Archives of Hydro-Engineering and Environmental Mechanics
Vol. 53 (2006), No. 4, pp. 381–387

Technical Note

Simple Water Waves in Lagrangian Description

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(Received August 08, 2006; revised November 21, 2006)

Abstract

Details of the model of long water waves in the Lagrangian description are presented. The equation of motion is derived from variational formulation of the problem. Only two important cases are considered: when the water depth changes uniformly in space or the depth is constant. For quasilinear hyperbolic system obtained in this description the Riemann invariants and equation of simple waves are found. For constant depth, the Riemann invariants are exactly the same as in the Euler description, however, the velocity of wave propagation is different. In case of uniform slope the velocity, as well as the Riemann invariants are different. In the Lagrangian description the free surface is described in parametric form.

Key words: long waves, Riemann invariants, simple wave, Lagrangian description

1. Theory

Let us consider two-dimensional motion of an infinite layer of inviscid fluid. In order to describe the motion let us introduce the Cartesian co-ordinate system such that vertical and horizontal co-ordinates denote particles at rest. It is assumed, that the fluid for the time $t \leq 0$ is at rest and the corresponding particle co-ordinates are named $a, b, (-h(a) \leq b \leq 0)$, where h denotes depth, and free surface elevation is described as $b = 0$. The motion of the fluid is described by the mapping of the names into the positions occupied by the points at time t. Let us assume that horizontal displacement is independent of the vertical co-ordinate. Thus, the mapping is given as:

$$\begin{aligned} x &= x(a, b, t) = a + u(a, t), \\ y &= y(a, b, t) = b + w(a, b, t). \end{aligned} \tag{1}$$

The incompressibility condition

$$\frac{\partial(x, y)}{\partial(a, b)} = \frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a} = 1, \tag{2}$$

leads to the following relation:

$$\frac{\partial w}{\partial b} = \frac{-h(a)u_a(a,t)}{1 + u_a(a,t)}. \tag{3}$$

The solution of equation (3) reads

$$w(a,b,t) = \frac{(b + h(a))\, u_a(a,t)}{1 + u_a(a,t)} + f(a,t) \tag{4}$$

and continuity equation is satisfied for any function $f(a,t)$. This function can be specified from boundary condition on the bottom, which in the Lagrangian description can be formulated as follows: the water particle originally rested on the bottom should stay there. It is easy to verify that function

$$f(a,t) = -h\,(a + u\,(a,t)) + h\,(a) \tag{5}$$

guarantees, that this condition is fulfilled.

The equations of motion are derived from a principle of stationary action (Herivel 1955):

$$\delta \int_t \int_b \int_a L da db dt = 0, \tag{6}$$

with Lagrangian function

$$L = \rho \frac{1}{2}\left(x_t^2 + y_t^2\right) + p\left(\frac{\partial(x,y)}{\partial(a,b)} - 1\right) - \rho g y. \tag{7}$$

Let us additionally assume, that vertical velocity is small and can be neglected. In the considered case determinant of Jacobi matrix (2) is always equal to unity and Lagrangian function, presents the difference between the kinetic and potential energy, given as:

$$L = \rho \frac{1}{2}\left(x_t^2\right) - \rho g y. \tag{8}$$

This function can be integrated analytically with respect to the vertical co-ordinate b from $-h(a)$ to 0:

$$\delta \iiint \frac{\rho u_t^2}{2} - \rho g\left[(b + h\,(a))\frac{\partial u}{\partial a}\left(1 + \frac{\partial u}{\partial a}\right)^{-1} + b + h(a) - h\,(a + u)\right] da db dt = 0. \tag{9}$$

The Euler equation reads

$$\frac{\partial^2 u}{\partial t^2} - gh\frac{\frac{\partial^2 u}{\partial a^2}}{\left(1+\frac{\partial u}{\partial a}\right)^3} = gh'(a+u) - \frac{gh'(a)}{\left(1+\frac{\partial u}{\partial a}\right)^2}, \tag{10}$$

where $h' = \partial h/\partial a$. Equation (9) represents the basic equation of the long waves theory in the Lagrangian description. In this paper only two important cases are considered: first when the depth is constant and secondly when the long waves move over a uniform slope.

2. Constant Depth and Riemann Invariants

It is obligatory that function (1) be always invertible, so $1 + u_a(a,t) > 0$. For constant water depth, the right side of Eq. (10) vanishes and the equation of motion is a hyperbolic differential equation of the second order. Let us introduce two new functions defined as:

$$\upsilon = \frac{\partial u}{\partial t}, \quad \vartheta = \frac{\partial u}{\partial a}, \tag{11}$$

where $\upsilon(a,t)$ denotes horizontal velocity and $\vartheta(a,t)$ is the spatial derivative of horizontal displacement. From Eqs. (10) and (11) we have the equivalent system of equations of the first order:

$$\frac{\partial \vartheta}{\partial t} - \frac{\partial \upsilon}{\partial a} = 0, \tag{12}$$

$$\frac{\partial \upsilon}{\partial t} - \frac{gh}{(1+\vartheta)^3}\frac{\partial \vartheta}{\partial a} = 0. \tag{13}$$

Let us reformulate the equations (12–13) with vertical displacement on free surface $\eta(a,t)$ in place ϑ. Setting $b = 0$ to Eq. (4) we receive

$$\eta(a,t) = -\frac{hu_a(a,t)}{1+u_a(a,t)} = -\frac{h\vartheta(a,t)}{1+\vartheta(a,t)} \Rightarrow \vartheta(a,t) = -\frac{\eta(a,t)}{h+\eta(a,t)}. \tag{14}$$

Since $\partial\vartheta/\partial t = -h\partial\eta/\partial t\,(h+\eta)^{-2}$ and $\partial\vartheta/\partial a = -h\partial\eta/\partial a\,(h+\eta)^{-2}$ one finds readily

$$\frac{\partial \eta}{\partial t} + \frac{(h+\eta)^2}{h}\frac{\partial \upsilon}{\partial a} = 0, \tag{15}$$

$$\frac{\partial \upsilon}{\partial t} + \frac{g(h+\eta)}{h}\frac{\partial \eta}{\partial a} = 0. \tag{16}$$

In a matrix form system Eqs. (15–16) can be written in the form:

$$A\begin{bmatrix} \eta_t \\ \upsilon_t \end{bmatrix} + B\begin{bmatrix} \eta_a \\ \upsilon_a \end{bmatrix} = 0, \tag{17}$$

where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & \dfrac{(h+\eta)^2}{h} \\ \dfrac{g(h+\eta)}{h} & 0 \end{bmatrix}$.

Using standard methods of hyperbolic equations we may write this system in characteristic form. First let us look for eigenvalues. Equation

$$|A - \lambda B| = 0 \tag{18}$$

has two solutions

$$\lambda_1 = -\sqrt{gh\left(1+\frac{\eta}{h}\right)^3}, \quad \lambda_2 = \sqrt{gh\left(1+\frac{\eta}{h}\right)^3} \tag{19}$$

and corresponding left eigenvectors $\mathbf{l}_i$ have the form

$$\mathbf{l}_1 = \left(1, -\frac{\sqrt{h+\eta}}{\sqrt{g}}\right), \quad \mathbf{l}_2 = \left(1, \frac{\sqrt{h+\eta}}{\sqrt{g}}\right). \tag{20}$$

We may write this system in characteristic form:

$$\mathbf{l}_i A - \mathbf{l}_i B = \mathbf{l}_i C, \tag{21}$$

or

$$\left(\frac{\partial\eta}{\partial t} + \sqrt{gh\left(1+\frac{\eta}{h}\right)^3}\frac{\partial\eta}{\partial a}\right) + \frac{\sqrt{(h+\eta)}}{\sqrt{g}}\left(\frac{\partial\upsilon}{\partial t} + \sqrt{g}h\left(1+\frac{\eta}{h}\right)^3\frac{\partial\upsilon}{\partial a}\right) = 0, \tag{22}$$

$$\left(\frac{\partial\eta}{\partial t} - \sqrt{g}h\left(1+\frac{\eta}{h}\right)^3\frac{\partial\eta}{\partial a}\right) - \frac{\sqrt{(h+\eta)}}{\sqrt{g}}\left(\frac{\partial\upsilon}{\partial t} - \sqrt{gh\left(1+\frac{\eta}{h}\right)^3}\frac{\partial\upsilon}{\partial a}\right) = 0. \tag{23}$$

As can be readily verified, the integrating factor has the form $\pm\sqrt{g}\Big/\sqrt{h+\eta}$ and we may write

$$\frac{\partial I_1}{\partial t} + \sqrt{gh\left(1+\frac{\eta}{h}\right)^3}\frac{\partial I_1}{\partial a} = 0, \tag{24}$$

$$\frac{\partial I_2}{\partial t} - \sqrt{gh\left(1+\frac{\eta}{h}\right)^3}\frac{\partial I_2}{\partial a} = 0, \tag{25}$$

where

$$I_1 = 2\left(\sqrt{g\,(h+\eta)} - \sqrt{gh}\right) + \upsilon, \tag{26}$$

$$I_2 = 2\left(\sqrt{g\,(h+\eta)} - \sqrt{gh}\right) - \upsilon, \tag{27}$$

are Riemann invariants of the system. The constant $\sqrt{gh}$ is chosen this way, so the velocity of water is equal to zero, when the exceeding of free surface vanishes. The interpretation of the system is well known: first equation of the system states that the function I_1 is constant on the curve C_1 given by the ordinary differential equation

$$C_1: \frac{\partial a}{\partial t} = \sqrt{gh\left(1+\frac{\eta}{h}\right)^3} \tag{28}$$

and I_2 is constant on C_2

$$C_2: \frac{\partial a}{\partial t} = -\sqrt{gh\left(1+\frac{\eta}{h}\right)^3} \tag{29}$$

from the second equation. Equation (24) (or (25)) is named the equation of simple wave. Let us note, that Riemann invariants are exactly the same as in the Eulerian description, but the velocity of the propagation of waves is different. Let us remember that velocity of simple wave in the Eulerian system equals $3\sqrt{g\,(h+\eta)} - 2\sqrt{gh}$ (Stoker 1957, Whitham 1974) while in the considered case $\sqrt{gh\,(1+\eta/h)^3}$. Comparison of both velocities has been shown in Figure 1.

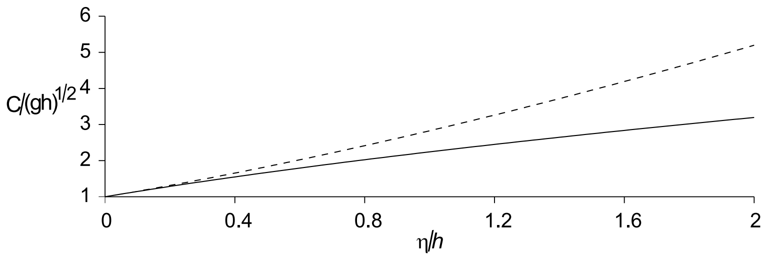

Fig. 1. Comparison of simple wave velocities in the Eulerian description – solid line and in the Lagrangian description – dashed line

3. Uniform Slope

Let us assume that the bottom is described by the equation:

$$b = -\beta a, \tag{30}$$

so the depth of the water is $h(a) = \beta a$. Equations (10), (11) and (30) lead to

$$\frac{\partial \vartheta}{\partial t} - \frac{\partial \upsilon}{\partial a} = 0, \tag{31}$$

$$\frac{\partial \upsilon}{\partial t} - \frac{gh}{(1+\vartheta)^3}\frac{\partial \vartheta}{\partial a} = g\beta\left(1 - \frac{1}{(1+\vartheta)^2}\right). \tag{32}$$

In this case, vertical displacements on free surface take the form

$$\eta(a,t) = -\frac{bh(a)\vartheta(a,t)}{1+\vartheta(a,t)} - \beta u(a,t) \tag{33}$$

and introducing variable η is impossible without an extension system. The eigenvalues and eigenvectors can be found using the same method as before:

$$\lambda_1 = -\frac{\sqrt{gh}}{(1+\vartheta)^{3/2}}, \quad \lambda_2 = \frac{\sqrt{gh}}{(1+\vartheta)^{3/2}}, \tag{34}$$

$$\mathbf{l}_1 = \left(1, \quad -\frac{(1+\vartheta)^{3/2}}{\sqrt{gh}} \right), \quad \mathbf{l}_2 = \left(1, \quad \frac{(1+\vartheta)^{3/2}}{\sqrt{gh}} \right). \tag{35}$$

Therefore we may write system (31–32) in the characteristic form:

$$\frac{\partial \vartheta}{\partial t} + \frac{\sqrt{gh}}{(1+\vartheta)^{3/2}}\frac{\partial \vartheta}{\partial a} - \frac{(1+\vartheta)^{3/2}}{\sqrt{gh}}\left(\frac{\partial \upsilon}{\partial t} + \frac{\sqrt{gh}}{(1+\vartheta)^{3/2}}\frac{\partial \upsilon}{\partial a}\right) = -\beta\sqrt{g}\frac{\vartheta(2+\vartheta)}{\sqrt{h(1+\vartheta)}}, \tag{36}$$

$$\frac{\partial \vartheta}{\partial t} - \frac{\sqrt{gh}}{(1+\vartheta)^{3/2}}\frac{\partial \vartheta}{\partial a} + \frac{(1+\vartheta)^{3/2}}{\sqrt{gh}}\left(\frac{\partial \upsilon}{\partial t} - \frac{\sqrt{gh}}{(1+\vartheta)^{3/2}}\frac{\partial \upsilon}{\partial a}\right) = +\beta\sqrt{g}\frac{\vartheta(2+\vartheta)}{\sqrt{h(1+\vartheta)}}. \tag{37}$$

Multiplying the first equation by $-(1+\vartheta)^{3/2}\Big/\sqrt{gh}$ and second by $(1+\vartheta)^{3/2}\Big/\sqrt{gh}$ we may write

$$\frac{\partial I_1}{\partial t} + \frac{\sqrt{gh}}{(1+\vartheta)^{3/2}}\frac{\partial I_1}{\partial a} = 0, \tag{38}$$

$$\frac{\partial I_2}{\partial t} - \frac{\sqrt{gh}}{(1+\vartheta)^{3/2}}\frac{\partial I_2}{\partial a} = 0, \tag{39}$$

where

$$I_1 = \upsilon + 2\frac{\sqrt{gh}}{\sqrt{1+\vartheta}} - \beta g t, \tag{40}$$

$$I_2 = \upsilon - 2\frac{\sqrt{gh}}{\sqrt{1+\vartheta}} - \beta g t, \tag{41}$$

are Riemann invariants of the system. Let us note that in this case Riemann invariants are different than those obtained in the Eulerian description. Introducing to Eqs. (38–39) a new variable η defined by Eq. (14) we obtain the same expression as in the Eulerian description, but in this case η is not the free surface elevation.

4. Conclusions

1. For the long waves, the assumptions introduced in the description of the geometry of motion, lead to the expressions for vertical displacement.
2. The application of the standard variational calculus to the action integral, leads directly to the partial differential equation of the problem.
3. The equation of motion is the hyperbolic differential equation.
4. The Riemann invariants of the system are found.
5. For constant depth the Riemann invariants are the same as in the Eulerian description. The velocity of propagation in the Lagrangian description differs from that in the Eulerian description.
6. In the case in which the bottom slope is constant, the Riemann invariants differ from those in the Eulerian description.
7. It should be noted, that a simple wave solution is the rigorous analytical solution and may served as a reference for accuracy estimation of a numerical results of a discrete problem.

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