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Pointwise completeness and pointwise degeneracy of 2D standard and positive Fornasini-Marchesini models with state-feedbacks*

by

Tadeusz Kaczorek

Faculty of Electrical Engineering
Białystok University of Technology
Białystok, Poland

Abstract: Necessary and sufficient conditions are established for the pointwise completeness of 2D standard and positive Fornasini-Marchesini models with state-feedbacks. Similar relations are obtained for the pointwise degeneracy of the 2D models with state-feedbacks. It is shown that if the positive 2D model is pointwise complete then there exists a gain matrix of the state-feedback such that the closed-loop system is pointwise degenerated if both matrices B1 and B2 of the 2D Fornasini-Marchesini model are nonzero. The considerations are illustrated by numerical examples.

Keywords: pointwise completeness, degeneracy, standard, positive, Fornasini-Marchesini model, state-feedbacks.

1. Introduction

In positive systems, inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of the state of the art in positive systems theory is given in Farina and Rinaldi (2000), and Kaczorek (2002).

The most popular models of two-dimensional (2D) linear systems are the discrete models introduced by Roesser (1975), Fornasini and Marchesini (1976,

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1978), and Kurek (1985). These models have been extended for positive systems. An overview of positive 2D system theory has been given in Kaczorek (2002).

A dynamical system described by homogenous equation is called pointwise complete if every given final state of the system can be reached by a suitable choice of its initial state. A system which is not pointwise complete, is called pointwise degenerate.

Pointwise completeness and pointwise degeneracy belong to the basic concepts of the modern control theory of 2D linear systems and they play an especially important role in positive 2D linear systems.

The pointwise completeness and pointwise degeneracy of linear continuous-time system with delays have been investigated in Choundhury (1972), Olbrot (1972), Popov (1972), and Weiss (1970), of discrete-time and continuous-time systems of fractional order – in Busłowicz (2008) and Kaczorek and Busłowicz (2009), and of positive discrete-time systems with delays in Busłowicz, Kociszewski and Trzasko (2006). The pointwise completeness of linear discrete-time cone-systems with delays has been analyzed in Trzasko, Busłowicz and Kaczorek (2007). The pointwise completeness and pointwise degeneracy of standard and positive linear systems with state-feedbacks have been investigated in Kaczorek (2009, 2010a).

The pointwise completeness and pointwise degeneracy of 2D standard and positive Fornasini-Marchesini models have been addressed in Kaczorek (2010b).

In this paper the pointwise completeness and pointwise degeneracy of standard and positive 2D Fornasini-Marchesini models with state-feedbacks will be addressed.

The structure of the paper is as follows. In Section 2 the pointwise completeness of 2D standard Fornasini-Marchesini models with state-feedback is addressed. The pointwise degeneracy of the same class of models is investigated in Section 3. The pointwise completeness of 2D positive Fornasini-Marcheisni models with state-feedback is analyzed in Section 4 and the pointwise degeneracy in Section 5. Concluding remarks are given in Section 6. In the Appendix two lemmas are presented which are used in the proof of the main result of the paper.

2. Pointwise completeness of standard 2D Fornasini-Marchesini models

2.1. Preliminaries and problem formulation

Let $\Re^{n\times m}$ be the set of $n \times m$ real matrices and $\Re^n = \Re^{n\times 1}$. Consider the autonomous 2D (second) Fornasini-Marchesini model, Fornasini and Marchesini (1976, 1978), Kaczorek (1985):

$$x_{i+1,j+1} = A_1 x_{i,j+1} + A_2 x_{i+1,j} \quad i,j \in Z_+ = \{0,1,,...\} \tag{2.1}$$

where $x_{ij} \in \Re^n$ is the state vector at the point (i,j) and $A_k \in \Re^{n\times n}$, $k = 1,2$.

Boundary conditions for (2.1) are given by

$$x_{i,0} \quad \text{for } i = 1, 2, \dots \text{ and } x_{0,j} \text{ for } j = 1, 2, \dots \tag{2.2}$$

The transition matrix T_{ij} of the model (2.1) is defined as follows, Fornasini and Marchesini (1976, 1978), Kaczorek (1985):

$$T_{ij} = \begin{cases} I_n(\text{ the identity matrix}) & \text{for } i = j = 0 \\ A_1 T_{i-1,j} + A_2 T_{i,j-1} & \text{for } i, j \in Z_+ \text{ and } i + j > 0 \\ 0(\text{ the zero matrix}) & \text{for } i < 0 \text{ or } j < 0 \end{cases} \tag{2.3}$$

The solution of the equation (2.1) with boundary conditions (2.2) has the form, Fornasini and Marchesini (1976, 1978), Kaczorek (1985):

$$\begin{aligned} x_{ij} &= \sum_{l=1}^{j} T_{i-1,j-l} A_1 x_{0,l} + \sum_{k=1}^{i} T_{i-k,j-1} A_2 x_{k,0} \\ &= T_1(i,j) \begin{bmatrix} x_{01} \\ \vdots \\ x_{0j} \end{bmatrix} + T_2(i,j) \begin{bmatrix} x_{10} \\ \vdots \\ x_{i0} \end{bmatrix} \end{aligned} \tag{2.4a}$$

where

$$\begin{aligned} T_1(i,j) &= [\ T_{i-1,j-1}A_1 \quad \dots \quad T_{i-1,0}A_1\ ], \\ T_2(i,j) &= [\ T_{i-1,j-1}A_2 \quad \dots \quad T_{0,j-1}A_2\ ]. \end{aligned} \tag{2.4b}$$

Definition 2.1 *The model* (2.1) *is called pointwise complete at the point* (p, q) *if for every final state* $x_f \in \Re^n$ *there exist boundary conditions*

$$x_{i,0} \text{ for } i = 1, \dots, p \text{ and } x_{0,j} \text{ for } j = 1, \dots, q \tag{2.5}$$

such that $x_{pq} = x_f$.

Theorem 2.1 (Kaczorek, 2010b) *The model* (2.1) *is pointwise complete if and only if*

$$\operatorname{rank} [\ T_1(p,q) \quad T_2(p,q)\ ] = n \tag{2.6}$$

where $T_1(p,q)$ *and* $T_2(p,q)$ *are defined by* (2.4b) *for* $i = p,\ j = q$.

Theorem 2.2 *The model* (2.1) *is not pointwise complete at the point* (p, q) *if*

$$\operatorname{rank} [\ A_1 \quad A_2\ ] < n. \tag{2.7}$$

Proof. From (2.3) we have

$$T_{ij} = [\ A_1 \quad A_2\ ] \begin{bmatrix} T_{i-1,j} \\ T_{i,j-1} \end{bmatrix}, \quad i, j \in Z_+,\ \ i + j > 0 \tag{2.8}$$

Condition (2.7) implies rank $T_{ij} < n$ and from (2.4b) we obtain

$$\text{rank} \begin{bmatrix} T_1(p,q) & T_2(p,q) \end{bmatrix} < n \tag{2.9}$$

and by Theorem 2.1 the model (2.1) is not pointwise complete at the point (p, q). ■

Consider the 2D (second) Fornasini-Marchesini model

$$x_{i+1,j+1} = A_1 x_{i,j+1} + A_2 x_{i+1,j} + B_1 u_{i,j+1} + B_2 u_{i+1,j}, \quad i, j \in Z_+ \tag{2.10}$$

with the state-feedback

$$u_{ij} = K x_{ij}, \quad i, j \in Z_+ \tag{2.11}$$

where $x_{ij} \in \Re^n$, $u_{ij} \in \Re^m$ are the state and input vectors, $A_k \in \Re^{n\times n}$, $B_k \in \Re^{n\times m}$, $k = 1, 2$, and $K \in \Re^{m\times n}$ is a gain matrix.

Assume that the model (2.10) is not pointwise complete at the point (p, q). We are looking for a gain matrix K such that the closed-loop system

$$x_{i+1,j+1} = (A_1 + B_1 K)x_{i,j+1} + (A_2 + B_2 K)x_{i+1,j}, \quad i, j \in Z_+ \tag{2.12}$$

is pointwise complete at the point (p, q).

2.2. Problem solution

The following two cases will be considered. In both cases it is assumed that (2.7) holds, i.e. the model (2.10) is not pointwise complete.

Case 1. It is assumed that

$$\text{rank} \begin{bmatrix} A_1 & B_1 \end{bmatrix} = n \tag{2.13}$$

or

$$\text{rank} \begin{bmatrix} A_2 & B_2 \end{bmatrix} = n \tag{2.14}$$

Case 2. It is assumed that

$$\text{rank} \begin{bmatrix} A_k & B_k \end{bmatrix} < n \ \text{ for } \ k = 1, 2 \tag{2.15}$$

but

$$\text{rank} \begin{bmatrix} A_1 & A_2 & B_1 & B_2 \end{bmatrix} = n \tag{2.16}$$

First we shall consider Case 1, when the assumption (2.13) is satisfied.

THEOREM 2.3 *Let the condition* (2.7) *be satisfied and* (2.14) ((2.13)) *be not satisfied. There exists a gain matrix* $K \in \Re^{m\times n}$ *of the state-feedback* (2.11) *such that the closed-loop system* (2.12) *is pointwise complete at the point* (p, q) *if and only if the condition* (2.13) ((2.14)) *is met.*

Proof. From Lemma A.1 for $q = n$ it follows that there exists a gain matrix K such that the matrix

$$\bar{A}_1 = A_1 + B_1 K \tag{2.17}$$

is nonsingular if and only if the condition (2.13) is met.

Let $\bar{T}_{ij}$ be the transition matrix defined by (2.3) for the pair $(\bar{A}_1, \bar{A}_2)$. If $\bar{A}_1$ is nonsingular then from (2.4a) we have

$$\operatorname{rank} \left[\begin{array}{cccc} \bar{T}_{p-1,q-1}\bar{A}_1 & \dots & \bar{T}_{p-1,1}\bar{A}_1 & \bar{A}_1^p \end{array} \right] = n \tag{2.18}$$

since $\det \bar{A}_1^p \neq 0$ and by Theorem 2.1 the closed-loop system (2.12) is pointwise complete at the point (p, q). In a similar way we may show the dual theorem. ■

Example 2.1 Consider the model (2.10) with the matrices

$$A_1 = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ -2 & 4 & -2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & -2 & -4 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \tag{2.19}$$

It is easy to check that the model with matrices (2.19) satisfies the assumption of Theorem 2.3 since

$$\operatorname{rank} [\, A_1 \;\; A_2 \,] = \operatorname{rank} \begin{bmatrix} 1 & -2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 & 0 & 1 \\ -2 & 4 & -2 & 0 & -2 & -4 \end{bmatrix} = 2 < n = 3$$

and

$$\operatorname{rank} [\, A_1 \;\; B_1 \,] = \operatorname{rank} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ -2 & 4 & -2 & 1 \end{bmatrix} = 3$$

$$\operatorname{rank} [\, A_2 \;\; B_2 \,] = \operatorname{rank} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix} = 2 < n = 3.$$

Now, we want to find a gain matrix $K = [\, k_1 \;\; k_2 \;\; k_3 \,]$ such that the closed-loop system is pointwise complete at the point $(p, q) = (2, 1)$, and to find initial conditions for a given final state x_f.

We choose the gain matrix so that the matrix

$$\begin{aligned} \bar{A}_1 = A_1 + B_1 K &= \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ -2 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} [\, k_1 \;\; k_2 \;\; k_3 \,] \\ &= \begin{bmatrix} 1 & -2 & 1 \\ 2 + k_1 & 1 + k_2 & k_3 \\ k_1 - 2 & 4 + k_2 & k_3 - 2 \end{bmatrix} \end{aligned} \tag{2.20}$$

is nonsingular.

For $K = [\,2\ \ -3\ \ 2\,]$ the matrix (2.20) has the form

$$\bar{A}_1 = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -2 & 2 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.21}$$

and

$$\bar{A}_2 = A_2 + B_2 K = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & -2 & -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} [\,2\ \ -3\ \ 2\,] = \begin{bmatrix} 0 & 1 & 2 \\ 5 & -6 & 5 \\ 0 & -2 & -4 \end{bmatrix}. \tag{2.22}$$

Using (2.4b) and (2.3) for $\bar{A}_1$ and $\bar{A}_2$ we obtain

$$\bar{T}_1(2,1) = [\bar{A}_1^2] = \begin{bmatrix} -7 & 3 & -3 \\ -4 & -2 & 0 \\ 4 & -2 & 2 \end{bmatrix},$$

$$\bar{T}_2(2,1) = [\ \bar{A}_1\bar{A}_2 \quad \bar{A}_2\ ] = \begin{bmatrix} -10 & 15 & -12 & 0 & 1 & 2 \\ -10 & 12 & -10 & 5 & -6 & 5 \\ 5 & -6 & 5 & 0 & -2 & -4 \end{bmatrix}.$$

From (2.4a) for $i = 2$, $j = 1$ we have

$$x_f = x_{21} = [\ \bar{T}_1(2,1)\ \ \bar{T}_2(2,1)\ ] \begin{bmatrix} x_{01} \\ x_{10} \\ x_{20} \end{bmatrix} = [\ \bar{A}_1^2\ \ \bar{A}_1\bar{A}_2\ \ \bar{A}_2\ ] \begin{bmatrix} x_{01} \\ x_{10} \\ x_{20} \end{bmatrix}. \tag{2.23}$$

Assuming $x_{10} = x_{20} = 0$ we can compute the initial condition

$$x_{01} = \bar{A}_1^{-2} x_f = \begin{bmatrix} -1 & 0 & -1,5 \\ 2 & -0,5 & 3 \\ 4 & -0,5 & 6,5 \end{bmatrix} x_f \tag{2.24}$$

for any given final state x_f.

Now let us consider Case 2.

Theorem 2.4 *Let conditions* (2.7) *and* (2.15) *be satisfied. There exists a gain matrix* $K \in \Re^{m\times n}$ *of the state-feedback* (2.11) *such that the closed-loop system* (2.12) *is pointwise complete at the point* (p, q) *if and only if the condition* (2.16) *is met.*

Proof. From comparison (A.2) and

$$[\ \bar{A}_1\ \ \bar{A}_2\ ] = [\ A_1\ \ A_2\ ] + [\ B_1\ \ B_2\ ] \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}, \tag{2.25}$$

as well as Lemma A.1, it follows that there exists a gain matrix K such that the matrix $[\ \bar{A}_1 \quad \bar{A}_2\ ]$ has full row rank if and only if the condition (2.16) is satisfied. This, by Lemma A.2, implies the condition (2.18) and the closed-loop system (2.12) is pointwise complete at the point (p, q). ■

Example 2.2 For the model (2.10) with matrices

$$A_1 = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.26}$$

find a gain matrix of the state-feedback (2.11) such that the closed-loop system is pointwise complete at the point $(p, q) = (1, 2)$.

For the model (2.10) with (2.26), the conditions (2.7), (2.15) and (2.16) are satisfied, since

$$\begin{aligned}
\operatorname{rank}[\, A_1 \ \ A_2 \,] &= \operatorname{rank} \begin{bmatrix} 1 & -2 & -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & -1 & -2 \end{bmatrix} = 2 < n = 3 \\
\operatorname{rank}[\, A_1 \ \ B_1 \,] &= \operatorname{rank} \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ -1 & 2 & 1 & 0 \end{bmatrix} = 2 < n = 3 \\
\operatorname{rank}[\, A_2 \ \ B_2 \,] &= \operatorname{rank} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} = 2 < n = 3
\end{aligned} \tag{2.27}$$

and

$$\operatorname{rank}[\, A_1 \ \ A_2 \ \ B_1 \ \ B_2 \,] = \operatorname{rank} \begin{bmatrix} 1 & -2 & -1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & -1 & -2 & 0 & 1 \end{bmatrix} = 3. \tag{2.28}$$

We are looking for a gain matrix $K = [\, k_1 \ \ k_2 \ \ k_3 \,]$ such that the matrix

$$\begin{aligned}
[\, \bar{A}_1 \ \ \bar{A}_2 \,] &= [\, A_1 + B_1 K \quad A_2 + B_2 K \,] \\
&= \begin{bmatrix} 1 & -2 & -1 & 0 & 1 & 2 \\ k_1 & k_2 + 1 & k_3 + 2 & 0 & 0 & 0 \\ -1 & 2 & 1 & k_1 & k_2 - 1 & k_3 - 2 \end{bmatrix}
\end{aligned} \tag{2.29}$$

has full row rank. For $k_1 = 1$, $k_2 = -1$, $k_3 = 2$, from (2.29) we obtain the matrix

$$[\, \bar{A}_1 \ \ \bar{A}_2 \,] = \begin{bmatrix} 1 & -2 & -1 & 0 & 1 & 2 \\ 1 & 0 & 4 & 0 & 0 & 0 \\ -1 & 2 & 1 & 1 & -2 & 0 \end{bmatrix} \tag{2.30}$$

with full row rank.

In this case the matrix

$$[\ \bar{T}_1(1,2)\ \ \bar{T}_2(1,2)\ ] = [\ \bar{A}_2\bar{A}_1\ \ \bar{A}_1\ \ \bar{A}_2^2\ ] = \begin{bmatrix} -1 & 4 & 6 & 1 & -2 & -1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 & 0 \\ -1 & -2 & -9 & -1 & 2 & 1 & 0 & 1 & 2 \end{bmatrix} \tag{2.31}$$

has also full row rank, and by Theorem 2.1 the closed-loop system is pointwise complete at the point (1,2).

3. Pointwise degeneracy of standard 2D Fornasini-Marchesini models

DEFINITION 3.1 *The model* (2.1) *is called pointwise degenerate at the point* (p, q) *in the direction* ν *if there exists a non-zero vector* $\nu \in \Re^n$ *such that for all boundary conditions* (2.5) *the solution* (2.4) *for* $i = p$, $j = q$ *satisfies the condition* $\nu^T x_{pq} = 0$, *where* T *denotes the transpose.*

From Theorem 2.1 we have the following remark.

REMARK 3.1 *The model* (2.1) *is pointwise degenerate at the point* (p, q) *in the direction* ν *if and only if*

$$\text{rank}\ [\ T_1(p,q) \quad T_2(p,q)\ ] < n \tag{3.1}$$

where $T_1(p,q)$ *and* $T_2(p,q)$ *are defined by* (2.4b).

The vector ν can be found from the equation

$$\nu^T[\ T_1(p,q) \quad T_2(p,q)\ ] = 0. \tag{3.2}$$

From Theorem 2.2 we have the following corollary.

COROLLARY 3.1 *The model* (2.1) *is pointwise degenerate at the point* (p, q) *in the direction* ν *if*

$$\text{rank}\ [\ A_1 \quad A_2\ ] < n \tag{3.3}$$

and the vector ν *is determined by the equation*

$$\nu^T[\ A_1 \quad A_2\ ] = 0. \tag{3.4}$$

Consider the 2D Fornasini-Marchesini model (2.10) with the state-feedback (2.11). Let the model (2.10) be pointwise degenerate at the point (p, q). We are looking for a gain matrix $K \in \Re^{m\times n}$ of the state-feedback (2.11) such that the closed-loop system (2.12) is pointwise complete at the point (p, q).

From Theorems 2.4 and 2.2 we have the following theorem.

THEOREM 3.1 *Let the model* (2.10) *be pointwise degenerate at the point* (p, q). *There exists a gain matrix* $K \in \Re^{m\times n}$ *of the state-feedback* (2.11) *such that the closed-loop system* (2.12) *is pointwise complete at the point* (p, q) *if and only if the condition* (2.16) *is satisfied.*

Proof is similar to the proof of Theorem 2.4.

4. Pointwise completeness of positive 2D Fornasini-Marchesini models

Let $\Re_+^{n\times m}$ be the set of $n \times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$.

DEFINITION 4.1 *The model* (2.10) *is called positive, if* $x_{ij} \in \Re_+^n$, $i,j \in Z_+$ *for any boundary conditions* $x_{i0} \in \Re_+^n$, $i \in Z_+$, $x_{0j} \in \Re_+^n$, $j \in Z_+$ *and all input sequences* $u_{ij} \in \Re_+^m$, $i,j \in Z_+$.

THEOREM 4.1 (KACZOREK, 2002) *The model* (2.10) *is positive if and only if*

$$A_k \in \Re_+^{n\times n} \text{ and } B_k \in \Re_+^{n\times m} \text{ for } k = 1, 2. \tag{4.1}$$

A matrix $A \in \Re_+^{n\times n}$ is called monomial if and only if its every row and column have only one positive entry and the remaining entries are zero.

DEFINITION 4.2 *The positive model* (2.10) *is called pointwise complete at the point* (p,q) *if for every final state* $x_f \in \Re_+^n$ *there exist boundary conditions*

$$x_{i0} \in \Re_+^n \text{ for } i = 1, ..., p \text{ and } x_{0j} \in \Re_+^n \text{ for } j = 1, ..., q$$

such that $x_{pq} = x_f$.

THEOREM 4.2 (KACZOREK, 2010b) *The positive model* (2.10) *is pointwise complete at the point* (p,q) *if and only if the matrix*

$$\left[\begin{array}{cc} T_1(p,q) & T_2(p,q) \end{array}\right] \tag{4.2}$$

contains n linearly independent monomial columns, where the matrices $T_1(p,q)$ *and* $T_2(p,q)$ *are defined by* (2.4b).

THEOREM 4.3 (KACZOREK, 2010b) *The positive model* (2.10) *is pointwise complete at any point* (p,q) *only if the matrix*

$$\left[\begin{array}{cc} A_1 & A_2 \end{array}\right] \tag{4.3}$$

contains n linearly independent monomial columns.

THEOREM 4.4 (KACZOREK, 2010b) *The positive model* (2.10) *is pointwise complete at the point* $(1,q)$ *for* $q > 1$ ($(p,1)$ for $p > 1$) *if and only if the matrix* (4.3) *contains n linearly independent monomial columns.*

The ith column $a_i \in \Re_+^n$ of the matrix $A = \left[\begin{array}{ccc} a_1 & ... & a_n \end{array}\right]$ is called the standard monomial column if $a_i = c_i e_i$, where e_i is the ith column of I_n and $c_i > 0$, $i = 1, \ldots, n$.

THEOREM 4.5 (KACZOREK, 2010b) *Let the matrix* (4.3) *contain n linearly independent monomial columns. The positive model* (2.1) *is pointwise complete at the point* (p, q) *for* $p > 1$, $q > 1$ *if one of the following conditions is satisfied:*

1) *at least one of the matrices* A_1, A_2 *is a monomial matrix,*
2) *the matrix* A_1 *contains* k ($k = 1, 2, \dots, n-1$) *different standard monomial columns and the matrix* A_2 *contains* $n - k$ *different standard monomial columns such that the matrix composed of all these columns is a monomial matrix.*

Consider the positive 2D Fornasini-Marchesini model (2.10) with the state-feedback (2.11). Let the model be pointwise degenerate at the point (p, q). We are looking for a gain matrix $K \in \Re^{m\times n}$ of the state-feedback (2.11) such that the closed-loop system (2.12) is positive and pointwise complete at the point (p, q).

Further on, it is assumed that the following assumptions are satisfied.

ASSUMPTION 1 *The positive model* (2.10) *is not pointwise complete at the point* (p, q) *but the condition* (2.16) *is satisfied.*

ASSUMPTION 2 *The matrix* (4.3) *contains at least* $n - 1$ *columns with zero entries in all rows corresponding to zero rows in the matrix*

$$\begin{bmatrix} B_1 & B_2 \end{bmatrix} \in \Re_+^{n\times 2m}. \tag{4.4}$$

REMARK 4.1 *If Assumption 2 is not satisfied then there does not exist a gain matrix* $K \in \Re^{m\times n}$ *such that the matrix*

$$\begin{bmatrix} A_1 + B_1 K & A_2 + B_2 K \end{bmatrix} \tag{4.5}$$

has n monomial columns.

Note that if Assumption 2 is satisfied, then we can choose from the matrix (4.3) n columns

$$A_{1i_1}, A_{1i_2}, ..., A_{1i_k}, A_{2i_{k+1}}, ..., A_{2i_n}, \quad k \in \{0, 1, ..., n-1\} \tag{4.6}$$

having zero entries in all rows corresponding to zero rows in the matrix (4.4). For $k = 0$ the columns of A_2 and for $k = n-1$ the columns of A_1 are selected. The columns (4.6) will be called the columns of the desired monomial matrix

$$\bar{A}_m = \begin{bmatrix} \bar{A}_{1m} & \bar{A}_{2m} \end{bmatrix} \tag{4.7a}$$

where

$$\bar{A}_{1m} = \begin{bmatrix} A_{1i_1} & \dots & A_{1i_k} \end{bmatrix} \in \Re^{n\times k}, \bar{A}_{2m} = \begin{bmatrix} A_{2i_{k+1}} & \dots & A_{2i_n} \end{bmatrix} \in \Re^{n\times(n-k)} \tag{4.7b}$$

By $\tilde{A}_{12}$ ($\tilde{A}_{21}$) we will denote the matrix composed of the columns of A_1 (A_2) which are not included in $\bar{A}_{1m}$ ($\bar{A}_{2m}$).

THEOREM 4.6 *Let Assumptions 1 and 2 be satisfied. There exists a gain matrix* $K \in \Re^{m\times n}$ *of the state-feedback* (2.11) *such that the closed-loop system* (2.12) *is positive and pointwise complete at the point* (p, q) *if and only if for a given matrix composed of standard monomial columns*

$$\hat{A}_m = [\ \hat{A}_{1m} \quad \hat{A}_{2m}\ ] = [\ a_1 \quad \ldots \quad a_k \quad a_{k+1} \quad \ldots \quad a_n\ ] \tag{4.8}$$

the following holds

$$\operatorname{rank}[\ B_j \quad \hat{A}_{jm} - \bar{A}_{jm}\ ] = \operatorname{rank} B_j \text{ for } j = 1, 2 \tag{4.9}$$

and

$$\tilde{A}_{12} + B_1\tilde{K}_1 \in \Re_+^{n\times(n-k)}, \quad \tilde{A}_{21} + B_2\tilde{K}_2 \in \Re_+^{n\times k} \tag{4.10}$$

where $\tilde{K}_1$ *(*$\tilde{K}_2$*) is the matrix composed of those columns of* K*, which correspond to the columns of the matrix* $\tilde{A}_{12}$ *(*$\tilde{A}_{21}$*).*

Proof. By the Kronecker-Cappeli theorem, the equations

$$B_jK_j = \hat{A}_{jm} - \bar{A}_{jm}, \quad j = 1, 2 \tag{4.11}$$

have solution K_1, K_2 if and only if condition (4.9) is satisfied. Note that the matrix A_1^p contains k different standard monomial columns if and only if matrix A_1 contains such columns, since if $a_i = c_ie_i$ is the ith column of A_1 then $c_ie_i^p$ is the ith column of A_1^p. Similarly, matrix A_2^q contains $n-k$ different standard monomial columns if and only if A_2 contains such columns. In this case from (2.4b) it follows that the matrix $[\ \bar{T}_1(p,q) \quad \bar{T}_2(p,q)\ ]$ contains a monomial matrix and by Theorem 4.2 the closed-loop system (2.12) is pointwise complete at the point (p, q). The matrices

$$\bar{A}_j = A_j + B_jK \in \Re_+^{n\times n} \text{ for } j = 1, 2 \tag{4.12}$$

if and only if the conditions (4.10) are satisfied. In this case the closed-loop system (2.12) is positive and pointwise complete at the point (p, q). ■

REMARK 4.2 *If the model* (2.10) *is positive (*$A_k \in \Re_+^{n\times n}$*,* $B_k \in \Re_+^{n\times m}$ *for* $k = 1, 2$*) and not pointwise complete, then there does not exist* $K \in \Re_+^{m\times n}$ *such that the closed-loop system is positive and pointwise complete at the point* (p, q)*.*

EXAMPLE 4.1 For the positive model (2.10) with the matrices

$$A_1 = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 4 & 3 \\ 0 & 0 & 2 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{4.13}$$

find a gain matrix

$$K = [\ k_1 \ \ k_2 \ \ k_3\ ] \tag{4.14}$$

of the state-feedback (2.11) such that the closed-loop system is positive and pointwise compete at the point $(p, q) = (1, 2)$.

It is easy to check that the matrices (4.13) satisfy Assumptions 1 and 2. In this case we choose as the columns of the desired monomial matrix (4.7) the second column A_{12} and the third column A_{13} of the matrix A_1 and the first column] A_{21} of the matrix A_2, i.e.

$$\bar{A}_m = [\ \bar{A}_{1m} \quad \bar{A}_{2m}\ ] = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.15}$$

and

$$\tilde{A}_{12} = A_{11} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \tilde{A}_{21} = [\ A_{22} \quad A_{23}\ ] = \begin{bmatrix} 2 & 4 \\ 4 & 3 \\ 0 & 2 \end{bmatrix},$$
$$K_1 = [\ k_1 \quad k_2\ ], \quad K_2 = [k_3] \tag{4.16}$$

We choose

$$\hat{A}_{1m} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{A}_{2m} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}. \tag{4.17}$$

Conditions (4.9) are satisfied since

$$\operatorname{rank} [\ B_1 \ \ \hat{A}_{1m} - \bar{A}_{1m}\ ] = \operatorname{rank} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{rank} B_1 = \operatorname{rank} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$$
$$\operatorname{rank} [\ B_2 \ \ \hat{A}_{2m} - \bar{A}_{2m}\ ] = \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} = \operatorname{rank} B_2 = \operatorname{rank} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1. \tag{4.18}$$

Equations (4.11) have the forms

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} K_1 = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} K_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \tag{4.19a}$$

and

$$K_1 = [\ k_1 \ \ k_2\ ] = [\ -2 \ \ -2\ ], \quad K_2 = [k_3] = [-1] \tag{4.19b}$$

Conditions (4.10) are satisfied, since the matrices

$$\tilde{A}_{12} + B_1 K_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [-1] = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\tilde{A}_{21} + B_2 K_2 = \begin{bmatrix} 2 & 4 \\ 4 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [\, -2 \quad -2 \,] = \begin{bmatrix} 2 & 4 \\ 4 & 1 \\ 0 & 2 \end{bmatrix}$$

have nonnegative entries.

Using (2.3) and (2.4b) for the closed-loop system we obtain

$$[\, \bar{T}_1(1,2) \quad \bar{T}_2(1,2) \,] = [\, \bar{A}_2 \bar{A}_1 \quad \bar{A}_1 \quad \bar{A}_2^2 \,] = \begin{bmatrix} 14 & 2 & 4 & 2 & 0 & 0 & 9 & 10 & 22 \\ 5 & 2 & 1 & 2 & 1 & 0 & 0 & 4 & 4 \\ 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 4 \end{bmatrix}. \quad (4.20)$$

Matrix (4.20) contains three linearly independent columns and by Theorem 4.2 the closed-loop system is positive and pointwise complete at the point $(1, 2)$.

5. Pointwise degeneracy of the positive 2D Fornasini-Marchesini models

DEFINITION 5.1 *The positive model* (2.1) *is called pointwise degenerate at the point* (p, q) *in the direction* ν *if there exists a nonzero vector* $\nu \in \Re^n$ *such that for all boundary conditions* $x_{i0} \in \Re_+^n$, $i = 1, \ldots, p$; $x_{0j} \in \Re_+^n$, $j = 1, \ldots, q$ *the solution* (2.4) *for* $i = p$, $j = q$ *satisfies the condition* $\nu^T x_{pq} = 0$.

THEOREM 5.1 (KACZOREK, 2010b) *The positive model* (2.1) *is pointwise degenerate at the point* (p, q) *in the direction* ν *if the condition*

$$\operatorname{rank} [\; T_1(p,q) \quad T_2(p,q) \;] < n \quad (5.1)$$

is met, where $T_1(p,q)$ *and* $T_2(p,q)$ *are defined by* (2.4b). *The vector* ν *can be found from the equation*

$$\nu^T [\; T_1(p,q) \quad T_2(p,q) \;] = 0. \quad (5.2)$$

THEOREM 5.2 (KACZOREK, 2010b) *The positive model* (2.1) *is pointwise degenerate at the point* (p, q) *in the direction* ν *if the matrix* (4.3) *does not contain* n *linearly independent monomial columns.*

Consider the positive 2D Fornasini-Marchesini model (2.10) with the state-feedback (2.11). Let the model be pointwise complete at the point (p, q). We are looking for a gain matrix $K \in \Re^{m \times n}$ of state-feedback (2.11), such that the closed-loop system (2.12) is pointwise degenerate at the point (p, q) in the direction ν.

THEOREM 5.3 *Let the positive 2D model* (2.10) *be pointwise complete at the point* (p,q). *There exists a (nonzero) gain matrix* $K \in \Re^{m\times n}$ *of the state-feedback* (2.10) *such that the closed-loop system* (2.12) *is pointwise degenerate at the point* (p,q) *in the direction* ν *if both matrices* B_1 *and* B_2 *are nonzero.*

Proof. By Theorem 4.3 the positive model (2.10) is pointwise complete at the point (p,q) only if the matrix (4.3) contains n linearly independent monomial columns. If B_1 and B_2 are nonzero matrices, then there exists a nonzero gain matrix K such that the matrix (4.5) has less than n linearly independent monomial columns. ■

EXAMPLE 5.1 Consider the positive 2D model (2.10) with matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{5.3}$$

The model (2.10) with (5.3) is pointwise complete at the point $(p,q) = (2,2)$ since the matrix

$$[\, T_{11}A_2 \;\; T_{01}A_2 \;\; T_{11}A_1 \;\; T_{10}A_1 \,] = \begin{bmatrix} 21 & 5 & 8 & 6 & 0 & 3 & 5 & 11 & 6 & 1 & 1 & 0 \\ 3 & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 30 & 6 & 12 & 6 & 0 & 3 & 8 & 16 & 8 & 0 & 4 & 4 \end{bmatrix} \tag{5.4}$$

contains the monomial matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \tag{5.5}$$

We are looking for the gain matrix

$$K = [\, k_1 \;\; k_2 \;\; k_3 \,] \tag{5.6}$$

such that the matrix

$$\begin{aligned} [\, \bar{A}_1 \;\; \bar{A}_2 \,] &= [\, A_1 + B_1K \quad A_2 + B_2K \,] \\ &= \begin{bmatrix} k_1+1 & k_2+1 & k_3 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & k_1+2 & k_2 & k_3+1 \end{bmatrix} \end{aligned} \tag{5.7}$$

does not contain three linearly independent monomial columns and the closed-loop system is degenerate at the point $(2,2)$. For example for $k_1 = 0$, $k_2 = k_3 = 1$, we obtain the matrix

$$[\, \bar{A}_1 \quad \bar{A}_2 \,] = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 1 & 2 \end{bmatrix} \tag{5.8}$$

with only one monomial column.

The positive closed-loop system is degenerate at the point $(2,2)$ since the matrix

$$[\,\bar{T}_{11}\bar{A}_2 \;\; \bar{T}_{01}\bar{A}_2 \;\; \bar{T}_{11}\bar{A}_1 \;\; \bar{T}_{10}\bar{A}_1\,] = \begin{bmatrix} 39 & 16 & 22 & 6 & 1 & 4 & 8 & 30 & 22 & 1 & 4 & 3 \\ 6 & 3 & 3 & 3 & 1 & 1 & 1 & 4 & 3 & 0 & 0 & 0 \\ 48 & 22 & 28 & 9 & 3 & 6 & 8 & 36 & 28 & 0 & 4 & 4 \end{bmatrix} \quad (5.9)$$

contains only one monomial column.

6. Concluding remarks

The pointwise completeness and pointwise degeneracy of 2D standard and positive Fornasini-Marchesini models with state-feedbacks have been addressed. Necessary and sufficient conditions have been established for the pointwise completeness of the 2D standard and positive Fornasini-Marchesini models with state-feedbacks. It has been shown that there exists a gain matrix K of the state-feedback such that the standard closed-loop system is pointwise complete at the point (p,q) if and only if the condition (2.13) is met. Similar result has been also established for the 2D positive Fornasini-Marchesini models. It has been also shown that if the standard model (2.10) is pointwise degenerate at the point (p,q), then there exists a gain matrix K of the state-feedback such that the closed-loop system (2.12) is pointwise complete at the point (p,q) if and only if the condition (2.16) is satisfied.

If the model (2.10) is positive, but not pointwise complete, then there exists a gain matrix K such that the closed-loop system (2.12) is positive and pointwise complete at the point (p,q) if and only if for a given matrix composed of standard monomial columns conditions (4.9) and (4.10) are satisfied. If the positive 2D model (2.10) is pointwise complete, then there exists a gain matrix K such that the closed-loop system is pointwise degenerate if both matrices B_1, B_2 are nonzero.

These considerations can be extended to 2D linear systems described by the standard and positive Roesser models and by the general Kurek type models.

An open problem is constituted by the extension of these considerations to 2D standard and positive systems of fractional orders.

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Appendix

DEFINITION A.1 *The following operations will be called elementary operations on a real matrix* $A \in \Re^{n\times m}$*:*

1) *addition to any* i*th row of the* j*th row multiplied by any nonzero number* c*. This operation will be denoted by* $L[i + j \times c]$*,*
2) *the interchange of the* i*th and* j*th rows. This operation will be denoted by* $L[i, j]$*,*

It is well known, see Kaczorek (2007), that using those elementary operations it is possible to reduce a real matrix $A \in \Re^{n\times m}$ with rank $A = p < n$ to the form $\begin{bmatrix} A_1 \\ 0 \end{bmatrix}$, where rank $A_1 = p$.

LEMMA A.1 *Let* $A \in \Re^{n\times q}$ $(n \le q)$*,* $B \in \Re^{n\times m}$ $(m < n)$ *and*

$$\operatorname{rank} A = p < n \tag{A.1}$$

There exists $K \in \Re^{m\times q}$ *such that the matrix*

$$\bar{A} = A + BK \in \Re^{n\times q} \tag{A.2}$$

has full row rank if and only if

$$\operatorname{rank} \begin{bmatrix} A & B \end{bmatrix} = n \tag{A.3}$$

Proof. Necessity. From (A.2) written in the form

$$\bar{A} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I_q \\ K \end{bmatrix} \tag{A.4}$$

it follows that matrix $\bar{A}$ has full row rank only if the condition (A.3) is satisfied.

Sufficiency. The proof of sufficiency will be constructive. It is well known, Kaczorek (2007) that if condition (A.3) is met, then there exists a nonsingular matrix $P \in \Re^{n\times n}$ of elementary row operations such that for $p + m = n$

$$P\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix} \tag{A.5}$$

where $A_1 \in \Re^{p\times n}$ and $B_1 \in \Re^{(n-p)\times m}$ have full row ranks. From (A.2) and (A.4) we have

$$P\bar{A} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} K = \begin{bmatrix} A_1 \\ B_1 K \end{bmatrix} \tag{A.6}$$

and

$$\operatorname{rank} \bar{A} = \operatorname{rank} P\bar{A} = \operatorname{rank} \begin{bmatrix} A_1 \\ B_1 K \end{bmatrix}. \tag{A.7}$$

If $m > n - p$ then we choose first $p + m - n$ rows of K equal to zero. Therefore, if (A.3) holds, then there exists K such that $\bar{A}$ has full row rank. ∎

From the proof of sufficiency we have the following procedure for computation of the gain matrix K such that the matrix (A.2) has full row rank.

Procedure A.1

Step 1. Using elementary operation (L) perform the reduction

$$[\,A \;\; B \;\; I_n\,] \xrightarrow{L} \begin{bmatrix} A_1 & 0 & P \\ 0 & B_1 & \end{bmatrix} \tag{A.8}$$

and find the matrices A_1, B_1 and P.

Step 2. Choose $K \in \Re^{m\times q}$ such that the matrix $\begin{bmatrix} A_1 \\ B_1K \end{bmatrix}$ has full row rank

Step 3. Using

$$\bar{A} = P^{-1}\begin{bmatrix} A_1 \\ B_1K \end{bmatrix} \tag{A.9}$$

find the matrix $\bar{A}$.

Example A.1 Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & -2 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}. \tag{A.10}$$

In this case $n = 3$, $q = 4$, $m = 1$ and rank $A = p = 2$. It is easy to check that for the matrices (A.10) the condition (A.3) is satisfied. We are looking for a matrix $K = [\,k_1 \quad k_2 \quad k_3 \quad k_4\,]$ such that the matrix

$$\bar{A} = A + BK = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & -2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [\,k_1 \;\; k_2 \;\; k_3 \;\; k_4\,] \tag{A.11}$$

has full row rank.

Using Procedure A.1 we obtain the following:

Step 1. Applying the elementary operations $L[3 + 1 \times (-2)]$, $L[1 + 3 \times (-1)]$, $L[2 + 3 \times (-2)]$ to the matrix

$$[\,A \;\; B \;\; I_n\,] = \begin{bmatrix} 1 & -1 & 0 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & -2 & 0 & 4 & 3 & 0 & 0 & 1 \end{bmatrix} \tag{A.12}$$

we obtain

$$\begin{bmatrix} A_1 & 0 & P \\ 0 & B_1 & \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 & 3 & 0 & -1 \\ 2 & 1 & 1 & 2 & 0 & 4 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix} \tag{A.13}$$

and

$$A_1 = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix}, \quad B_1 = [1], \quad P = \begin{bmatrix} 3 & 0 & -1 \\ 4 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix} \tag{A.14}$$

Step 2. We choose $K = [\ k_1 \quad k_2 \quad k_3 \quad k_4\ ]$ such that the matrix

$$\begin{bmatrix} A_1 \\ B_1K \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & 2 \\ k_1 & k_2 & k_3 & k_4 \end{bmatrix} \tag{A.15}$$

has full row rank, for example

$$K = [\ 1 \ \ 0 \ \ 0 \ \ 0\ ]. \tag{A.16}$$

Step 3. Using (A.9) for (A.14) and (A.16) we obtain

$$\bar{A} = P^{-1} \begin{bmatrix} A_1 \\ B_1K \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 4 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 2 \\ 4 & 1 & 1 & 2 \\ 5 & -2 & 0 & 4 \end{bmatrix}. \tag{A.17}$$

Lemma A.2 *Let* $A_k \in \Re^{n\times n}$, $k = 1, 2$ *and*

$$T(p,q) = [\ T_1(p,q) \ \ T_2(p,q)\ ] \tag{A.18a}$$

where

$$\begin{aligned} T_1(p,q) &= [\ T_{p-1,q-1}A_1 \ \ \ldots \ \ T_{p-1,0}A_1\ ] \in \Re^{n\times qn} \\ T_2(p,q) &= [\ T_{p-1,q-1}A_2 \ \ \ldots \ \ T_{0,q-1}A_2\ ] \in \Re^{n\times np} \end{aligned} \tag{A.18b}$$

and T_{ij}, $i, j \in Z_+$ *is defined by* (2.3). *Then*

$$\operatorname{rank} T(p,q) = n \tag{A.19}$$

if and only if

$$\operatorname{rank} [\ A_1 \ \ A_2\ ] = n. \tag{A.20}$$

Proof. Necessity of (A.20) follows immediately from (2.8). If $\det A_1 \neq 0$ then the matrix $T_{p-1,0}A_1 = A_1^p$ is nonsingular and the condition (A.19) is satisfied, since $\det A_1^p \neq 0$. For $\det A_2 \neq 0$ the proof is similar. Now we assume that $\det A_1 = 0$ and $\det A_2 = 0$ but the condition (A.20) is met. From (A.18) we have

$$T(p,q) = \hat{T}_{p-1,q-1} D(A_1, A_2) \tag{A.21}$$

where

$$\hat{T}_{p-1,q-1} = [\ T_{p-1,q-1}\ \ \dots\ \ T_{p-1,0}\ \ T_{p-1,q-1}\ \ \dots\ \ T_{0,q-1}\ ] \in \Re^{n\times(p+q)n} \tag{A.22}$$

$$D(A_1, A_2) = \text{blockdiag}\ [\ \underbrace{A_1\ \ \dots\ \ A_1}_{q}\ \ \underbrace{A_2\ \ \dots\ \ A_2}_{p}\ ] \in \Re^{(p+q)n\times(p+q)n} \tag{A.23}$$

For $p = q = 1$ the hypothesis is true since from (A.21) we have

$$\text{rank}\, T(1,1) = \text{rank}\ [\ T_1(1,1)\ \ T_2(1,1)\ ] = \text{rank}\ [\ A_1\ \ A_2\ ] \tag{A.24}$$

Using (A.21) it is easy to show that

$$T(2,1) = [\ T_1(2,1)\ \ T_2(2,1)\ ] = [\ A_1\ \ A_2\ ] \begin{bmatrix} A_1 & A_2 & 0 \\ 0 & 0 & I_n \end{bmatrix} \tag{A.25a}$$

and

$$T(1,2) = [\ T_1(1,2)\ \ \ T_2(1,2)\ \ ] = [\ A_1\ \ \ A_2\ ] \begin{bmatrix} 0 & I_n & 0 \\ A_1 & 0 & A_2 \end{bmatrix}. \tag{A.25b}$$

From (A.25) we have

$$\text{rank}\, T(2,1) = \text{rank}\, T(1,2) = \text{rank}\ [\ A_1\ \ A_2\ ] \tag{A.26}$$

since

$$\text{rank}\, T(2,1) = \text{rank}\ [\ A_1\ \ A_2\ ] + \text{rank} \begin{bmatrix} A_1 & A_2 & 0 \\ 0 & 0 & I_n \end{bmatrix} - 2n = \text{rank}\ [\ A_1\ \ A_2\ ] \tag{A.27}$$

if and only if (A.20) holds.

We get a similar result for $T(1,2)$. Using (A.21) it is easy to show that

$$T(p,1) = [\ A_1\ \ A_2\ ] \begin{bmatrix} T(p-1,1) & 0 \\ 0 & I_n \end{bmatrix} \quad \text{for} \quad p = 2,3\dots \tag{A.28}$$

and

$$\text{rank}\, T(p,1) = \text{rank}\ [\ A_1\ \ A_2\ ] \tag{A.29}$$

since

$$\text{rank} \begin{bmatrix} T(p-1,1) & 0 \\ 0 & I_n \end{bmatrix} = 2n \tag{A.30}$$

if and only if (A.20) holds.

Similarly

$$\text{rank}\, T(1,q) = \text{rank}\ [\ \ A_1\ \ \ \ A_2\ \ ] \quad \text{for} \quad p = 2,3\dots. \tag{A.31}$$

In general case

$$\text{rank}\, T(p,q) = \text{rank}\ [\ \ A_1\ \ \ \ A_2\ \ ]. \tag{A.32}$$

■