-\frac{1}{\rho}\frac{\partial p}{\partial z} &= g + \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2\right),
\end{aligned}
\tag{13}
$$

where ρ is the fluid density, and g is the gravity acceleration.

The substitution of equations (13) into equation (12) gives

$$
\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2\right) + \left(\frac{\partial h}{\partial x} - m\right)\left[\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2\right) + g\right]\Bigg|_{z=h-h_b} = 0. \tag{14}
$$

This formula describes the dynamic boundary condition for the upper surface of the fluid. At the same time, the kinematic boundary condition for the free surface reads

$$
\frac{\partial h}{\partial t} + \left(\frac{\partial h}{\partial x} - m\right)u - v\Bigg|_{z=h-h_b} = 0. \tag{15}
$$

These boundary conditions enable us to derive the final differential equations of the problem discussed. On account of equations (1) and (11), the dynamic boundary condition assumes the form

$$
\begin{aligned}
&\sum_{n=0}^{\infty} h^n\frac{\partial f_n}{\partial t} + (h'-m)\left[g - \sum_{n=1}^{\infty}\left(mh^{n-1}\frac{\partial f_{n-1}}{\partial t} + \frac{h^n}{n}\frac{\partial f'_{n-1}}{\partial t}\right)\right] +\\
&+\left(\sum_{n=0}^{\infty} h^n f_n\right)\cdot\sum_{n=1}^{\infty} h^{n-1}\left(f'_{n-1} + nh' f_n\right) + \left[\sum_{n=1}^{\infty}\left(mh^{n-1}f_{n-1} + \frac{h^n}{n}f'_{n-1}\right)\right]\times\\
&\times\sum_{n=1}^{\infty} h^{n-1}\left\{(h'-m)f'_{n-1} - \left[1 - m(h'-m)\right]nf_n\right\} = 0.
\end{aligned}
\tag{16}
$$

In a similar way, the following kinematic boundary condition is obtained:

$$
\frac{\partial h}{\partial t} + (h'-m)\cdot\sum_{n=0}^{\infty} h^n f_n + \sum_{n=1}^{\infty}\left(mh^{n-1}f_{n-1} + \frac{h^n}{n}f'_{n-1}\right) = 0. \tag{17}
$$

Up to this point no approximations have been introduced. The differential equations derived have a complicated structure. In order to obtain final equations of the problem, all functions f_n $(n = 1, 2, \ldots)$ together with their derivatives, entering equations (16) and (17), should be expressed in terms of the function $f_0(x,t)$ and its derivatives. Such a procedure may be carried out by means of recurrence formulae (10). Our further discussion will be confined to long water waves, propagating in a fluid of small depth with a small bottom slope, for which it is justifiable to neglect higher order terms in the description mentioned above.

3. Approximations in the Description of Long Waves

Water waves propagating in a fluid of small depth are characterised by two important parameters: $\mu = h_0/l$, i.e. the ratio of the still water depth to a typical wave length, and $\varepsilon = a_0/h_0$, i.e. the ratio of the wave amplitude to the water depth (Nwogu 1993, Dingemans 1997). For long waves, the first of these parameters is assumed to be a small quantity – typically smaller than 1/10, whereas the second parameter is a quantity of order one (Dingemans 1997). At the same time, the ratio h/h_0 of the water depth to a typical water depth is also a quantity of order one. For uneven bottoms, it is assumed that the bottom slope $|dh_b/dx| = O(\mu)$ is a small quantity, i.e. appreciable changes in the water depth may occur only in a region of a typical water wave length.

In deriving an approximate description of long waves, it is desirable to examine first the range of magnitudes of the subsequent terms in equations (1). In order to simplify the further discussion, we confine our attention to the case of a constant bottom slope $m =$ const. For this case, the following relations are obtained from recurrence formulae (10):

$$\begin{aligned}
& && \varphi_0 = -mf_0, \\
f_1 &= -\frac{2m}{1+m^2}f_0', && \varphi_1 = -\frac{1-m^2}{1+m^2}f_0', \\
f_2 &= -\frac{1}{2}\frac{1-3m^2}{(1+m^2)^2}f_0'', && \varphi_2 = \frac{1}{2}\frac{3m}{(1+m^2)^2}\left(1-\frac{m^2}{3}\right)f_0'', \\
f_3 &= \frac{1}{6}\frac{4m}{(1+m^2)^3}(1-m^2)f_0''', && \varphi_3 = \frac{1}{6}\frac{1}{(1+m^2)^3}\left(1-6m^2+m^4\right)f_0''' \quad \ldots .
\end{aligned} \tag{18}$$

Knowing that $m = O(\mu)$ and neglecting the square and higher powers of the bottom slope, equations (18) are reduced to the following form:

$$\begin{aligned}
& f_0, && \varphi_0 = -mf_0, \\
& f_1 \simeq -2mf_0', && \varphi_1 \simeq -f_0', \\
& f_2 \simeq -\frac{1}{2}f_0'', && \varphi_2 \simeq -\frac{3}{2}mf_0'', \\
& f_3 \simeq \frac{2}{3}mf_0''', && \varphi_3 \simeq \frac{1}{6}f_0''', \ \ldots \\
& f_4 \simeq \frac{1}{24}f_0'''', && \varphi_4 \simeq -\frac{5}{24}mf_0'''', \ \ldots
\end{aligned} \tag{19}$$

From the approximations, it may be seen that every other consecutive term in the formulation depends directly on the bottom slope. Moreover, the respective components of the functions f_n and φ_n $(n = 1, 2, \ldots)$, with or without the bottom slope term ($m$in the relations), are shifted by one. This means that in the description of the velocity field in shallow water of insignificantly variable depth (in the limit – a constant water depth), the most important terms in the description of the horizontal component of the velocity are those with even powers of the water depth, while in the case of the vertical components, the most important are terms with odd powers of the depth. An important feature of the description is that higher components of the velocity depend on higher order derivatives of the fundamental function describing the horizontal velocity at the fluid bottom. Thus, from the above relations it follows that

$$\begin{aligned} f_{2n} &\approx (-1)^n f_0^{(2n)}, \\ f_{2n-1} &\approx (-1)^n m f_0^{(2n-1)}, \;\; n = 1, 2, \ldots, \end{aligned} \tag{20}$$

where $f_0^{(2n)}$ ($f_0^{(2n-1)}$) means the $2n$ ($2n-1$) derivative with respect to the horizontal coordinate of the function $f_0(x,t)$.

For the long waves considered, the derivative with respect to the horizontal coordinate of the function $f_0(x,t)$ is a small quantity. $hf_0'O(\mu)$. This means that, on account of equations (20), it is justifiable to confine our attention to a few lowest order terms in describing the long wave phenomenon. In addition, with respect to relations (19) for a small bottom slope, the multipliers of the water depth powers in equation (16) may be further approximated as follows:

$$\begin{aligned} h^0 :\; & \alpha \frac{\partial f_0}{\partial t} + \beta g + f_0(f_0' + h' f_1) + m f_0(\beta f_0' - \alpha f_1) = \\ & \approx \alpha \frac{\partial f_0}{\partial t} + \beta g + \alpha f_0 f_0' \approx \frac{\partial f_0}{\partial t} + (h' - m) g + f_0 f_0', \end{aligned} \tag{21}$$

$$\begin{aligned} h^1 :\; & \alpha \frac{\partial f_1}{\partial t} - \beta \frac{\partial f_0'}{\partial t} + f_0(f_1' + 2h' f_2) + f_1(f_0' + h' f_1) + \sim + m f_0(\beta f_1' - 2\alpha f_2) + \\ & + (m f_1 + f_0')(\beta f_0' - \alpha f_1) = \approx -(h' + m) \frac{\partial f_0'}{\partial t} - (h' + m) f_0 f_0'' + \\ & + (h' - m)(f_0')^2 \approx -(h' + m) \left(\frac{\partial f_0'}{\partial t} + f_0 f'' \right), \end{aligned} \tag{22}$$

$$\begin{aligned} h^2 :\; & \alpha \frac{\partial f_2}{\partial t} - \frac{1}{2} \beta \frac{\partial f_1'}{\partial t} + f_0(f_2' + 3h' f_3) + f_1(f_1' + 2h' f_2) + f_2(f_0' + h' f_1) + \\ & + (m f_0)(\beta f_2' - 3\alpha f_3) + (m f_1 + f_0')(\beta f_{12}' - 2\alpha f_2) + \\ & + \left(m f_2 + \frac{1}{2} f_1' \right) \left(\beta f_0' - \alpha f_1 \right) = \approx \left(-\frac{1}{2} \alpha + m\beta \right) \frac{\partial f_0''}{\partial t} + \frac{1}{2} (f_0' f_0'' - f_0 f_0''' \;) \approx \\ & \approx \frac{1}{2} \left(-\frac{\partial f_0''}{\partial t} + f_0' f_0'' - f_0 f''' \right), \end{aligned} \tag{23}$$

$$
\begin{aligned}
h^3:\ & \alpha\frac{\partial f_3}{\partial t} - \frac{1}{3}\beta\frac{\partial f_2'}{\partial t} + f_0(f_3' + 4h'f_4) + f_1(f_2' + 3h'f_3) + f_2(f_1' + 2h'f_2) + \\
& + f_3(f_0' + h'f_1) + (mf_0)(\beta f_3' - 4\alpha f_4) + (mf_1 + f_0')(\beta f_2' - 3\alpha f_3) + \\
& + \left(mf_2 + \frac{1}{2}f_1'\right)(\beta f_1' - 2\alpha f_2) + \left(mf_3 + \frac{1}{3}f_2'\right)(\beta f_0' - \alpha f_1) = \\
& \approx \left(\frac{2}{3}m\alpha + \frac{1}{6}\beta\right)\frac{\partial f_0'''}{\partial t} + \frac{1}{6}(3m + h')f_0 f_0'''' - \frac{2}{3}h'f_0'f_0''' + \\
& + \frac{1}{2}(h' - m)f_0''f'' \approx \frac{1}{6}(3m + h')\left(\frac{\partial f_0'''}{\partial t} + f_0 f''''\right),
\end{aligned}
\tag{24}
$$

where $\alpha = 1 - m(h' - m)$ and $\beta = h' - m$.

With respect to the above approximations, the dynamic boundary condition for the free surface of the fluid is assumed in the following form:

$$
\begin{aligned}
& \frac{\partial f_0}{\partial t} - h(h' + m)\frac{\partial f_0'}{\partial t} - \frac{1}{2}h^2\frac{\partial f_0''}{\partial t} + \frac{1}{6}h^3(h' + 3m)\frac{\partial f_0'''}{\partial t} + g(h' - m) + f_0 f_0' + \\
& - h(h' + m)f_0 f_0'' + \frac{1}{2}h^2(f_0'f_0'' - f_0 f_0''') + \frac{1}{6}h^3(h' + 3m)f_0 f_0'''' = 0.
\end{aligned}
\tag{25}
$$

Similarly, the kinematic boundary condition for the free surface is

$$
\begin{aligned}
& \frac{\partial h}{\partial t} + h'f_0 + h(h'f_1 + f_0') + h^2\left(h'f_2 + \frac{1}{2}f_1'\right) + h^3\left(h'f_3 + \frac{1}{3}f_2'\right) = \\
& \approx \frac{\partial h}{\partial t} + (hf_0)' - \frac{1}{2}h^2(h' + 2m)f_0'' - \frac{1}{6}h^3 f_0'''(1 - 4h'm) = 0.
\end{aligned}
\tag{26}
$$

For specific cases considered, some of the terms entering equations (25) and (26) may be ignored. In our further discussion, we neglect products of the space derivatives in these equations and confine our attention to the second order powers in the dynamic boundary condition and to the third order powers in the kinematic boundary condition. At this point a remark is needed. As mentioned above, the most important in describing horizontal velocity components are terms corresponding to even powers of the water depth, while for the vertical component the most important are odd powers. Therefore, in equation (26) the third order power term is retained. For a specific case considered, this term may also be ignored. Thus, taking these assumptions into account, we arrive at a system of Boussinesq-like equations for long waves propagating in shallow water

$$
\begin{aligned}
& \frac{\partial f_0}{\partial t} - \frac{1}{2}h^2\frac{\partial^3 f_0}{\partial t\partial x^2} + \frac{\partial}{\partial x}\left[g(h - h_b) + \frac{1}{2}f_0^2\right] - \frac{1}{2}h^2 f_0\frac{\partial^3 f_0}{\partial x^3} = 0\ , \\
& \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hf_0) - \frac{1}{6}h^3\frac{\partial^3 f_0}{\partial x^3} = 0.
\end{aligned}
\tag{27}
$$

This system of equations is similar to the one presented in Dingemans's monograph (1997). The linearization of these equations for a constant still water depth (h_b = const) leads to the system of equations

$$\begin{aligned} &\frac{\partial f_0}{\partial t} - \frac{1}{2} h_0^2 \frac{\partial^3 f_0}{\partial t \partial x^2} + g \frac{\partial \eta}{\partial x} = 0 \ , \\ &\frac{\partial \eta}{\partial t} + h_0 \frac{\partial f_0}{\partial x} - \frac{1}{6} h_0^3 \frac{\partial^3 f_0}{\partial x^3} = 0, \end{aligned} \tag{28}$$

where $h(x,t) = h_0 + \eta(x,t)$.

Let us consider a small amplitude periodic wave with frequency ω and wave number k

$$\begin{aligned} f_0 &= u_0 \exp\left[i(kx - \omega t\right], \\ \eta &= a_0 \exp\left[i(kx - \omega t\right], \end{aligned} \tag{29}$$

where u_0 and a_0 denote amplitudes of respective variables.

From the substitution of (29) into (28) the following dispersion relation is obtained

$$\omega^2 = gk(kh_0) \frac{1 + \frac{1}{6}(kh_0)^2}{1 + \frac{1}{2}(kh_0)^2} \cong gk \left[(kh_0) - \frac{1}{3}(kh_0)^3 \right]. \tag{30}$$

The dispersion derived is close to the standard linear dispersion formula

$$\omega^2 = gk \tanh(kh_0) = gk \left[(kh_0) - \frac{1}{3}(kh_0)^3 + \frac{2}{15}(kh_0)^5 - \cdots \right]. \tag{31}$$

On account of equation (30), one may calculate the associated phase speed $c_f = \omega/k$ of the periodic wave. On the other hand, for relatively long waves with finite amplitudes propagating in water of constant depth, equations (27) may be simplified to the form inherent in the assumption that the pressure is given as in hydrostatics (Stoker 1948)

$$\begin{aligned} &\frac{\partial f_0}{\partial t} + \frac{\partial}{\partial x} \left(gh + \frac{1}{2} f_0^2 \right) = 0, \\ &\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (h f_0) = 0 \ . \end{aligned} \tag{32}$$

For the latter case, one may show that the wave speed $c_f = \sqrt{gh}$, and

$$f_0 = 2\left(\sqrt{gh} - \sqrt{gh_0}\right) = 2\sqrt{gh_0}\left(\sqrt{1 + \eta/h_0} - 1\right) \cong \sqrt{gh_0}\eta/h_0, \tag{33}$$

where h_0 is the still water depth, η is the free surface elevation, and $h = h_0 + \eta$.

In our further discussion, however, we shall confine our attention to non-linear equations (27), which are assumed to properly describe the main features of the phenomenon of long water waves propagating in a fluid of insignificantly variable depth.

4. Reduction of the Problem to a System of Ordinary Differential Equations

In the preceding section, we derived the system of partial differential equations (27), describing the plane problem of waves propagating in water of non-uniform depth. This system was derived under assumptions that enabled us to confine our attention to the lowest-order terms of the expansion procedure with respect to the water depth. However, even if the lowest-order terms are taken into account, the resulting equations are still non-linear partial differential equations, and thus, they are difficult to solve analytically. Therefore, in order to solve the problem, we resort to an approximate method allowing us to replace the partial differential equations with a system of ordinary differential equations. In the derivation of the latter equations, the continuous fluid domain is replaced with a set of nodal points, and the spatial derivatives in equations (27) are replaced with finite difference quotients written at these points. In the case of an infinite fluid domain, however, the formal approach to the problem leads to an infinite set of ordinary differential equations. In order to overcome this difficulty, we may confine our attention to a finite part of the infinite fluid domain with appropriate boundary conditions assumed at the boundary between the finite and infinite parts. The boundary conditions should allow approaching waves to pass through the boundary without any reflection. This procedure leads to a finite system of difference equations for nodal values of the variables, dependent on time. To make the further discussion clear, let us consider the generation of waves in a finite layer of a fluid of variable depth. It is assumed that the fluid, initially at rest, is forced to move by a piston-type generator placed at $x = 0$ and starting to move at a certain point in time. Such a case corresponds directly to experiments carried out in a laboratory flume. At the generator face $x = x_g(t)$, we have the boundary condition that the fluid velocity $f_0(x_g, t)$ equals the generator velocity $\dot{x}_g(t)$. At the boundary $x = L$, a condition should be imposed that waves are not reflected from the boundary. For the initial value problem considered, the generated wave will reach the right boundary $x = L$ after a finite time measured from the starting point. This means that for a sufficiently long L and a relatively short time, it is reasonable to assume that at the right boundary the fluid it at rest. In accordance with the finite differences formulation, instead of the continuous functions $f_0(x, t)$, $h(x, t)$ and $h_b(x)$, we consider a finite set of their values at selected nodal points: $x_j = j \cdot a$ ($j = 1, 2, \ldots, M$), where j denotes the node number and a is a constant spacing of the nodal points. In the discrete formulation, we have to replace the space derivatives, entering equations (27), with appropriate finite difference quotients. Such a procedure is not unique. In particular, in deriving equations for the first nodal point $x_1 = a$, we have to take into account the boundary condition at the moving boundary $x = x_g(t)$. With regard to the first and second space derivatives of the velocity $f_0(x = a, t)$ at the first nodal point, we resort to the finite difference approximations

$$
\begin{aligned}
\frac{\partial f_0}{\partial x} &\approx -\frac{a}{b(a+b)} f^0 + \frac{a-b}{ab} f^1 + \frac{b}{a(a+b)} f^2, \\
\frac{\partial^2 f_0}{\partial x^2} &\approx \frac{2}{b(a+b)} f^0 - \frac{2}{ab} f^1 + \frac{2}{a(a+b)} f^2 \ ,
\end{aligned}
\tag{34}
$$

derived with the help of the Taylor formula for a non-uniform spacing of points, i.e. for $b = a - x_g(t)$ and a, and for the nodal values $f^0 = \dot{x}_g(t),\ f^1, f^2$ of the function f_0 at subsequent points. The third derivative may be obtained with a forward difference formula (Hirsch 1992). On the other hand, in order to calculate the first derivative of the water depth at the first nodal point, it is convenient to use the Gregory-Newton extrapolation formula (Chan and Street 1970)

$$
f^r = \frac{11}{3} f^{r+1} - 5 f^{r+2} + 3 f^{r+3} - \frac{2}{3} f^{r+4} \tag{35}
$$

where the superscripts denote the numbers of consecutive nodal points.

In particular, the above formula has also been used for calculating the third derivative of the velocity at the first nodal point. For the remaining points ($1 < j < M$), the space derivatives are obtained from central difference formulae.

In accordance with these finite difference approximations, instead of the continuous functions $f_0(x,t)$ and $h(x,t)$, we operate with the time-dependent vectors $[\mathbf{f_0}(t)]$ and $[\mathbf{h}(t)]$, and, instead of the partial differential equations, we have the system of the ordinary differential equations

$$
\begin{aligned}
&\mathbf{RA}(f_j, h_j, t) \frac{d\mathbf{f_0}}{dt} + \mathbf{FA}(f_j, h_j, t) = \mathbf{0}, \\
&\frac{d\mathbf{h}}{dt} + \mathbf{GA}(f_j, h_j, t) = \mathbf{0},
\end{aligned}
\tag{36}
$$

where **RA** is a square matrix, and **FA** and **GA** are vectors resulting from equations (27).

For the initial value problem considered, the equations in the first row of (36) depend on the boundary conditions at the boundary $x = x_g(t)$. For the displacement $x_g(t)$ of the generator plate, the tri-diagonal matrix **RA** in (36) is written in the form

$$
\mathbf{RA} = \begin{bmatrix}
K^\bullet & -\alpha^\bullet & & & & \\
-\alpha & K & -\alpha & & & \\
 & -\alpha & K & -a & & \\
 & & \cdot & \cdot & \cdot & \\
 & & & \cdot & \cdot & \cdot \\
 & & & & -\alpha & K
\end{bmatrix}, \tag{37}
$$

where $\alpha^{\bullet} = \frac{(h_1)^2}{a(a+b)}$, $K^{\bullet} = 1 + \frac{(h_1)^2}{ab}$, $\alpha = \frac{1}{2}\left(\frac{h_j}{a}\right)^2$, and $K = 1 + \left(\frac{h_j}{a}\right)^2$ with $j = 1, 2, \cdots, N$, and $b = a - x_g(t)$. Now, from equations (36), it follows that

$$\begin{aligned} \frac{d\mathbf{f_0}}{dt} &= \mathbf{FA}^{\times}, \\ \frac{d\mathbf{h}}{dt} &= \mathbf{GA}^{\times}, \end{aligned} \tag{38}$$

where $\mathbf{FA}^{\times} = -\mathbf{RA}^{-1} \cdot \mathbf{FA}$, and $\mathbf{GA}^{\times} = -\mathbf{GA}$.

It should be noted here that equations (38) are non-linear (the vectors $\mathbf{FA}^{\times}$ and $\mathbf{GA}^{\times}$ depend on the unknown vectors $\mathbf{f_0}$ and $\mathbf{h}$). Therefore, in order to find a solution of the equations, it will be necessary to resort to an approximate numerical integration in the time domain.

5. Numerical Solutions and Experiments in a Laboratory Flume

In order to learn more about this formulation and to estimate its applicability, in this section we shall integrate equations (38) for specific problems corresponding directly to experiments in a laboratory flume. In this way we shall estimate the accuracy of the approximate description of the non-linear problem considered. Let us then consider the initial-value problem of the generation of waves in a wave flume, as shown schematically in Fig. 2. The waves are generated by a piston-type wave maker (the rigid vertical wall in the figure), which starts to move at a certain point in time. The generated waves propagate over a rigid underwater obstacle (two inclined ramps with a segment of constant, small water depth) installed at a distance of 9 m from the generator plate. The motion of the wave maker is assumed in the following form (Wilde and Wilde 2001):

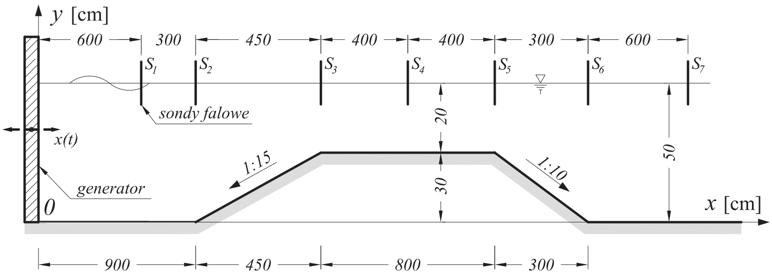

Fig. 2. Generation of waves in water of non-uniform depth. The vertical dashes indicate the distribution of wave gauges

$$x_g(t) = A_g \left[A(\tau)\cos(\omega t) + D(\tau)\sin(\omega t)\right], \tag{39}$$

where A_g is the amplitude of the generation, ω is the angular frequency, $\tau = \eta t$ is the non-dimensional time factor, η is a parameter responsible for a growth in time of the generator displacement, and the relevant terms in the square brackets are defined as follows:

$$
\begin{aligned}
A(\tau) &= \frac{1}{3!}\tau^3 \exp(-\tau), \\
D(\tau) &= 1 - \left(1 + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3\right) \exp(-\tau).
\end{aligned} \tag{40}
$$

On the basis of the latter equations, it is a simple task to calculate the generator velocity $\dot{x}_g(t)$ and its acceleration $\ddot{x}_g(t)$. One can check that for $\eta = 2$, assumed in our calculations, the generator motion approaches a steady-state harmonic motion within the first few periods of time. At the same time, all important parameters, i.e. displacement, velocity and acceleration of the generator face, are equal to zero at the initial point in time, i.e. at $t = 0^+$. For the assumed generator frequency ω, the length λ of the surface wave may be obtained from equation (30).

In order to find a solution of equations (38), we resort to a numerical integration of the equations in the time domain by the fourth order Runge-Kutta method. With this method, the intermediate parameters of the discrete integration read (Björk and Dahlquist 1983)

$$
\begin{aligned}
\mathbf{k}_1 &= t\mathbf{FA}^{\times}\left(t^n, \mathbf{f}_0^n, \mathbf{h}^n\right), \\
\mathbf{k}_2 &= t\mathbf{FA}^{\times}\left(t^n + \frac{t}{2}, \mathbf{f}_0^n + \frac{k_1}{2}, \mathbf{h}^n + \frac{\mathbf{l}_1}{2}\right), \\
\mathbf{k}_3 &= t\mathbf{FA}^{\times}\left(t^n + \frac{t}{2}, \mathbf{f}_0^n + \frac{k_2}{2}, \mathbf{h}^n + \frac{\mathbf{l}_2}{2}\right), \\
\mathbf{k}_4 &= t\mathbf{FA}^{\times}\left(t^n + t, \mathbf{f}_0^n + \mathbf{k}_3, \mathbf{h}^n + \mathbf{l}_3\right),
\end{aligned} \tag{41}
$$

and

$$
\begin{aligned}
\mathbf{l}_1 &= t\mathbf{GA}^{\times}\left(t^n, \mathbf{f}_0^n, \mathbf{h}^n\right), \\
\mathbf{l}_2 &= t\mathbf{GA}^{\times}\left(t^n + \frac{t}{2}, \mathbf{f}_0^n + \frac{k_1}{2}, \mathbf{h}^n + \frac{\mathbf{l}_1}{2}\right), \\
\mathbf{l}_3 &= t\mathbf{GA}^{\times}\left(t^n + \frac{t}{2}, \mathbf{f}_0^n + \frac{k_2}{2}, \mathbf{h}^n + \frac{\mathbf{l}_2}{2}\right), \\
\mathbf{l}_4 &= t\mathbf{GA}^{\times}\left(t^n + t, \mathbf{f}_0^n + \mathbf{k}_3, \mathbf{h}^n + \mathbf{l}_3\right),
\end{aligned} \tag{42}
$$

where $t = t^{n+1} - t^n$ denotes the time step length.

The solution at the successive point in time, i.e. at t^{n+1}, is obtained from the formulae

$$
\begin{aligned}
\mathbf{f}_o^{n+1} &= \mathbf{f}_o^n + \frac{1}{6}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right), \\
\mathbf{h}^{n+1} &= \mathbf{h}^n + \frac{1}{6}\left(\mathbf{l}_1 + 2\mathbf{l}_2 + 2\mathbf{l}_3 + \mathbf{l}_4\right).
\end{aligned} \tag{43}
$$

Numerical calculations were performed for the initial problem of the generation of waves in a finite fluid domain of length $L = 60$ m and depth $h = 0.5$ m. In the discrete approach, the horizontal length was replaced with equally spaced nodal points. The distance between consecutive points of the net was chosen to be $a = 0.1$ m. At the same time, the distance between the first nodal point and the generator face depended on time $b(t) = a - x_g(t)$. In this way, instead of a continuous description in space, we have a description of a finite set of $N = 600$ nodal points in time. In the Runge-Kutta numerical integration it is assumed that the time step is $\Delta t = 0.01$ s. This time step satisfies the Courant condition (Toro 1997), which requires that the ratio of the wave celerity to the 'net velocity' be less than one, i.e. $c_f \Delta t / a < 1$.

With respect to the initial-value problem considered and the assumed condition that the fluid is at rest at the boundary $x = L$, one can obtain the maximum time allowed in the numerical integration. As already mentioned, in addition to numerical calculations, experiments in a laboratory flume were carried out. The experiments were conducted in a wave flume of the Institute of Hydro-Engineering of the Polish Academy of Sciences in Gdańsk. The wave channel of a rectangular cross-section (0.6 m wide and 1.4 m high) with glass sides is 64 m long. At one end of the channel a piston-type wave generator is installed. At the opposite end, there is an inclined artificial ramp, which absorbs the energy of incoming waves. The experiments were conducted for an assumed set of amplitudes and generation frequencies. Some of the results obtained in numerical computations and recorded in the experiments are presented in Figures 3, 3a, 3b. The figures show the evolution in time of surface waves calculated and recorded in the experiments at selected points of the hydraulic flume. The consecutive figures correspond to waves of growing lengths.

The plots in Figure 3 correspond to the generation frequency $\omega = 2.6201$ s^{-1} and the generator displacement amplitude $A_g = 2.64$ cm. The first plot in the figure illustrates the generator motion with the solid line showing experimental results obtained in the laboratory flume and the dashed line representing the theoretical solution. In the subsequent plots theoretical results are compared with data obtained in the experiments. From the graphs it may be seen that changes in the water depth induce significant changes in the surface elevation, where emerging higher components become more important (gauges S4, S5 and S6 in the figure). The comparison shows that the theoretical model provides reliable, fairly accurate results. Fig. 3a illustrates the case of wave generation with $\omega = 1.9252$ s^{-1} and the generator amplitude $A_g = 4.62$ cm. The last figure (3b) shows the water elevation corresponding to the frequency $\omega = 1.5159$ s^{-1} and the generator amplitude $A_g = 6.54$ cm.

From the plots in all figures, it can be seen that the wave phases obtained in numerical solutions are practically equal to those measured in the laboratory experiments. Small discrepancies between the theory and the experiments seem to result from the hydraulic system steering the generator motion. The highest elevations of the waves occur in the area of the smallest water depth. It may also be seen that the results of computations exceed the levels of waves measured in the

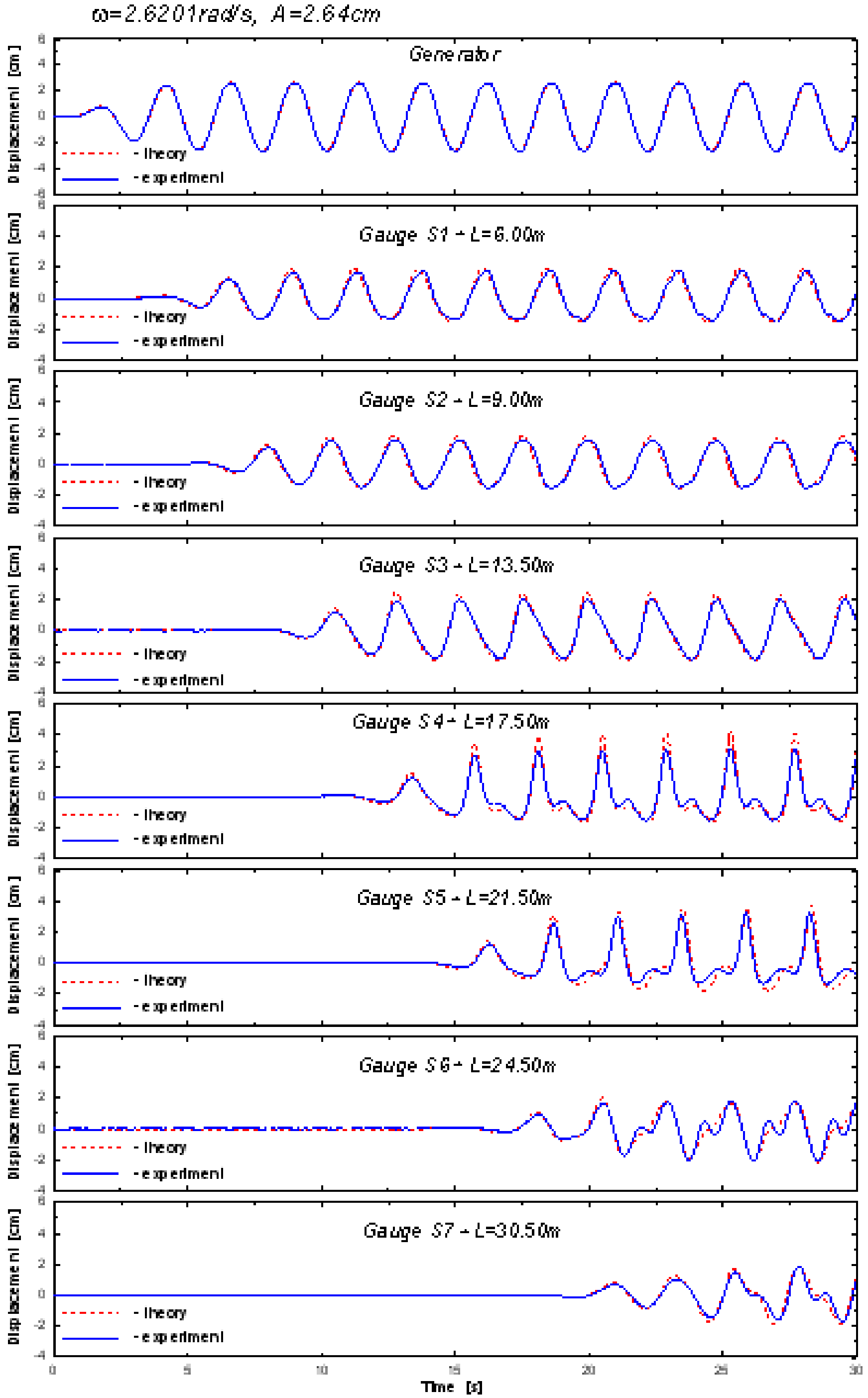

Fig. 3. Comparison of theoretical results with data recorded in experiments

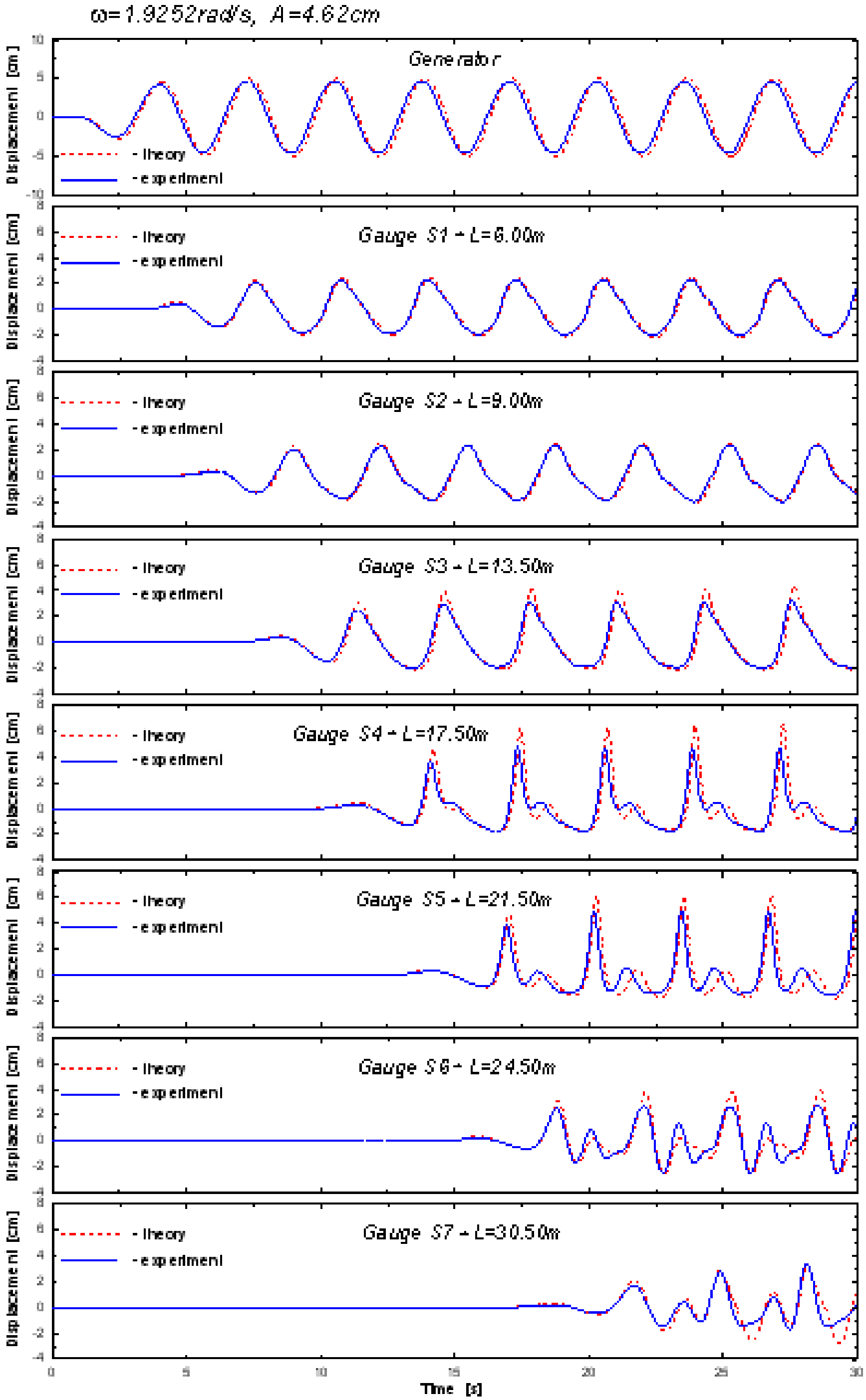

Fig. 3a. Continued

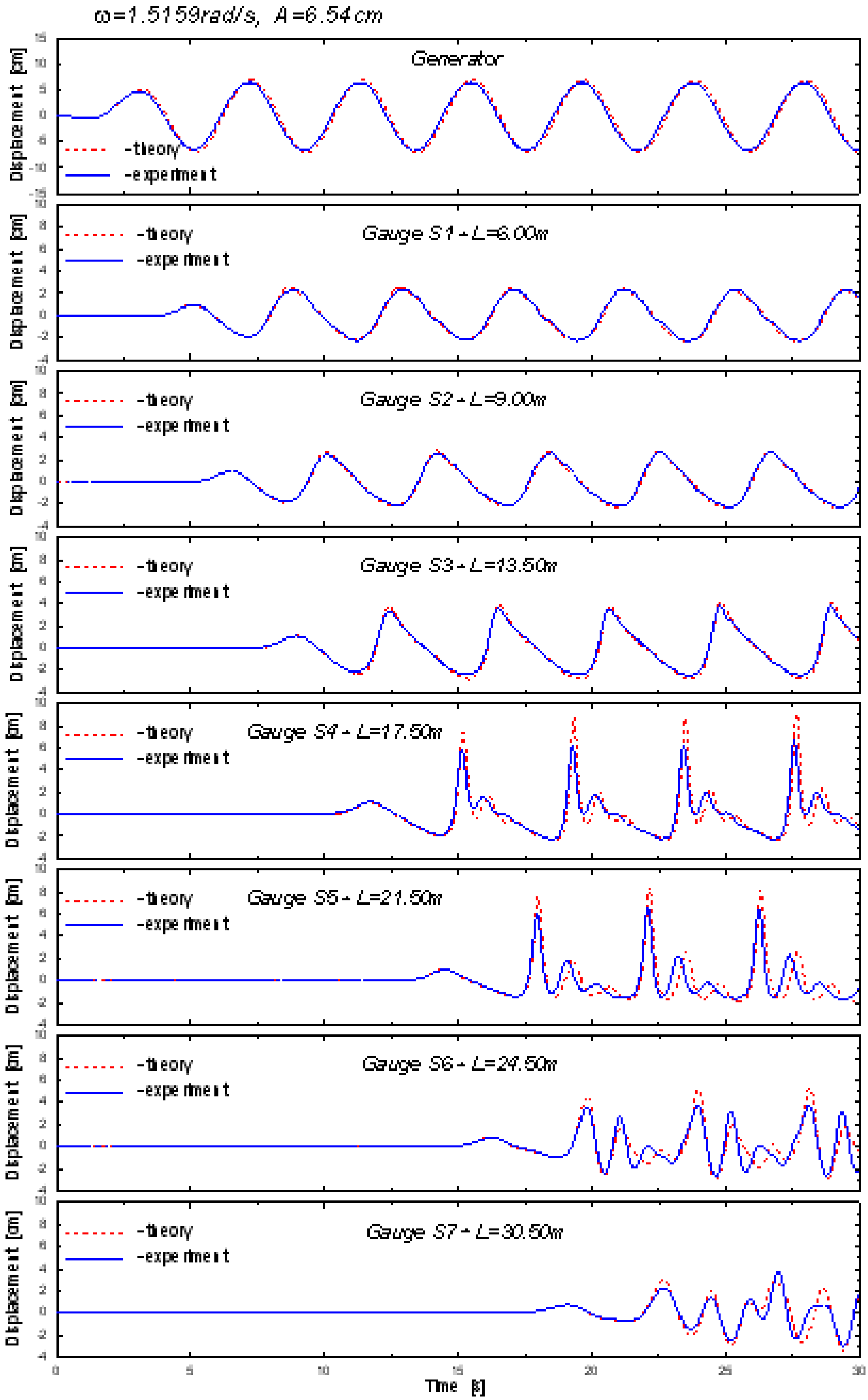

Fig. 3b. Continued

laboratory experiments, which is due to the lack of a dissipation mechanism in the theoretical model developed above.

6. Concluding Remarks

The theoretical description of long waves propagating in water of variable depth, presented above, is based on the fundamental assumption that the waves may be accurately described by means of a few lowest order terms of the power series expansion with respect to the water depth. The final differential equations derived correspond to the conservation of mass and momentum of a vertical column of water. A validation of the proposed theoretical model was performed by comparing its results with data obtained in experiments carried out in a laboratory flume. The comparison shows that the formulation describes well the main features of waves propagating in water of insignificantly variable depth. In a rather formal way, it is possible to take into account higher order terms in the formulation. With these terms, however, one can face more difficulties in integrating resulting differential equations of the problem considered.

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