

# On the ambiguity function for accelerating target in FMCW radar

Rafał Rytel-Andrianik

**Abstract**— In the paper, we are concerned with FMCW radar detection of an accelerating target, echo of which is buried in an additive white Gaussian noise. We derive and analyze three-dimensional generalized ambiguity function for target range, velocity and acceleration. We interpret known properties of this function and obtain new ones, which allows us to specify resolutions and regions of unambiguity for range, velocity and acceleration. The obtained resolutions we express in terms of corresponding Cramer-Rao bounds.

**Keywords**— FMCW radar, detection, ambiguity function.

## 1. Introduction

In the paper, we are concerned with a linearly frequency modulated continuous wave radar (L-FMCW radar) that transmits  $T$ -periodic constant amplitude signal with frequency linearly rising in each period. If we denote the carrier frequency by  $f_c$  and frequency modulation slope as  $\alpha$ , the instantaneous frequency of a transmitted signal is  $f_c + \alpha t$  for one period  $-T/2 < t < T/2$ , and frequency deviation is  $\alpha T$ . An illuminated target at range  $r(t)$  backscatters the transmitted signal to the radar, where it is received and mixed with a copy of the transmitted signal.

The obtained beat signal is block processed, each block corresponds to  $2K + 1$  modulation periods. We assume in the paper, that within the time  $(2K + 1)T$  of a signal block (the measurement time or coherent integration time), the illuminated target is moving with constant acceleration with respect to the radar, thus the range is:

$$r(t) = r_0 + v_0 t + 0.5 a t^2 \quad (1)$$

during the measurement time

$$-(2K + 1)T/2 < t < (2K + 1)T/2. \quad (2)$$

In the range Eq. (1), the parameter  $r_0 = r(0)$  is target range in the middle of the measurement time Eq. (2),  $v_0$  is target radial velocity for  $t = 0$ , and  $a$  is target radial acceleration.

The beat signal is sampled with sampling frequency  $f_s$  omitting transients at the beginning of each modulation period. We denote the obtained discrete beat signal as  $y(m, k)$  for discrete time within each modulation period  $m = -M, \dots, M$  and modulation periods  $k = -K, \dots, K$ . This signal is sum of the useful component  $Ae^{j\phi_0}x(m, k)$  (that is “template”  $x(m, k)$  and complex amplitude  $Ae^{j\phi_0}$ ) and

an additive complex circular white Gaussian noise  $n(m, k)$  of variance  $\sigma^2$ :

$$y(m, k) = Ae^{j\phi_0}x(m, k) + n(m, k) \quad (3a)$$

for

$$k = -K, \dots, K \quad \text{and} \quad m = -M, \dots, M \quad (3b)$$

Neglecting range walk, the template  $x(m, k)$  of the useful signal can be approximated (see [3, 5]) as:

$$x(m, k) = \exp\{j(\theta_r m + b_v k + b_a k^2)\}, \quad (4)$$

where the parameters  $\theta_r, b_v, b_a$  are normalized range, normalized velocity, and normalized acceleration, respectively:

$$\begin{aligned} \theta_r &= 2\pi \frac{2\alpha}{c f_s} r_0, \\ b_v &= 2\pi \frac{2f_c T}{c} v_0, \\ b_a &= 2\pi \frac{f_c T^2}{c} a, \end{aligned} \quad (5)$$

where  $c$  is speed of light. From Eqs. (4) and (5) we see that the beat signal is (a) linear-phase with respect to the “fast time”  $m$ , and (b) quadratic-phase with respect to modulation period index, or “slow time”,  $k$ .

In order to use a vector notation, we define the vector  $\mathbf{y}$  containing all samples of the measured beat signal:

$$\mathbf{y} = [y(-M, -K), \dots, y(M, -K), \dots, \dots, y(-M, K), \dots, y(M, K)]. \quad (6)$$

Analogously, we define vector  $\mathbf{x}$  of the useful signal template Eq. (4), and vector  $\mathbf{n}$  of a noise component. The norm of vector  $\mathbf{x}$  is  $\|\mathbf{x}\|^2 = (2M + 1)(2K + 1)$ .

## 2. Detection and ambiguity function

In the detection problem we need to decide if the target echo is present in the received signal (hypothesis  $H_1$ :  $\mathbf{y} = Ae^{j\phi_0}\mathbf{x} + \mathbf{n}$  as in Eq. (3)) or target echo is not present (hypothesis  $H_0$ :  $\mathbf{y} = \mathbf{n}$ ). For the Neyman-Pearson criterion [4], we do the optimal test by calculating the test statistic  $D$  defined as:

$$D = \frac{1}{\|\mathbf{x}\|^2 \sigma^2} |\mathbf{x}^H \mathbf{y}|^2 \quad (7)$$

and compare it with a threshold  $\gamma$ . If the threshold is exceeded we decide “target present” ( $H_1$ ), if not we decide otherwise ( $H_0$ ), that is:

$$D \underset{H_0}{\overset{H_1}{\geq}} \gamma. \quad (8)$$

The threshold is a function of designed probability of false alarm:  $\gamma = -\ln P_{fa}$ .

Calculation of the test statistic  $D$  (Eq. (7)) requires knowledge of  $\mathbf{x}$  and, according to Eq. (4), knowledge of target parameters  $r_0, v_0, a$ . If these parameters are unknown, then we do multiple tests for a discrete set of hypothetical target motion parameters, specified on a certain grid. The question that here appears is how dense and how wide should this grid be. In order to answer this question, we use the ambiguity function concept.

We denote unknown true normalized target parameters as  $[\theta_r^+, b_v^+, b_a^+]$ , and useful signal template corresponding to these parameters as  $\mathbf{x}(\theta_r^+, b_v^+, b_a^+)$ . Using this notation and assuming that the target is present, we have  $\mathbf{y} = Ae^{j\theta_0} \mathbf{x}(\theta_r^+, b_v^+, b_a^+) + \mathbf{n}$ , and the detection statistic Eq. (7), calculated for hypothetical target parameters  $[\theta_r, b_v, b_a]$ , is

$$D(\theta_r, b_v, b_a) = \frac{1}{\|\mathbf{x}\|^2 \sigma^2} \left| \mathbf{x}^H(\theta_r, b_v, b_a) (Ae^{j\theta_0} \mathbf{x}(\theta_r^+, b_v^+, b_a^+) + \mathbf{n}) \right|^2. \quad (9)$$

Defining the ambiguity function as:

$$H(\theta_r - \theta_r^+, b_v - b_v^+, b_a - b_a^+) = \left| \frac{\mathbf{x}^H(\theta_r^+, b_v^+, b_a^+) \mathbf{x}(\theta_r, b_v, b_a)}{\|\mathbf{x}\|^2} \right|^2 \quad (10)$$

and signal to noise ratio as  $\text{SNR} = A^2 \|\mathbf{x}\|^2 / \sigma^2$ , we may rewrite the test statistic Eq. (9) as:

$$D(\theta_r, b_v, b_a) = \left| \sqrt{\text{SNR} \cdot H(\theta_r - \theta_r^+, b_v - b_v^+, b_a - b_a^+) + n_1} \right|^2, \quad (11)$$

where  $n_1$  represents complex random value with Gaussian pdf  $CN(0, 1)$ . The last equation means that probability of detection depends only on SNR, shape of the ambiguity function, and how close the hypothetical parameters  $[\theta_r, b_v, b_a]$  are to the unknown true parameters  $[\theta_r^+, b_v^+, b_a^+]$ . The width of the main lobe of the ambiguity function tells us how dense should the hypothetical parameter grid be (radar resolution) and period of this function specifies size of the grid (region of unambiguous parameters). In next section we analyze the ambiguity function Eq. (10).

### 3. Analysis of the ambiguity function

Since in Eq. (11) we have only differences  $\theta_r - \theta_r^+, b_v - b_v^+$  and  $b_a - b_a^+$ , we may assume for simplicity that true target parameters are all zeros, that is  $[\theta_r^+, b_v^+, b_a^+] = [0, 0, 0]$ . According to Eq. (4), we can rewrite the ambiguity function as:

$$H(\theta_r, b_v, b_a) = \left| \frac{1}{(2M+1)(2K+1)} \sum_{m=-M}^M \sum_{k=-K}^K \exp\{j\theta_r m\} \times \exp\{j(b_v k + b_a k^2)\} \right|^2 = H_r(\theta_r) \cdot H_b(b_v, b_a),$$

where

$$H_r(\theta_r) = \left| \frac{1}{2M+1} \sum_{m=-M}^M \exp\{j\theta_r m\} \right|^2 \quad (12)$$

and

$$H_b(b_v, b_a) = \left| \frac{1}{2K+1} \sum_{k=-K}^K \exp\{j(b_v k + b_a k^2)\} \right|^2. \quad (13)$$

We see, that the ambiguity function is a product of function  $H_r(\theta_r)$  dependant only on range, and function  $H_b(b_v, b_a)$ , dependent only on the movement parameters  $b_v, b_a$ . Thus, radar movement characteristics are, in this sense, independent of range characteristics. We will call functions  $H_r(\theta_r)$  and  $H_b(b_v, b_a)$  *range ambiguity function* and *movement ambiguity function*, respectively.

#### 3.1. Ambiguity function with respect to range

The range ambiguity function  $H_r(\theta_r)$  from Eq. (12) is a squared modulus of a rectangular window spectrum normalized in such a way that its maximum is equal to one:

$$H_r(\theta_r) = \left| \frac{\sin[(2M+1)\theta_r/2]}{(2M+1)\sin[\theta_r/2]} \right|^2. \quad (14)$$

The zero-to-zero width of the main lobe of this function is equal to  $4\pi/(2M+1)$ . Defining the range resolution  $\Delta\theta_r$  as half of this value, we obtain:

$$\Delta\theta_r = 2\pi/(2M+1). \quad (15)$$

The density of the range grid on which the detection tests are done should not be smaller than  $\Delta\theta_r$ . Furthermore, the function  $H_r(\theta_r)$  is  $2\pi$ -periodic, thus the maximal unambiguous range is  $0 \leq \theta_r < 2\pi$ .

#### 3.2. Ambiguity function with respect to velocity and acceleration

The movement ambiguity function  $H_b(b_v, b_a)$  defined in Eq. (13) is depicted in Fig. 2. It is more complicated than previously analyzed range ambiguity function  $H_r(\theta_r)$ . We cannot express this function in a simple form, but a few interesting properties can be derived directly from its definition Eq. (13).

*Property 1 (maximum):* The function  $H_b(b_v, b_a)$  acquires maximum for  $b_v = b_a = 0$ , and this maximum is equal to 1. To prove it, we can easily check that  $H_b(b_v, b_a) = 1$ , and using the Schwartz inequality we have:

$$H_b(b_v, b_a) = \left| \sum_{k=-K}^K \frac{1}{2K+1} \exp\{-j(b_v k + b_a k^2)\} \right|^2 \leq \sum_{k=-K}^K \left| \frac{1}{2K+1} \right|^2 \times \sum_{k=-K}^K |\exp\{-j(b_v k + b_a k^2)\}|^2 = 1.$$

This property together with Eq. (11) means that for high SNR, the statistic  $D(\theta_r, b_v, b_a)$  acquires maximum in the vicinity of true parameters  $[\theta_r, b_v, b_a] = [\theta_r^+, b_v^+, b_a^+]$  and this maximum is approximately (because of noise) equal to SNR.

*Property 2 (symmetry):* It was shown in [1], that:

$$H_b(b_v, b_a) = H_b(-b_v, -b_a) = H_b(b_v, -b_a). \quad (16)$$

The property is illustrated in Fig. 1. Its intuitive interpretation is that positive and negative values of motion parameters (velocity and acceleration) are not much distinct from each other.

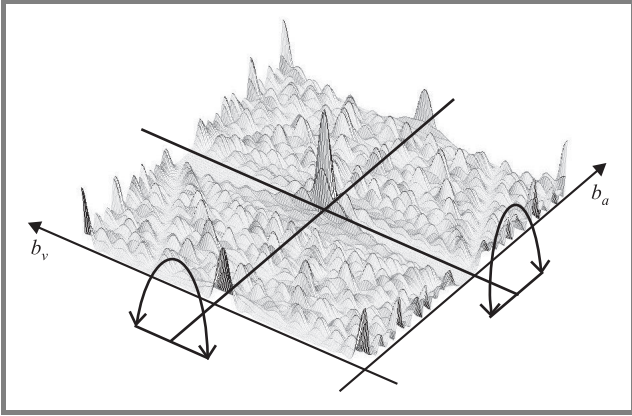


Fig. 1. Symmetry of the motion ambiguity function  $H_b(b_v, b_a)$ .

*Property 3 (periodicity):* It was also shown in [1], that

$$H_b(b_v, b_a) = H_b(b_v + \pi n_1, b_a + \pi n_2) \quad (17)$$

for integers  $n_1$  and  $n_2$  such that  $n_1 + n_2$  is even.

This property means that periods of the ambiguity function are  $(2\pi, 0)$ ,  $(0, 2\pi)$ ,  $(\pi, \pi)$ ,  $(\pi, -\pi)$ ,  $(-\pi, \pi)$ ,  $(-\pi, -\pi)$ ,  $(0, -2\pi)$ ,  $(-2\pi, 0)$ . This periodicity is visible in Fig. 2a.

Property 3 allows us to find a region of unambiguous velocity and acceleration  $b_v, b_a$ . We do it by observing that if  $[b_v, b_a]$  is in this region, then  $[b_v + \pi n_1, b_a + \pi n_2]$  is not. We may notice that shape of this region is not unique. A few possible regions of unambiguous velocity and acceleration are depicted in Figs. 2b-e. For example in the Fig. 2b we have unambiguous velocity-acceleration region such that:

$$\begin{aligned} -\pi &\leq b_v < \pi, \\ -\pi/2 &\leq b_a < \pi/2. \end{aligned} \quad (18)$$

We may notice that in Eq. (18), the maximal unambiguous velocity  $\pm\pi$  is the same as in the constant velocity case when  $b_a \equiv 0$ . In other words, extending target range model to include acceleration, does not affect ambiguity of velocity measurement.

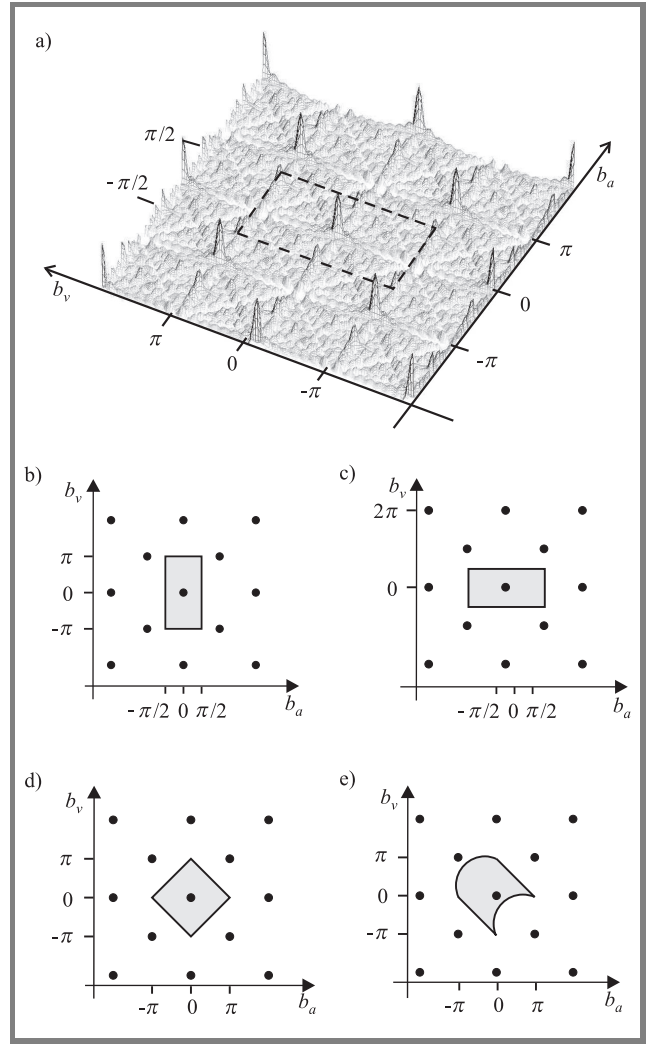


Fig. 2. (a) Plot of the ambiguity function  $H_b(b_v, b_a)$  with marked unambiguity region corresponding to (b). Figures (b)-(e): a few examples of unambiguity regions for parameters  $b_v, b_a$  (dots symbolize periodically repeated maxima of  $H_b(b_v, b_a)$ ).

It is worth noting that sidelobes for  $b_a \approx \pi/2$  are very high and it would be difficult to use the whole range of unambiguous acceleration  $-\pi/2 \leq b_a < \pi/2$  in a multi-target detection. Hence, the radar parameters should be chosen to assure that acceleration of a typical target is much smaller than  $\pi/2$ .

*Property 4 (intersection for  $b_a = 0$ ):* Intersection of  $H_b(b_v, b_a)$  for  $b_a = 0$ , that is  $H_b(b_v, 0)$ , is the squared modulus of the rectangular window spectrum.

This property can be derived directly from definition of function  $H_b(\cdot)$ . Thanks to this property we know that radar velocity resolution is

$$\Delta b_v = 2\pi/(2K + 1) \quad (19)$$

and is the same as in the case of constant velocity target. Radar velocity resolution is illustrated in Fig. 3.

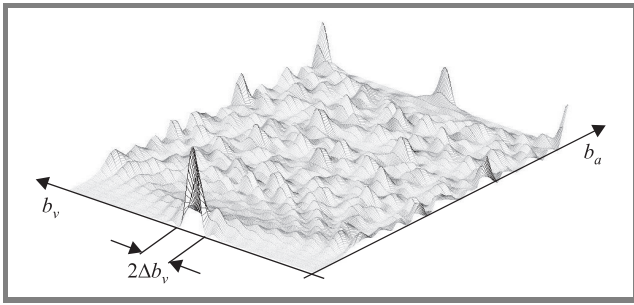


Fig. 3. Cross-section of  $H_b(b_v, b_a)$  for  $b_a = 0$  and radar velocity resolution  $\Delta b_v$ .

Property 5 (intersection for  $b_v = 0$ ): If  $b_a$  is small, and  $K$  is big enough, then  $H_b(0, b_a)$  is a function of  $b_a(2K + 1)^2$  only (not independently of  $b_a$  and  $K$ ).

This property can be proved by approximating the function  $H_b(0, b_a)$  with the corresponding integral:

$$\begin{aligned}
 H_b(0, b_a) &= \left| \frac{1}{2K + 1} \sum_{k=-K}^K e^{jb_a k^2} \right|^2 \\
 &\approx \left| \frac{1}{2K + 1} \int_{-K-1/2}^{K+1/2} e^{jb_a t_1^2} dt_1 \right|^2 \\
 &= \left| \frac{1}{(2K + 1)\sqrt{b_a}} \int_{-(2K+1)\sqrt{b_a}/2}^{(2K+1)\sqrt{b_a}/2} e^{jt^2} dt \right|^2. \quad (20)
 \end{aligned}$$

The last equality was obtained by setting  $b_a t_1^2 = t^2$ . The main result of this property is that half of the main lobe width, that is radar acceleration resolution (see Fig. 4) is

$$\Delta b_a = 2\pi \frac{c_a}{(2K + 1)^2} \quad (21)$$

for a constant  $c_a$ . Using computer simulations we showed that for half of a *minimum-to-minimum* main lobe width (we can call it radar acceleration resolution)  $c_a \approx 3.676$ . Hence, the non-normalized acceleration resolution is (from Eq. (5)):  $\Delta a = 3.68\lambda / [(2K + 1)T]^2$ , where  $\lambda = c/f_c$  is a wavelength. We may use this equation to decide if target acceleration should be taken into account in the de-

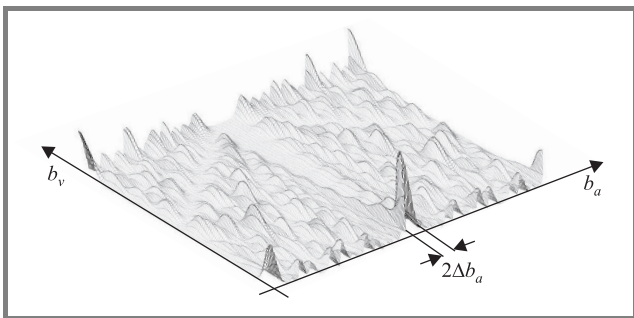


Fig. 4. Cross-section of  $H_b(b_v, b_a)$  for  $b_v = 0$  and radar acceleration resolution  $\Delta b_a$ .

tection test design. If expected maximal acceleration of a typical useful target is smaller than  $\Delta a$  then we may neglect acceleration and assume constant velocity what corresponds to having only one acceleration cell centered at  $a = 0$ . If the acceleration is greater, then more acceleration cells should be created, otherwise, velocity spectrum would be smeared (see [6]) considerably reducing probability of detection.

Property 6 (relation to the Cramer-Rao bounds): Velocity and acceleration resolutions  $\Delta b_v$  and  $\Delta b_a$  are related to respective Cramer-Rao bounds  $\text{CRB}\{b_v\}$  and  $\text{CRB}\{b_a\}$  according to the equations:

$$\sqrt{\text{CRB}\{b_v\}} = \frac{0.39}{\sqrt{\text{SNR}}} \Delta b_v, \quad (22)$$

$$\sqrt{\text{CRB}\{b_a\}} = \frac{0.41}{\sqrt{\text{SNR}}} \Delta b_a. \quad (23)$$

According to [3] and [2], the Cramer-Rao bounds for normalized velocity and acceleration are:

$$\text{CRB}\{b_v\} = \frac{6}{\text{SNR} \cdot (2K + 1)^2}, \quad (24)$$

$$\text{CRB}\{b_a\} = \frac{90}{\text{SNR} \cdot (2K + 1)^4}. \quad (25)$$

The Cramer-Rao bounds  $\text{CRB}\{b_v\}$  and  $\text{CRB}\{b_a\}$  are lower bounds on variance of any unbiased estimator of parameters  $b_v$  and  $b_a$ , respectively. Hence, Eqs. (22) and (23) reveal proportionality of bounds on standard deviations to radar resolutions obtained from the analysis of the ambiguity function. It is interesting, that although cross sections of  $H_b(b_v, b_a)$  across velocity and acceleration dimensions are quite different, the two proportionality coefficients  $0.39/\sqrt{\text{SNR}}$  and  $0.41/\sqrt{\text{SNR}}$  are almost the same.

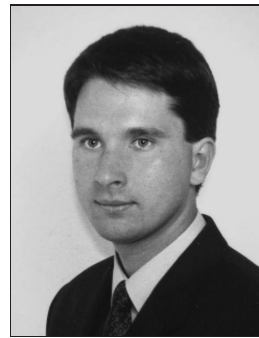
## 4. Conclusions

We analyzed the ambiguity function for accelerating target. This allowed us to calculate radar resolutions and specify regions of unambiguous range, velocity and acceleration. We showed that due to choosing measurement time Eq. (2) symmetrical around  $t = 0$ , maximal unambiguous velocity and velocity resolution are the same for an accelerating target as would be in a constant velocity case. We also related radar resolutions to corresponding Cramer-Rao bounds.

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