

# On Vertical Variations of Wave-Induced Radiation Stress Tensor

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## Abstract

The flux of momentum generated by an incident wave field, commonly known as the radiation stress, plays an important role in near-shore water circulation. Many researchers use the concept of radiation stress in the calculation of cross-shore and long-shore circulation. In this paper, the traditional concept is extended to the case of vertical variation of radiation stress, and analytical expressions for the vertical profile of radiation stress are derived. The distributions of the wave-induced radiation stress tensor with depth are studied by linear wave theory. The application of radiation stress with vertical variation is expected to play an important role in further studies of the near-shore system.

Information regarding the vertical distribution of the radiation stress components ( $S_{xx}$ ,  $S_{yy}$  and  $S_{xy}$ ) resulting from obliquely incident, shoaling waves is provided. The results show that the vertical variations of the wave-induced radiation stress tensor are significant as regards of wave propagation.

**Key words:** radiation stress, linear water waves, three-dimensional flow

## 1. Introduction

The near-shore coastal region is a dynamic and complex system. The long-shore and cross-shore currents formed by incident waves, along with the coupling that takes place between the incident wave field and the currents, generate the near-shore water circulation. The concept of radiation stress is usually used to identify the appropriate forcing mechanisms that initiate cross-shore and long-shore circulation. Usually, the two-dimensional case is considered, and the depth-integrated value of radiation stress is taken into account.

The term “radiation stress” describes the excess flow of momentum due to propagating water waves (Longuet-Higgins and Stewart 1964, Longuet-Higgins 1970). Radiation stresses are the forces per unit area that arise because of the excess momentum flux due to the presence of waves. Also, the resulting analytical expressions presented in Longuet-Higgins and Stewart (1964) are depth-integrated quantities that ignore potentially important vertical information. While depth-averaged

near-shore current models are still widely used today, advancements in technology have permitted the adaptation of three-dimensional modeling techniques to the study of flow properties of complex near-shore circulation systems.

Traditionally, linear wave theory is used to approximate radiation stresses. This theory is generally sufficient to explain such phenomena as the wave set-down and set-up, and the generation of long-shore and cross-shore currents.

The simulation of near-shore currents with three-dimensional numerical models is gaining popularity (e.g. Tsanis et al 1996, Ezer and Mellor 2004, Blumberg and Mellor 1987). Recently, analytical expressions describing the vertical structure of radiation stress components have appeared in literature (see Mellor 2003, Xia et al 2004, Grusza 2007).

Considering the importance of the vertical structure of radiation stress, in the present paper the concept of traditional radiation stress is extended to its vertical profile. A simple method for calculating vertical distribution of radiation stress components ( $S_{xx}$ ,  $S_{xy}$  and  $S_{yy}$ ) resulting from obliquely incident, linear shoaling waves is proposed. The main idea is the same as that used by Dolata and Rosenthal (1984), but the accurate expression for a pressure function is used. This feature distinguishes results of the work presented here from the expressions obtained by Mellor (2003).

## 2. Theoretical Description

The calculation of depth-dependent formulae of the radiation stress will be carried out by employing the theory of small-amplitude waves. It is assumed that the  $x$  axis is located at mean water level (MWL), with its positive direction coinciding with the wave propagation direction; the vertical  $z$ -axis points upwards, and  $h$  denotes the water depth below MWL. The surface elevation and horizontal and vertical velocities are given by:

$$\begin{aligned} \eta &= a \cos(kx - \omega t), \\ u &= a\omega \cos(kx - \omega t) \frac{\cosh(k(h+z))}{\sinh(kh)}, \\ w &= a\omega \sin(kx - \omega t) \frac{\cosh(k(h+z))}{\sinh(kh)}, \end{aligned} \quad (1)$$

where  $t$  is time,  $u$  and  $w$  are the velocity components in  $x$  and  $z$  directions, respectively;  $k = 2\pi/L$  is the wavenumber,  $L$  is the wavelength;  $\omega = 2\pi/T$  is the angular frequency,  $T$  is the period;  $a$  is the wave amplitude, and  $\rho$  is the density of water. The radiation stress defined by Longuet-Higgins and Stewart (1964) is expressed as:

$$S_{xx} = \overline{\int_{-h}^{\eta} (p + \rho u^2) dz} - \int_{-h}^0 (p_0) dz, \quad (2)$$

where  $p$  is pressure and the overbar denotes an operation of averaging over a wave period defined by

$$\overline{\int_{-h}^{\eta} (p + \rho u^2) dz} = \frac{1}{T} \int_0^T \left( \int_{-h}^{\eta} (p + \rho u^2) dz \right) dt. \quad (3)$$

If a particle of water at rest is located at the depth of  $z = z_0$ , then at time  $t$  it has a position on the line  $z_0 + W(x, z_0, t)$ , where  $W$  is a function given by

$$W = \frac{a \sinh(k(z_0 + h)) \cos(kx - \omega t)}{\sinh(kh)}. \quad (4)$$

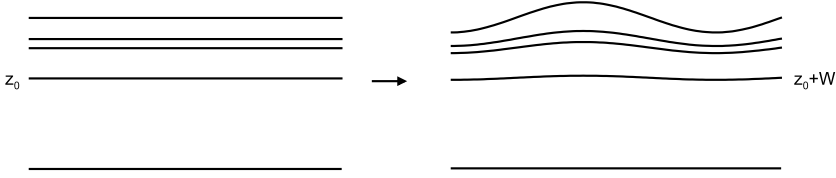
We consider the excess of flow of momentum due to propagating water waves of the lower layer of water, which at rest is described by the inequalities  $-h < z < z_b$ . In accordance with Eq. (2), we have

$$P_{xx}(z_b) = \overline{\int_{-h}^{z_b+W(x,z_b,t)} (p + \rho u^2) dz} - \int_{-h}^{z_b} (p_0) dz. \quad (5)$$

If we consider the entire layer of liquid, we obtain exactly the same expression as Lonquet-Higgins and Stewart (1964). We define the function  $S_{xx}(z)$  as the derivative of the function  $P_{xx}(z)$ :

$$S_{xx}(z) = \frac{\partial P_{xx}(z)}{\partial z}. \quad (6)$$

We recall that “radiation stress”, as defined by Lonquet-Higgins and Stewart (1964) is actually not a “stress” (force per unit area) but a depth integrated stress, the density (vertical profile)  $S_{xx}(z)$  defined by Eq. (6) is however a “true” radiation stress. The dimension of the density of radiation stress  $S_{xx}(z)$  has indeed the dimension of the stress (force per unit area). Comparison of the expressions of (2), (5) and (6), as well as their physical meanings, shows that  $S_{xx}(z)$  is a natural extension of the traditional radiation stress. Integration of the density radiation stress from  $-h$  to  $z_b$  immediately gives Eq. (5). The physical meaning is as follows: the radiation stress of a thin layer of fluid, at rest, described by the inequalities  $z_0 < z < z_0 + dz$ , is associated with momentum transport of the same particles of water, but their actual locations are taken into account (Fig. 1). The concept of calculating the function of the density of the radiation stress is similar to that used by Xia et al (2004), but there is a significant difference between the two methods: in the present work the



**Fig. 1.** Physical meaning of the radiation stress – positions of particles which at rest have positions at the line  $z = z_0$

real locations of water particles are taken into account. Based on the method used by Xia et al (2004), we use the expression for the function of pressure, in which all nonlinear terms are considered. The wave movement satisfies the general fluid dynamics equations, and the Euler equation of motion in the vertical direction has the form:

$$\rho \left( \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} u + \frac{\partial w}{\partial z} w \right) + \frac{\partial p}{\partial z} + g = 0. \quad (7)$$

Integrating this equation with respect to the vertical variable  $z$  from  $z$  to  $\eta$ , and using the continuity equation, one obtains:

$$\int_z^\eta \rho \left( \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} u \right) dz + \frac{1}{2} (w^2(\eta) - w^2(z)) + p(\eta) - p(z) + g(\eta - z) = 0. \quad (8)$$

From the above equation we can determine the pressure function and, because the pressure at the free surface is equal to zero, we obtain

$$p(z) = \frac{1}{2} (w^2(\eta) - w^2(z)) + g(\eta - z) + \int_z^\eta \rho \left( \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} u \right) dz. \quad (9)$$

We notice that the integral in the above equation can be calculated analytically. Using equation (1), this integral is expressed as:

$$I_1 = \int_z^\eta \rho \frac{\partial w}{\partial t} dz = \frac{\rho a \omega^2 \cosh(k(z+h))}{k \sinh(kh)} \cos(kx - \omega t) + \frac{\rho a \omega^2 \cosh(k(\eta+h))}{k \sinh(kh)} \cos(kx - \omega t), \quad (10)$$

and

$$I_2 = \int_z^\eta \rho \frac{\partial w}{\partial x} u dz = \frac{1}{4} \frac{\rho a^2 \omega^2 \cosh(2k(\eta + h))}{\sinh^2(kh)} \cos^2(kx - \omega t) + \quad (11)$$

$$-\frac{1}{4} \frac{\rho a^2 \omega^2 \cosh(2k(z + h))}{\sinh^2(kh)} \cos^2(kx - \omega t).$$

Using equations (9), (10) and (11), we can calculate expression (9)

$$P_{xx}(z_b) = \frac{\rho a^2 g}{16 \sinh^2(kh) \cosh(kh)} \times \quad (12)$$

$$\times (2 \cosh(kh) - 3 \cosh(k(h + 2z_b)) + \cosh(k(3h + 2z_b))) +$$

$$+ 8k(h + z_b) \sinh(kh) + O(a^3).$$

By setting  $z_b = 0$  in the above equation, we obtain the integrated radiation stress. In this case Eq. (12), this simplifies to the following relation

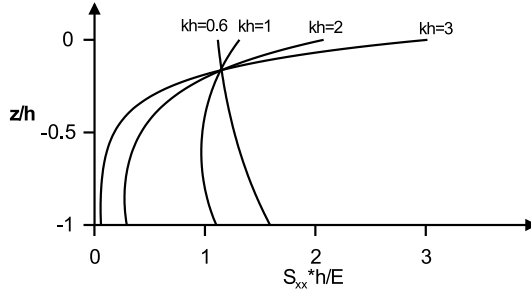
$$P_{xx}(0) = \frac{\rho A^2 g}{4} \left( 1 + \frac{4kh}{\sinh(2hk)} \right). \quad (13)$$

Of course, the above expression is consistent with the result obtained by Longuet-Higgins and Stewart (1964). It can be seen again that the proposed definition is the extension of traditional radiation stress. Differentiation of the function  $P_{xx}(z)$  with respect to the vertical co-ordinate yields:

$$S_{xx}(z) = \frac{\rho A^2 g k}{8 \sinh^2(kh) \cosh(kh)} \times \quad (14)$$

$$(4 \sinh(kh) - 3 \sinh(k(2z + h)) + \sinh(k(2z + 3h))).$$

Fig. 2 shows the vertical variation of  $S_{xx}(z)$  for different values of the parameter  $kh$ . It is seen that when  $kh$  is large, the radiation stress reaches its maximum value at the free surface and decreases rapidly with depth. This agrees with the attenuation rule for deep-water waves: velocity and the dynamic pressure decrease with depth. For small values of  $kh$ , the density of radiation stress is at its maximum at the bottom. This variability of radiation stress is similar to that obtained by Xia et al (2004). However, there are small differences: in contrast to the results obtained by the latter authors, values of the radiation stress are always positive. The same method can be followed to calculate the radiation stress in the normal direction. In



**Fig. 2.** Vertical variation of of the radiation stress density  $S_{xx}(z)$  for different values of the parameter  $kh$

this case the excess of the flow of momentum due to the propagating water waves of the lower water layer results in:

$$P_{yy}(z_b) = \overline{\int_{-h}^{z_b+W(x,z_b,t)} p dz} - \int_{-h}^0 (p_0) dz, \quad (15)$$

where, as before,  $W(x, z_b, t)$  denotes the vertical displacement at the level  $z = z_b$ .

The final result of calculations can be simplified to:

$$P_{yy}(z) = \frac{1}{8} \frac{\rho a^2 g}{\sinh^2(kh) \cosh(kh)} \times \quad (16)$$

$$\times (\cosh(kh) - \cosh(k(h+2z)) + 2k(h+z) \sinh(kh)),$$

and the corresponding density of the radiation stress is given by

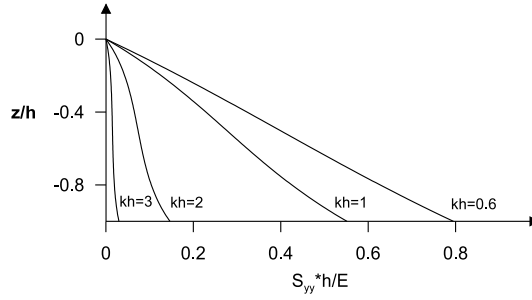
$$S_{yy}(z) = \frac{\rho a^2 g k}{4 \sinh^2(kh) \cosh(kh)} (\sinh(hk) - \sinh(k(h+2z))). \quad (17)$$

The vertical variation of  $S_{yy}(z)$  is shown in Fig. 3. For all values of  $kh$  the density of the radiation stress is at its maximum at the bottom. For shorter waves, the values of  $S_{yy}(z)$  are small. When there is an angle  $\theta$  between the  $x$  axis and the propagation direction of the wave, we can use the general rule of tensor transformation. Hence, the density of the radiation stress can be written as:

$$S_{xx}(z) = \frac{1}{2}(S_{rr} + S_{nn}) + \frac{1}{2}(S_{rr} - S_{nn}) \cos(2\theta) =$$

$$= \frac{a^2 g k}{8 \cosh(kh) \sinh(kh)} (\cos(2\theta) + 3) + \frac{a^2 g k}{16 \sinh^2(kh) \cosh(kh)} \times \quad (18)$$

$$\times (2 \cos^2(\theta) \sinh(k(3h+2z)) - (\cos(2\theta) + 5) \sinh(k(h+2z))),$$



**Fig. 3.** Vertical variation of the radiation stress density  $S_{yy}(z)$  for different values of the parameter  $kh$

$$\begin{aligned}
 S_{yy}(z) &= \frac{1}{2}(S_{rr} + S_{nn}) - \frac{1}{2}(S_{rr} - S_{nn}) \cos(2\theta) = \\
 &= \frac{a^2 g k}{16 \cosh(kh) \sinh(kh)} (6 - \cos(2\theta)) + \frac{a^2 g k}{16 \sinh^2(kh) \cosh(kh)} \times \\
 &\times (2 \sinh(k(3h + 2z)) \sin^2(\theta) + (\cos(2\theta) - 5) \sinh(k(h + 2z))),
 \end{aligned} \quad (19)$$

$$\begin{aligned}
 S_{xy}(z) &= S_{yx}(z) = \frac{1}{2}(S_{rr} - S_{nn}) \sin(2\theta) = \\
 &= \frac{a^2 g k}{2 \sinh(2kh)} \cosh^2(k(h + z)) \sin(2\theta),
 \end{aligned} \quad (20)$$

where  $S_{rr}(z)$  and  $S_{nn}(z)$  are components of the radiation stress in the direction of the wave propagation and normal direction given by the right-hand side of equations (15) and (18), respectively.

### 2.1. Forces Induced by Radiation Stress

More interesting are the vertical variations in the forces resulting from radiation stress. Wave induced forces are dependent on the radiation stress gradients:

$$F_x = \frac{\partial S_{xy}}{\partial y} + \frac{\partial S_{xx}}{\partial x}, \quad (21)$$

$$F_y = \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y}. \quad (22)$$

We consider the simple situation in which the slope of the bottom is gentle and the isobaths are parallel to the shore line  $x = 0$ . Let us assume that  $k_0$ ,  $a_0$  and  $\alpha_0$  denote the wavenumber, wave amplitude and the angle between a wave crest and a local isobath in deep water, respectively.

In such a situation, at each point the local wave number  $k$ , local amplitude  $a$  and the angle  $\alpha$  between a wave crest and a local isobath can be determined from the following relation:

$$\omega^2 = g k(x) \tanh(h(x) k(x)). \quad (23)$$

In the situation considered here of long-crested, harmonic waves approaching a straight beach (with parallel bottom contours), the direction of the wave is governed by the well-known Snell's law, by which, along the wave ray (the orthogonal to the wave crests) the relation holds

$$k \sin(\theta) = C_0, \quad (24)$$

where  $C_0$  is a constant dependent on the wavelength and the direction of wave propagation in deep water. Additionally,

$$a = a_0 \sqrt{\frac{C_{g0}}{C_g}} \sqrt{\frac{\cos(\theta_0)}{\cos(\theta)}}, \quad (25)$$

where  $\theta_0$ ,  $C_{g0}$  and  $a_0$  denote the angle between a wave crest and a local isobath, group velocity, and amplitude in deep water, respectively. The coefficients  $\sqrt{C_{g0}/C_g}$  and  $\sqrt{\cos(\theta_0)/\cos(\theta)}$  are named the shoaling coefficient and refraction factor.

In order to calculate the wave induced forces in the water of depth  $h$ , for a given frequency, the amplitude and the angle  $\alpha_0$  between the  $x$  axis and the propagation direction of the wave, the following methodology has been applied:

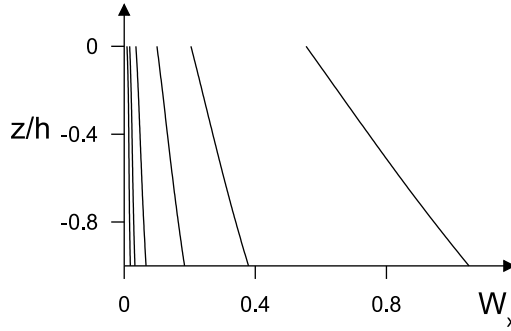
1. From dispersion relation (23) the local wave number is calculated.
2. From Snell's law (24) the angle  $\alpha$  between the propagation direction of the wave and the local isobath is determined.
3. From relation (25) the value of the wave amplitude is calculated.
4. The values of the radiation stress at two different nearby points are calculated.
5. Using a finite difference scheme and equations (21)–(22), the functions of wave-induced forces are obtained. These functions are proportional to the density of total energy  $E_0$  and bottom slope  $h'(x)$ , and can be expressed as:

$$\begin{aligned} F_x &= -E_0 h'(x) W_x(\alpha_0, k_0, h), \\ F_y &= -E_0 h'(x) W_y(\alpha_0, k_0, h), \end{aligned} \quad (26)$$

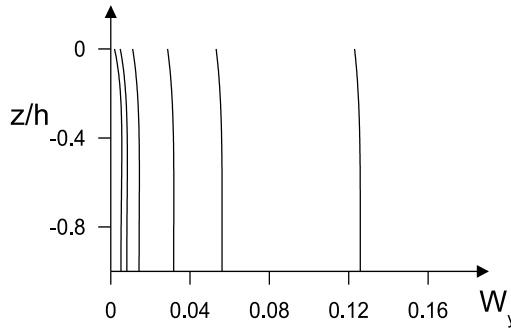
where  $\alpha_0$  and  $k_0$  denote the angle and wavenumber in deep water, respectively.

The vertical variation of  $W_x$  for non-breaking waves is presented in Fig. 4 for the following wave parameters: the angle  $\alpha_0 = \pi/4$  and the wave number  $k_0 = 1/4$ .





**Fig. 4.** Vertical variation of  $W_x$ . The lines (from left to right) denote values of  $W_x$  at depth 5, 4, 3, 2, 1.5 and 1 m, respectively



**Fig. 5.** Vertical variation of  $W_y$ . The lines (from left to right) denote values of  $W_y$  at depth 5, 4, 3, 2, 1.5 and 1 m, respectively

The variations of the wave-induced forces in the  $x$ -direction are almost linear. In each case, the force reaches its maximum at the bottom and the minimum at the free surface. The slope of the curve changes rapidly in the shallow water zone.

The wave induced force in the  $y$ -direction is smaller by one order than along the shore. The vertical variation of  $W_y$  is shown in Fig. 5.

As in the previous case, the force reaches its minimum at the free surface, though its vertical variation is very small.

### 3. A Note on Radiation Stress for Nonlinear Waves

We now consider the case in which nonlinear waves are taken into account. In place of equation (1) we should use the following expressions:

$$u = a\omega \frac{\cosh(k(h+z))}{\sinh(hk)} \cos(kx - \omega t) + a^2 w_u(z) \cos(2kx - 2\omega t) + O(a^3), \quad (27)$$

$$w = a\omega \sin(kx - \omega t) \frac{\cosh(k(h+z))}{\sinh(hk)} + a^2 w_w(z) \sin(2kx - 2\omega t) + O(a^3), \quad (28)$$

$$\eta = a \cos(kx - \omega t) + a^2 w_\eta \cos(2kx - 2\omega t) + O(a^3), \quad (29)$$

where the functions  $w_u(z)$ ,  $w_w(z)$  and the expression for  $w_\eta$  can be found in, e.g., Wehausen and Laitone (1960). For our purposes the detailed expressions are not needed. We should also modify the expression for the function  $W(x, y, t)$  defined by Eq. (4):

$$W = \frac{a \sinh(k(z_0 + h)) \cos(kx - \omega t)}{\sinh(kh)} + a^2 w_W(z) \cos(2kx - 2\omega t) + O(a^3). \quad (30)$$

It is easy to notice that the non-linear terms do not influence the final result followed from the term  $u^2$  in Eq. (2). Similarly, the pressure function defined by (9) has a general form and the non-linear terms do not affect the radiation stress. Hence, taking into consideration the non-linear terms does not change the results, obtained in Section 2.

#### 4. Conclusion

In this paper, the original concept of radiation stress in water waves has been extended to describe its vertical variation. A clear definition of density of radiation stress and the simple calculation formulae for the vertical profile of radiation stress are obtained. These can be used successfully in the analysis of 3-D wave-current coupling effects. The calculation shows that:

1. The long-shore force is greater by one order of magnitude than the cross-shore force.
2. The long-shore force can be assumed to be independent of the vertical co-ordinate.
3. The cross-shore force reaches its maximum value at the bottom.
4. In deeper water the long-shore and cross-shore forces are small and can be assumed to be independent of the vertical co-ordinate.
5. The obtained expressions for vertical variations of the radiation stress hold true for non-linear waves as well.

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