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Pseudomonotone semicoercive variational-hemivariational inequalities with unilateral growth condition

by

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Abstract: A variational-hemivariational inequality on a vector valued function space is studied with the nonlinear part satisfying the unilateral growth condition. The higher order term is assumed to be pseudo-monotone and semicoercive. The compatibility condition expressed in terms of a recession functional has been proposed and the existence result has been formulated in a form involving the notion of discontinuous subgradient.

Keywords: variational-hemivariational inequality, unilateral growth condition, discontinuous subgradient.

1. Introduction

The theory of hemivariational inequalities begun in the early eighties with the works of P. D. Panagiotopoulos (Panagiotopoulos 1981, 1983), and has as a main reason for its birth the need for the description of important problems in physics and engineering, where nonmonotone, multivalued boundary or interface conditions occur, or where some nonmonotone, multivalued relations between stress and strain, or reaction and displacement have to be taken into account. The theory of hemivariational inequalities (as the generalization of variational inequalities, see Duvaut and Lions (1972)) has been proved to be very useful in understanding of many problems of mechanics and engineering involving nonconvex, nonsmooth energy functionals. For the general study of hemivariational inequalities and their applications the reader is referred to Motreanu and Naniewicz (1996, 2001, 2002), Motreanu and Panagiotopoulos (1995, 1996, 1999), Naniewicz (1994a, 1995a,b, 1997), Naniewicz and Panagiotopoulos (1995), Panagiotopoulos (1985, 1993) and the references quoted there.

Some results related to variational-hemivariational inequalities on vector-

Panagiotopoulos (1995) and Pop et al., (1997). The semicoercive case with unilateral growth condition Naniewicz (1994a) has been studied in Naniewicz (2000) where the existence result has been obtained under some boundedness hypotheses concerning both the convex and nonconvex parts. The present paper is devoted to the study of the existence problem without the aforementioned boundedness hypotheses. Instead, the notion of a discontinuous subgradient is introduced by means of which the solution of variational-hemivariational inequality under consideration is characterized.

Let us consider the problem of finding $u \in V$ such as to satisfy a variational-hemivariational inequality

$$(P)\ \langle Au - g, v - u \rangle_V + \Phi(v) - \Phi(u) + \int_\Omega j^0(u; v-u)\,d\Omega \geq 0, \quad \forall\, v \in V,$$

where $A : V \to V^*$ is a pseudo-monotone semicoercive operator, $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ a convex, lower semicontinuous, proper function, $j : \mathbb{R}^s \to \mathbb{R}$ a locally Lipschitz function fulfilling the unilateral growth condition (Naniewicz, 1994a):

$$j^0(\xi; \eta - \xi) \leq \alpha(r)(1 + |\xi|^\sigma), \quad \forall\, \xi, \eta \in \mathbb{R}^s,\ |\eta| \leq r,\ r \geq 0.$$

Here $j^0(\cdot\,;\cdot)$ stands for the generalized Clarke differential given by Clarke (1983)

$$j^0(\xi; \eta) = \limsup_{\substack{h \to 0 \\ \lambda \to 0_+}} \frac{j(\xi + h + \lambda\eta) - j(\xi + h)}{\lambda}.$$

The main result ensures the existence of $u \in V$ such that both (P) and the inclusion below hold:

$$g - Au - l_\chi \in \widehat{\partial}\Phi(u),$$

where $\widehat{\partial}\varphi(u)$ is a $\mathcal{L}(\mathcal{V})$-subdifferential ($\mathcal{L}(\mathcal{V})$ being the set of linear densly defined functions on V) in the sense of Pallaschke and Rolewicz (1997), l_χ is a certain element of $\mathcal{L}(\mathcal{V})$. $\widehat{\partial}\varphi(u)$ will be referred to as a discontinuous subdifferential whose elements will be called discontinuous subgradients.

To prove the main result the theory of pseudomonotone multivalued mappings combined with the Lipschitz regularization of ∂j and some compactness arguments involving Dunford-Pettis criterion will be applied.

2. Statement of the problem and some preliminary results

Let $V = H^1(\Omega; \mathbb{R}^s)$, $s \geq 1$, be a vector valued Sobolev space of functions square integrable together with their first partial distributional derivatives in Ω, Ω being a bounded domain in $\mathbb{R}^m$, $m > 2$, with sufficiently smooth boundary Γ.

Kufner, John and Fučik, 1977). We write $\|\cdot\|_V$ and $\|\cdot\|_{L^p(\Omega;\mathbb{R}^s)}$ for the norms in V and $L^p(\Omega;\mathbb{R}^s)$, respectively. For the pairing over $V^* \times V$ the symbol $\langle\cdot,\cdot\rangle_V$ will be used, V^* being the dual of V.

Let $A : V \to V^*$ be a bounded, pseudo-monotone operator. It means that A maps bounded sets into bounded sets and that the following conditions hold Brézis (1968), Browder and Hess (1972):

(i) The effective domain of A, Dom(A), coincides with the whole V, i.e. Dom$(A) = V$;

(ii) If $u_n \to u$ weakly in V and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle_V \leq 0$ then $\liminf_{n\to\infty} \langle Au_n, u_n - v\rangle_V \geq \langle Au, u - v\rangle_V$ for any $v \in V$.

Note that from (i) and (ii) it follows that A is demicontinuous, i.e.

(iii) If $u_n \to u$ strongly in V then $Au_n \to Au$ weakly in V^*.

Moreover, we assume that V is endowed with a direct sum decomposition $V = \widehat{V} + V_0$, where V_0 is a finite dimensional linear subspace, with respect to which A is semicoercive, i.e. $\forall\, u \in V$ there exist $\widehat{u} \in \widehat{V}$ and $\theta \in V_0$ such that $u = \widehat{u} + \theta$ and

$$\langle Au, u\rangle_V \geq c(\|\widehat{u}\|_V)\|\widehat{u}\|_V, \tag{1}$$

$c : \mathbb{R}^+ \to \mathbb{R}$ stands for a coercivity function with $c(r) \to \infty$ as $r \to \infty$. Further, let $j : \mathbb{R}^s \to \mathbb{R}$ be a locally Lipschitz function fulfilling the unilateral growth conditions (Naniewicz, 1994a):

$$j^0(\xi; \eta - \xi) \leq \alpha(r)(1 + |\xi|^\sigma), \quad \forall\, \xi, \eta \in \mathbb{R}^s,\ |\eta| \leq r,\ r \geq 0, \tag{2}$$

$$j^0(\xi; -\xi) \leq \widehat{k}\,|\xi|, \quad \forall\, \xi \in \mathbb{R}^s, \tag{3}$$

where $1 \leq \sigma < p$, $\widehat{k}$ is a nonnegative constant and $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to be a nondecreasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$. Here $j^0(\cdot\,;\cdot)$ stands for the generalized Clarke differential (Clarke, 1983), i.e.

$$j^0(\xi; \eta) = \limsup_{\substack{h\to 0\\ \lambda \to 0_+}} \frac{j(\xi + h + \lambda\eta) - j(\xi + h)}{\lambda}. \tag{4}$$

Let $\Phi : V \to R \cup \{+\infty\}$ be a convex, lower semicontinuous and proper function, $g \in V^*$ be an element of V^*. Consider the problem (P) of finding $u \in V$ such as to satisfy the variational-hemivariational inequality

$$\langle Au - g, v - u\rangle_V + \Phi(v) - \Phi(u) + \int_\Omega j^0(u; v - u)\, d\Omega \geq 0, \quad \forall\, v \in V, \tag{5}$$

In fact, we establish a stronger result, namely, we show the existence of $u \in V$, $\chi \in L^1(\Omega;\mathbb{R}^s)$ and $\psi \in V^*$ with the properties that

$$\langle Au - g, v-u\rangle_V + \langle \psi, v-u\rangle_V + \int_\Omega \chi\cdot(v-u)d\Omega = 0,$$
$$\forall v \in V\cap L^\infty(\Omega;\mathbb{R}^s) \tag{6}$$
$$\chi \in \partial j(u) \text{ a.e. in } \Omega, \quad \chi\cdot u \in L^1(\Omega), \quad \psi\in\partial\Phi(u), \tag{7}$$

where the dot "$\cdot$" stands for the inner product in $\mathbb{R}^s$. From (6) and (7) we then derive the validity of variational-hemivariational inequality in the form (5).

For $R > 0$ define

$$\widetilde{j^0_R}(\xi;\eta) := \begin{cases} j^0(\xi;\eta) & \text{if } |\xi|\le R \\ j^0(R\frac{\xi}{|\xi|};\eta) & \text{if } |\xi| > R. \end{cases} \tag{8}$$

LEMMA 1. *The function* $\mathbb{R}^s\times\mathbb{R}^s \ni (\xi,\eta)\mapsto \widetilde{j^0_R}(\xi;\eta)$ *is upper semicontinuous and if* (2)–(3) *hold, then*

$$j^0_R(\xi;\eta-\xi) \le \widetilde{\alpha}(r)(1+|\xi|^\sigma), \quad \forall\,\xi\in\mathbb{R}^s,\ \forall\eta\in\mathbb{R}^s,\ |\eta|\le r,\ r\ge 0. \tag{9}$$
$$\widetilde{j^0_R}(\xi;-\xi) \le \widehat{k}\,|\xi|, \quad \forall\,\xi\in\mathbb{R}^s, \tag{10}$$

for some nondecreasing, independent of R function $\widetilde{\alpha}:\mathbb{R}^+\to\mathbb{R}^+$.

Proof. The upper semicontinuity of $\widetilde{j^0_R}(\cdot\,;\cdot)$ follows directly from the upper semicontinuity of $j^0(\cdot\,;\cdot)$. To establish (9) and (10) it is enough to consider the case $|\xi|>R$ and invoke the estimates

$$\widetilde{j^0_R}(\xi;\eta-\xi) = j^0(R\tfrac{\xi}{|\xi|};\eta-\xi) \le j^0(R\tfrac{\xi}{|\xi|};\eta-R\tfrac{\xi}{|\xi|}) + \tfrac{|\xi|-R}{R} j^0(R\tfrac{\xi}{|\xi|};-R\tfrac{\xi}{|\xi|})$$
$$\le \alpha(|\eta|)(1+R^\sigma) + \tfrac{|\xi|-R}{R}\widehat{k}R \le \alpha(r)(1+|\xi|^\sigma) + \widehat{k}\,|\xi|,$$
$$\forall\,\xi,\eta\in\mathbb{R}^s,\ |\eta|\le r,\ r\ge 0,$$

and

$$\widetilde{j^0_R}(\xi;-\xi) = j^0(R\tfrac{\xi}{|\xi|};-\xi) \le \tfrac{|\xi|}{R} j^0(R\tfrac{\xi}{|\xi|};-R\tfrac{\xi}{|\xi|}) \le \tfrac{|\xi|}{R}\widehat{k}R = \widehat{k}\,|\xi|,$$

from which the desired estimates easily follow. ∎

Define a recession function $j^\infty:\mathbb{R}^s\to\mathbb{R}\cup\{+\infty\}$ by (see Goeleven and Théra, 1995, Brezis and Nirenberg, 1978, Baiocchi et al., 1988, Goeleven, 1996)

$$j^\infty(\xi) = \liminf_{\substack{\eta\to\xi\\ t\to+\infty}} [-j^0(t\eta;-\eta)], \quad \xi\in\mathbb{R}^s. \tag{11}$$

From now on we assume that $g\in V^*$ fulfills the compatibility condition

$$\langle g,\theta\rangle_V < \Phi^\infty(\theta) + \int j^\infty(\theta)d\Omega, \quad \forall\,\theta\in V_0\setminus\{0\}, \tag{12}$$

where $\Phi^\infty : V \to \mathbb{R} \cup \{+\infty\}$, defined by

$$\Phi^\infty(u) = \lim_{\lambda \to +\infty} \frac{\Phi(u_0 + \lambda u) - \Phi(u_0)}{\lambda}, \quad u_0 \in \operatorname{Dom} \Phi,$$

stands for the recession functional of Φ (see Baiocchi et al., 1988, Goeleven, 1996).

LEMMA 2. *Suppose that* (1)-(3) *and* (12) *hold. Then there exists $R_0 > 0$ such that for any $R > R_0$ the set of all $u \in V$ with the property that*

$$\langle Au - g, u\rangle_V + \Phi(u) - \int_\Omega \widetilde{j_R^0}(u; -u)\, d\Omega \leq \mathfrak{C} \tag{13}$$

is bounded in V, i.e. there exists $M_R > 0$ such that (13) *implies*

$$\|u\|_V \leq M_R. \tag{14}$$

Proof. Suppose on the contrary that this claim is not true, i.e. there exists a sequence $\{u_n\}_{n=1}^\infty \subset V$ with the property that

$$\langle Au_n - g, u_n\rangle_V + \Phi(u_n) - \int_\Omega \widetilde{j_R^0}(u_n; -u_n)\, d\Omega \leq \mathfrak{C}, \tag{15}$$

where $\|u_n\|_V \to \infty$ as $n \to \infty$. By the hypothesis, each element u_n can be represented as

$$u_n = \widehat{u}_n + e_n\theta_n, \tag{16}$$

where $\widehat{u}_n \in \widehat{V}$, $e_n \geq 0$, $\theta_n \in V_0$, $\|\theta_n\|_V = 1$ and $\langle Au_n, u_n\rangle_V \geq c(\|\widehat{u}_n\|_V)\|\widehat{u}_n\|_V$. Since Φ is lower semicontinuous,

$$\Phi(u) \geq -a\|u\|_V - b, \quad \forall\, u \in V, \tag{17}$$

for some $a, b \in \mathbb{R}$. Therefore, taking into account (3) it follows that

$$\begin{aligned}
\mathfrak{C} &\geq \langle Au_n - g, u_n\rangle_V + \Phi(u_n) - \int_\Omega \widetilde{j_R^0}(u_n; -u_n) d\Omega \\
&\geq c(\|\widehat{u}_n\|_V)\|\widehat{u}_n\|_V - \|g\|_{V^*}\|\widehat{u}_n\|_V - e_n\langle g, \theta_n\rangle_V + \Phi(\widehat{u}_n + e_n\theta_n) \\
&\quad - \widehat{k}\int_\Omega |\widehat{u}_n + e_n\theta_n|\, d\Omega \\
&\geq c(\|\widehat{u}_n\|_V)\|\widehat{u}_n\|_V - \|g\|_{V^*}\|\widehat{u}_n\|_V - e_n\langle g, \theta_n\rangle_V + \Phi(\widehat{u}_n + e_n\theta_n) \\
&\quad - k\|\widehat{u}_n\|_V - e_n k\|\theta_n\|_V \\
&\geq \|\widehat{u}_n\|_V\left(c(\|\widehat{u}_n\|_V) - \|g\|_{V^*} - k\right) - e_n\left(\langle g, \theta_n\rangle_V + k\right) + \Phi(\widehat{u}_n + e_n\theta_n)
\end{aligned}$$

where $k = \text{const}$. The obtained estimates imply that $\{e_n\}$ is unbounded. Indeed, if it were not so, then due to the behavior of $c(\cdot)$ at infinity, $\{\widehat{u}_n\}$ would to be bounded. In such a case the contradiction with $\|u_n\|_V \to \infty$ as $n \to \infty$ results. Therefore, one can suppose without loss of generality that $e_n \to +\infty$ as $n \to \infty$. The next claim is that

$$\frac{1}{e_n}\widehat{u}_n \to 0 \text{ strongly in } V. \tag{19}$$

Indeed, if $\{\|\widehat{u}_n\|_V\}$ is bounded then (19) follows immediately. If $\|\widehat{u}_n\|_V \to \infty$ then $c(\|\widehat{u}_n\|_V) \to +\infty$. From (18) one has

$$k + a + \|g\|_{V^*} + \frac{\mathfrak{C} + b}{e_n} \geq \big(c(\|\widehat{u}_n\|_V) - \|g\|_{V^*} - k - a\big)\frac{\|\widehat{u}_n\|_V}{e_n}.$$

Thus, the boundedness of the sequence

$$\left\{\big(c(\|\widehat{u}_n\|_V) - \|g\|_{V^*} - k - a\big)\frac{\|\widehat{u}_n\|_V}{e_n}\right\}_{n=1}^{\infty}$$

results, which in view of

$$c(\|\widehat{u}_n\|_V) - \|g\|_{V^*} - k - a \to +\infty \text{ as } n \to \infty,$$

implies the assertion (19). The obtained results give rise to the following representation of u_n:

$$u_n = e_n\left(\frac{1}{e_n}\widehat{u}_n + \theta_n\right),$$

where $\frac{1}{e_n}\widehat{u}_n \to 0$ strongly in V and $\theta_n \to \theta$ in V_0 as $n \to \infty$ for some $\theta \in V_0$ with $\|\theta\|_V = 1$ (recall that V_0 has been assumed to be finite dimensional). Moreover, the compact imbedding $V \subset L^p(\Omega; \mathbb{R}^s)$ permits to suppose that

$$\tfrac{1}{e_n}\widehat{u}_n \to 0 \text{ and } \theta_n \to \theta \text{ a.e. in } \Omega. \tag{20}$$

Using (15) together with the semicoercivity of A leads to

$$\begin{aligned}
\mathfrak{C} &\geq \langle Au_n - g, u_n\rangle_V + \Phi(u_n) - \int_\Omega \widetilde{j_R^0}(u_n; -u_n)\,d\Omega \\
&\geq \big(c(\|\widehat{u}_n\|_V) - \|g\|_{V^*}\big)\|\widehat{u}_n\|_V - e_n\langle g, \theta_n\rangle_V \\
&+\Phi(\widehat{u}_n + e_n\theta_n) + e_n\int_\Omega -\widetilde{j_R^0}\Big(e_n(\tfrac{1}{e_n}\widehat{u}_n + \theta_n); -\tfrac{1}{e_n}\widehat{u}_n - \theta_n\Big)d\Omega.
\end{aligned}$$

Hence

$$\begin{aligned}
\langle g, \theta_n\rangle_V &\geq \big(c(\|\widehat{u}_n\|_V) - \|g\|_{V^*}\big)\frac{1}{e_n}\|\widehat{u}_n\|_V + \frac{1}{e_n}\Phi\big(e_n(\tfrac{\widehat{u}_n}{e_n} + \theta_n)\big) \\
&+ \int \widetilde{j^0}\big(e_n(\tfrac{\widehat{u}_n}{e_n} + \theta_n); -\tfrac{\widehat{u}_n}{e_n} - \theta_n\big)d\Omega - \frac{\mathfrak{C}}{[illegible]}
\end{aligned} \tag{21}$$

Now observe that either

$$\big(c(\|\widehat{u}_n\|_V) - \|g\|_{V^*}\big)\frac{1}{e_n}\|\widehat{u}_n\|_V \to 0 \quad \text{as } n \to \infty,$$

if $\{\|\widehat{u}_n\|_V\}$ is bounded, or

$$\big(c(\|\widehat{u}_n\|_V) - \|g\|_{V^*}\big)\frac{1}{e_n}\|\widehat{u}_n\|_V \geq 0$$

for sufficiently large n, if $\|\widehat{u}_n\|_V \to \infty$ as $n \to \infty$. Therefore, for each case

$$\liminf_{n\to\infty}\big(c(\|\widehat{u}_n\|_V) - \|g\|_{V^*}\big)\frac{1}{e_n}\|\widehat{u}_n\|_V \geq 0.$$

Moreover, by (10) the estimate follows

$$-\widetilde{j_R^0}\Big(e_n\big(\tfrac{\widehat{u}_n}{e_n} + \theta_n\big); -\tfrac{\widehat{u}_n}{e_n} - \theta_n\Big) \geq -\widehat{k}\left|\frac{\widehat{u}_n}{e_n} + \theta_n\right|. \tag{22}$$

This allows the application of Fatou's lemma in (21) from which one is led to

$$\begin{aligned}
\langle g, \theta\rangle_V &\geq \liminf_{n\to\infty}\frac{1}{e_n}\Phi\big(e_n\big(\tfrac{\widehat{u}_n}{e_n} + \theta_n\big)\big)\\
&+ \liminf_{n\to\infty}\int_\Omega\Big[-\widetilde{j_R^0}\big(e_n\big(\tfrac{\widehat{u}_n}{e_n} + \theta_n\big); -\tfrac{\widehat{u}_n}{e_n} - \theta_n\big)\Big]d\Omega\\
&\geq \liminf_{n\to\infty}\frac{1}{e_n}\Phi\big(e_n\big(\tfrac{\widehat{u}_n}{e_n} + \theta_n\big)\big)\\
&+ \int_\Omega\liminf_{n\to\infty}\Big[-\widetilde{j_R^0}\big(e_n\big(\tfrac{\widehat{u}_n}{e_n} + \theta_n\big); -\tfrac{\widehat{u}_n}{e_n} - \theta_n\big)\Big]d\Omega.
\end{aligned} \tag{23}$$

Taking into account (20) and upper semicontinuity of $j^0(\cdot\,,\cdot)$ one can easily verify that

$$\liminf_{n\to\infty}\Big[-\widetilde{j_R^0}\big(e_n\big(\tfrac{\widehat{u}_n}{e_n} + \theta_n\big); -\tfrac{\widehat{u}_n}{e_n} - \theta_n\big)\Big] \geq -j_{R+}^0(\theta; -\theta),$$

where

$$j_{R+}^0(\theta; -\theta) := \begin{cases} j^0\big(\frac{R}{|\theta|}\theta; -\theta\big) & \text{a.e. in } \{x \in \Omega\colon \theta(x) \neq 0\}\\ 0 & \text{a.e. in } \{x \in \Omega\colon \theta(x) = 0\}.\end{cases}$$

Further, for a convex, lower semicontinuous $\Phi : V \to \mathbb{R}\cup\{+\infty\}$ we have (Baiocchi et al., 1988)

$$\liminf \frac{1}{e_n}\Phi\big(e_n\big(\tfrac{\widehat{u}_n}{e_n} + \theta_n\big)\big) \geq \Phi^\infty(\theta)$$

Thus we arrive at

$$\langle g,\theta\rangle_V \geq \Phi^\infty(\theta) + \int_\Omega -j^0_{R+}(\theta;-\theta)\,d\Omega. \tag{24}$$

Since $j^\infty(\cdot)$ is lower semicontinuous and V_0 is finite dimensional, it follows from (12) that a $\delta > 0$ can be found such that for any $\theta \in V_0$ with $\|\theta\|_V = 1$,

$$\langle g,\theta\rangle_V + \delta < \Phi^\infty(\theta) + \int_\Omega j^\infty(\theta)d\Omega. \tag{25}$$

With the help of Fatou's lemma (permitted by (22)) we arrive at

$$\liminf_{R\to\infty} \int_\Omega -j^0_{R+}(\theta;-\theta)d\Omega \geq \int_\Omega j^\infty(\theta)d\Omega. \tag{26}$$

The upper semicontinuity of $j^0(\cdot\,\cdot)$ allows us to conclude the existence of $R_\theta > 0$ and $\varepsilon_\theta > 0$ such that

$$\int_\Omega -j^0_{R+}(\theta';-\theta')d\Omega \geq \int_\Omega j^\infty(\theta)d\Omega - \frac{\delta}{2}$$

for each $R > R_\theta$ and $\theta' \in V_0$ with $\|\theta-\theta'\|_V < \varepsilon_\theta$. As the sphere $\{v \in V_0 : \|v\|_V = 1\}$ is compact in V_0, there exists $R_0 > 0$ such that

$$\int_\Omega -j^0_{R+}(\theta;-\theta)d\Omega \geq \int_\Omega j^\infty(\theta)d\Omega - \frac{\delta}{2}, \tag{27}$$

for any $\theta \in V_0$ with $\|\theta\|_V = 1$, $R > R_0$. This combined with (24) contradicts (12). Accordingly, the existence of a constant $M_R > 0$ has been established such that (13) implies (14), provided $R > R_0$. The proof of Lemma 2 is complete. ■

3. Regularized semicoercive variational-hemivariational inequality

Define a mapping $\Gamma_R : L^p(\Omega;\mathbb{R}^s) \to 2^{L^q(\Omega;\mathbb{R}^s)}$ by the following formula

$$\Gamma_R(v) = \left\{\psi \in L^q(\Omega;\mathbb{R}^s) : \int_\Omega \psi\cdot w\,d\Omega \leq \int_\Omega \widetilde{j^0_R}(v;w)\,d\Omega,\ \ \forall\, w \in L^p(\Omega;\mathbb{R}^s)\right\}, \tag{28}$$

where $\frac{1}{p}+\frac{1}{q} = 1$. It is easy to verify that the restriction of Γ_R to V, $\Gamma_R|_V : V \to 2^{V^*}$, is pseudo monotone.

For any $R > 0$ the following regularization of the primal problem (P) can

PROBLEM (P_R). Find a triple $(u_R, \chi_R, \psi_R) \in V \times L^q(\Omega; \mathbb{R}^s) \times V^*$, such that

$$\begin{aligned} &\langle Au_R - g, v - u_R \rangle_V + \langle \psi_R, v - u_R \rangle_V + \int_\Omega \chi_R \cdot (v - u_R) d\Omega \\ &= 0, \quad \forall\, v \in V \qquad (29) \\ &\chi_R \in \Gamma_R(u_R), \quad \psi_R \in \partial\Phi(u_R). \qquad (30) \end{aligned}$$

PROPOSITION 3. *Assume that* (1)–(3), (12) *hold. Moreover, suppose*

$$0 \in \operatorname{Dom}\Phi. \qquad (31)$$

Then for any $R > R_0$ (with R_0 as in Lemma 2) there exist $u_R \in V$, $\chi_R \in L^q(\Omega; \mathbb{R}^s)$ and $\psi_R \in V^*$ such that (29) *and* (30) *hold, i.e. the problem (P_R) has solutions. Moreover, $\{u_R\}_{R>R_0}$ is bounded in V, i.e. there exists $M > 0$ independent of $R > R_0$, such that*

$$\|u_R\|_V \le M, \quad \forall\, R > R_0. \qquad (32)$$

Proof. Fix $R > R_0$. The sum of two pseudo-monotone mappings $A + \Gamma_R|_V$ is pseudo-monotone (Browder and Hess, 1972). Thus the variational inequality

$$\langle Au + \Gamma_R|_V(u_R) - g, v - u_R \rangle + \Phi(v) - \Phi(u_R) \ge 0, \quad \forall\, v \in B_{2M_R}, \qquad (33)$$

with M_R as in Lemma 2, $B_{2M_R} = \{v \in V : \|v\|_V \le 2M_R\}$ being a ball in V with the radius $2R$, admits a solution $u_R \in B_{2M_R}$ (see Browder and Hess, 1972). By substituting $v = 0$ into (33) we arrive at the inequality

$$\langle Au_R - g, u_R \rangle_V + \Phi(u_R) - \int_\Omega \widetilde{j_R^0}(u_R; -u_R)\, d\Omega \le \Phi(0)$$

which by Lemma 2 with $\mathfrak{C} = \Phi(0)$ allows the conclusion that $\|u_R\|_V \le M_R$. This together with the validity of (33) for any $v \in B_{2M_R}$ implies that, in fact, it holds for any $v \in V$. Thus we easily deduce the existence of a triple (u_R, χ_R, ψ_R) which is a solution of (P_R).

Let us proceed to the boundedness of $\{u_R\}_{R>R_0}$. To begin with let us note that from (29) and (30) the inequality follows

$$\begin{aligned} &\langle Au_R - g, v - u_R \rangle_V + \Phi(v) - \Phi(u_R) \\ &+ \int_\Omega \widetilde{j_R^0}(u_R; v - u_R) d\Omega \ge 0, \quad \forall\, v \in V, \qquad (34) \end{aligned}$$

which by substitution $v = 0$ leads to

$$\langle Au_R - g, u_R \rangle + \Phi(u_R) - \int \widetilde{j_R^0}(u_R; -u_R)\, d\Omega \le \Phi(0)$$

Now suppose on the contrary that the claim is not true. Then, there would exist a sequence $R_n \to \infty$ such that $\|u_{R_n}\|_V \to \infty$ as $n \to \infty$, and

$$\langle Au_{R_n} - g, u_{R_n}\rangle_V + \Phi(u_{R_n}) - \int_\Omega \widetilde{j^0_{R_n}}(u_{R_n}; -u_{R_n})d\Omega \le \Phi(0).$$

For simplicity of notations, instead of subscript "R_n" we write "n". Thus we have

$$\langle Au_n - g, u_n\rangle_V + \Phi(u_n) - \int_\Omega \widetilde{j^0_n}(u_n; -u_n)d\Omega \le \Phi(0). \tag{35}$$

Now we follow the lines of the proof of Lemma 2. Analogously, we arrive at the representation

$$u_n = e_n\left(\frac{1}{e_n}\widehat{u}_n + \theta_n\right),$$

with $e_n \to \infty$, $\frac{1}{e_n}\widehat{u}_n \to 0$ strongly in V and $\theta_n \to \theta$ in V_0 as $n \to \infty$ for some $\theta \in V_0$ with $\|\theta\|_V = 1$, and then at the inequality

$$\langle g, \theta\rangle_V \ge \Phi^\infty(\theta) + \liminf_{n\to\infty} \int_\Omega -\widetilde{j^0_n}\Big(e_n(\tfrac{1}{e_n}\widehat{u}_n + \theta_n); -\tfrac{1}{e_n}\widehat{u}_n - \theta_n\Big)d\Omega. \tag{36}$$

But

$$\widetilde{j^0_n}\big(e_n(\tfrac{1}{e_n}\widehat{u}_n + \theta_n); -\tfrac{1}{e_n}\widehat{u}_n - \theta_n\big) = \begin{cases} j^0\Big(e_n(\tfrac{1}{e_n}\widehat{u}_n + \theta_n); -\tfrac{1}{e_n}\widehat{u}_n - \theta_n\Big) & \text{if } |\widehat{u}_n + e_n\theta_n| \le R_n \\ j^0\Big(\frac{R_n}{\left|\frac{1}{e_n}\widehat{u}_n + \theta_n\right|}(\tfrac{1}{e_n}\widehat{u}_n + \theta_n); -\tfrac{1}{e_n}\widehat{u}_n - \theta_n\Big) & \text{if } |\widehat{u}_n + e_n\theta_n| > R_n, \end{cases}$$

with

$$e_n \to \infty \quad \text{and} \quad \frac{R_n}{\left|\frac{1}{e_n}\widehat{u}_n + \theta_n\right|} \to \infty \text{ a.e. in } \Omega,$$

because $\frac{1}{e_n}\widehat{u}_n + \theta_n \to \theta$ a.e. in Ω. Therefore, we easily conclude, using (11), that

$$\liminf_{n\to\infty} \widetilde{j^0_n}\Big(e_n(\tfrac{1}{e_n}\widehat{u}_n + \theta_n); -\tfrac{1}{e_n}\widehat{u}_n - \theta_n\Big) \ge j^\infty(\theta) \quad \text{a.e. in } \Omega.$$

Finally, by Fatou's lemma, permitted due to (3), we are led to the conclusion

$$\langle g, \theta\rangle_V \ge \Phi^\infty(\theta) + \int_\Omega j^\infty(\theta)\, d\Omega,$$

contrary to (12). This contradiction yields the boundedness of $\{u_R\}_{R>R_0}$. The proof of Proposition 3 is complete. ∎

The next result is related to the compactness property of $\{\chi_R\}_{R>R_0}$ in

PROPOSITION 4. *Let a triple* $(u_R, \chi_R, \psi_R) \in V \times L^q(\Omega; \mathbb{R}^s) \times V^*$ *be a solution of* (P_R). *Then the set* $\{\chi_R\}_{R>R_0}$ *is weakly precompact in* $L^1(\Omega; \mathbb{R}^s)$.

Proof. According to the Dunford-Pettis theorem (Ekeland and Temam, 1976) it suffices to show that for each $\varepsilon > 0$ a $\delta > 0$ can be determined such that for any $\omega \subset \Omega$ with $|\omega| < \delta$,

$$\int_\omega |\chi_R|\, d\Omega < \varepsilon, \quad R > R_0. \tag{37}$$

Fix $r > 0$ and let $\eta \in \mathbb{R}^s$ be such that $|\eta| \le r$. Then one has $\chi_R \cdot (\eta - u_R) \le \widetilde{j^0_R}(u_R; \eta - u_R)$ from which, by virtue of (9) it results that

$$\chi_R \cdot \eta \le \chi_R \cdot u_R + \tilde{\alpha}(r)(1 + |u_R|^\sigma) \tag{38}$$

a.e. in Ω. Let us set $\eta \equiv \frac{r}{\sqrt{s}}(\operatorname{sgn} \chi_{R_1}, \dots, \operatorname{sgn} \chi_{R_s})$, where χ_{R_i}, $i = 1, 2, \dots, s$, are the components of χ_R and where $\operatorname{sgn} y = 1$ if $y > 0$, $\operatorname{sgn} y = 0$ if $y = 0$, and $\operatorname{sgn} y = -1$ if $y < 0$. It is not difficult to verify that $|\eta| \le r$ for almost all $x \in \Omega$ and that $\chi_R \cdot \eta \ge \frac{r}{\sqrt{s}} |\chi_R|$. Therefore, by virtue of (38) one is led to the estimate

$$\frac{r}{\sqrt{s}} |\chi_R| \le \chi_R \cdot u_R + \tilde{\alpha}(r)(1 + |u_R|^\sigma).$$

Integrating this inequality over $\omega \subset \Omega$ yields

$$\int_\omega |\chi_R|\, d\Omega \le \frac{\sqrt{s}}{r} \int_\omega \chi_R \cdot u_R\, d\Omega + \frac{\sqrt{s}}{r} \tilde{\alpha}(r)\, |\omega| + \frac{\sqrt{s}}{r} \tilde{\alpha}(r)(|\omega|)^{\frac{p-\sigma}{p}} \|u_R\|^\sigma_{L^p(\Omega)}. \tag{39}$$

Thus, from (32) one obtains

$$\begin{aligned} \int_\omega |\chi_R|\, d\Omega &\le \frac{\sqrt{s}}{r} \int_\omega \chi_R \cdot u_R\, d\Omega + \frac{\sqrt{s}}{r} \tilde{\alpha}(r)\, |\omega| \\ &+ \frac{\sqrt{s}}{r} \tilde{\alpha}(r)(|\omega|)^{\frac{p-\sigma}{p}} \gamma^\sigma \|u_R\|^\sigma_V \\ &\le \frac{\sqrt{s}}{r} \int_\omega \chi_R \cdot u_R\, d\Omega + \frac{\sqrt{s}}{r} \tilde{\alpha}(r)\, |\omega| + \frac{\sqrt{s}}{r} \tilde{\alpha}(r)(|\omega|)^{\frac{p-\sigma}{p}} \gamma^\sigma M^\sigma \end{aligned} \tag{40}$$

$(\|\cdot\|_{L^p(\Omega;\mathbb{R}^s)} \le \gamma \|\cdot\|_V)$. The next claim is that

$$\int_\omega \chi_R \cdot u_R\, d\Omega \le C \tag{41}$$

for some positive constant C not depending on $\omega \subset \Omega$ and $R > R_0$. Indeed, from (10) one can easily deduce that

Thus it follows that

$$\int_\omega \left(\chi_R \cdot u_R + \widehat{k}\,|u_R|\right) d\Omega \le \int_\Omega \left(\chi_R \cdot u_R + \widehat{k}\,|u_R|\right) d\Omega,$$

and consequently

$$\int_\omega \chi_R \cdot u_R \, d\Omega \le \int_\Omega \chi_R \cdot u_R \, d\Omega + \widehat{k}_1 \|u_R\|_V.$$

But A maps bounded sets into bounded sets. Therefore, by means of (17), (29), (30) and (32),

$$\begin{aligned}\int_\Omega \chi_R \cdot u_R \; d\Omega &\le \langle Au_R - g, -u_R \rangle_V + \Phi(0) - \Phi(u_R) \\ &\le \|Au_R - g\|_{V^*} \|u_R\|_V + \Phi(0) + a\|u_R\|_V + b \le C, \quad C = \text{const}, \end{aligned} \tag{42}$$

and consequently, (41) easily follows. Further, from (40) and (41), for $r > 0$,

$$\int_\omega |\chi_R| \; d\Omega \le \frac{\sqrt{s}}{r} C + \frac{\sqrt{s}}{r} \widetilde{\alpha}(r)\,|\omega| + \frac{\sqrt{s}}{r} \widetilde{\alpha}(r)(|\omega|)^{\frac{p-\sigma}{p}} \gamma^\sigma M^\sigma. \tag{43}$$

This estimate is crucial for (37) to be obtained. Namely, let $\varepsilon > 0$. Fix $r > 0$ with

$$\frac{\sqrt{s}}{r} C < \frac{\varepsilon}{2} \tag{44}$$

and determine $\delta > 0$ small enough to fulfill

$$\frac{\sqrt{s}}{r} \widetilde{\alpha}(r)\,|\omega| + \frac{\sqrt{s}}{r} \widetilde{\alpha}(r)(|\omega|)^{\frac{p-\sigma}{p}} \gamma^\sigma M^\sigma \le \frac{\varepsilon}{2},$$

provided that $|\omega| < \delta$. Thus from (43) it follows that for any $\omega \subset \Omega$,

$$\int_\omega |\chi_R| \; d\Omega \le \varepsilon, \quad R > R_0, \tag{45}$$

whenever $|\omega| < \delta$. Finally, $\{\chi_R\}_{R > R_0}$ is equi-integrable and its precompactness in $L^1(\Omega; \mathbb{R}^s)$ has been proved (Ekeland and Temam, 1976). ■

4. Semicoercive variational-hemivariational inequality

Now the main result will be formulated.

THEOREM 5. *Let $A : V \to V^*$ be a pseudo-monotone, bounded operator, $j : \mathbb{R}^s \to \mathbb{R}$ a locally Lipschitz function. Let $\Phi : V \to R \cup \{+\infty\}$ be a convex, lower semicontinuous function with $0 \in \operatorname{Dom}\Phi$, $g \in V^*$ an element of V^*. Suppose that (1)–(3) and (12) hold. Moreover, assume that the following hypothesis*

(H) *For any* $v \in \operatorname{Dom}\Phi$ *there exists a sequence of functions* $\{\varepsilon_k\} \subset L^\infty(\Omega)$ *with* $0 \le \varepsilon_k \le 1$, $\{(1-\varepsilon_k)v\} \subset V \cap L^\infty(\Omega;\mathbb{R}^s)$ *and* $\widetilde{v}_k := (1-\varepsilon_k)v \to v$ *strongly in* V *such that*

$$\lim_{k\to\infty} \Phi(\widetilde{v}_k) = \Phi(v).$$

Then, the variational-hemivariational inequality

$$\langle Au - g, v - u\rangle_V + \Phi(v) - \Phi(u) + \int_\Omega j^0(u; v-u)\,d\Omega \ge 0, \quad \forall v \in V \quad (46)$$

admits solutions.

Proof. The proof is divided into a sequence of steps.
Step 1. By Proposition 3 and Proposition 4 it follows that from the set $\{u_R, \chi_R, \psi_R\}_{R>R_0}$ of solutions of (P_R) a sequence $\{u_{R_n}, \chi_{R_n}, \psi_{R_n}\}$ can be extracted with $R_n \to \infty$ as $n \to \infty$ (for simplicity of notations it will be denoted by $\{u_n, \chi_n, \psi_n\}$), such that

$$\begin{aligned}&\langle Au_n - g, v - u_n\rangle_V + \langle \psi_n, v - u_n\rangle_V\\ &+ \int_\Omega \chi_n \cdot (v - u_n)\,d\Omega = 0, \quad \forall\, v \in V,\end{aligned} \quad (47)$$

and

$$\left.\begin{aligned}&\chi_n \in \Gamma_n(u_n) \quad (\Gamma_n := \Gamma_{R_n})\\ &\psi_n \in \partial\Phi(u_n)\\ &u_n \to u \quad \text{weakly in } V\\ &\chi_n \to \chi \quad \text{weakly in } L^1(\Omega;\mathbb{R}^s).\end{aligned}\right\} \quad (48)$$

From (47) and (48) we get

$$\begin{aligned}&\langle Au_n - g, v - u_n\rangle_V + \Phi(v) - \Phi(u_n)\\ &+ \int_\Omega \widetilde{j}_n^0(u_n; v - u_n)\,d\Omega \ge 0, \quad \forall\, v \in V.\end{aligned} \quad (49)$$

Hence by setting $v = 0$ we obtain

$$\Phi(u_n) \le \|Au_n - g\|_{V^*}\|u_n\|_V + \Phi(0) + k\|u_n\|_V \le C, \quad C = \text{const},$$

which by (32), the boundedness of A, and lower semicontinuity of Φ yields $\Phi(u) \in \mathbb{R}$, i.e. $u \in \operatorname{Dom}\Phi$.

Step 2. Now we prove that $\chi \in \partial j(u)$ a.e in Ω. Since V is compactly imbedded into $L^p(\Omega;\mathbb{R}^s)$, due to (48) one may suppose that

$$u_n \to u \quad \text{strongly in } L^p(\Omega;\mathbb{R}^s) \quad (50)$$

This implies that for a subsequence of $\{u_n\}$ (again denoted by the same symbol) one gets $u_n \to u$ a.e. in Ω. Thus, Egoroff's theorem can be applied from which it follows that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with $\operatorname{mes}\omega < \varepsilon$ can be determined such that $u_n \to u$ uniformly in $\Omega \setminus \omega$ with $u \in L^\infty(\Omega \setminus \omega; \mathbb{R}^s)$. Let $v \in L^\infty(\Omega \setminus \omega; \mathbb{R}^s)$ be an arbitrary function. From the estimate

$$\int_{\Omega\setminus\omega} \chi_n \cdot v \, d\Omega \le \int_{\Omega\setminus\omega} \widetilde{j_n^0}(u_n; v)\, d\Omega = \int_{\Omega\setminus\omega} j^0(u_n; v)\, d\Omega, \quad \text{(for large } n)$$

(u_n remains bounded in $\Omega \setminus \omega$ as $n \to \infty$) combined with the weak convergence in $L^1(\Omega; \mathbb{R}^s)$ of χ_n to χ, (50) and with the upper semicontinuity of

$$L^\infty(\Omega \setminus \omega; \mathbb{R}^s) \ni u_n \longmapsto \int_{\Omega\setminus\omega} j^0(u_n; v)\, d\Omega$$

it follows that

$$\int_{\Omega\setminus\omega} \chi \cdot v\, d\Omega \le \int_{\Omega\setminus\omega} j^0(u; v)\, d\Omega, \quad \forall\, v \in L^\infty(\Omega \setminus \omega; \mathbb{R}^s).$$

But the last inequality amounts to saying that $\chi \in \partial j(u)$ a.e. in $\Omega \setminus \omega$. Since $|\omega| < \varepsilon$ and ε was chosen arbitrarily,

$$\chi \in \partial j(u) \quad \text{a.e. in } \Omega, \tag{51}$$

as claimed.

Step 3. Now it will be shown that

$$\limsup_{n\to\infty} \int_\Omega \widetilde{j_n^0}(u_n; v - u_n)\, d\Omega \le \int_\Omega j^0(u; v - u)\, d\Omega \tag{52}$$

holds for any $v \in V \cap L^\infty(\Omega; \mathbb{R}^s)$. It can be supposed that $u_n \to u$ a.e. in Ω, since $u_n \to u$ in $L^p(\Omega; \mathbb{R}^s)$. Fix $v \in L^\infty(\Omega; \mathbb{R}^s)$ arbitrarily. In view of $\chi_n \in \Gamma_n(u_n)$, Eq. (9) implies

$$\widetilde{j_n^0}(u_n; v - u_n) \le \widetilde{\alpha}(\|v\|_{L^\infty(\Omega;\mathbb{R}^s)})(1 + |u_n|^\sigma). \tag{53}$$

From Egoroff's theorem it follows that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with $\operatorname{mes}\omega < \varepsilon$ can be determined such that $u_n \to u$ uniformly in $\Omega \setminus \omega$. One can also suppose that ω is small enough to fulfill $\int_\omega \widetilde{\alpha}(\|v\|_{L^\infty(\Omega;\mathbb{R}^s)})(1+|u_n|^\sigma)\, d\Omega \le \varepsilon$, $n = 1, 2, \dots$, and $\int_\omega \widetilde{\alpha}(\|v\|_{L^\infty(\Omega;\mathbb{R}^s)})(1 + |u|^\sigma)\, d\Omega \le \varepsilon$. Hence

$$\int_\Omega \widetilde{j_n^0}(u_n; v - u_n)\, d\Omega \le \int_{\Omega\setminus\omega} \widetilde{j_n^0}(u_n; v - u_n)\, d\Omega + \varepsilon$$

$$= \int_{\Omega\setminus\omega} j^0(u_n; v - u_n)\, d\Omega + \varepsilon \quad \text{(for large } n),$$

which, by Fatou's lemma and upper semicontinuity of $j^0(\cdot\,;\cdot)$, yields

$$\limsup_{n\to\infty} \int_\Omega \widetilde{j}_n^0(u_n; v - u_n)\, d\Omega \le \int_\Omega j^0(u; v - u)\, d\Omega + 2\varepsilon.$$

By arbitrariness of $\varepsilon > 0$ one obtains (52), as required.

Step 4. Now we show that

$$\chi \cdot u \in L^1(\Omega) \tag{54}$$

$$\liminf_{n\to\infty} \int_\Omega \chi_n \cdot u_n\, d\Omega \ge \int_\Omega \chi \cdot u\, d\Omega. \tag{55}$$

For this purpose let $\widetilde{u}_k = (1-\varepsilon_k)u$ be as stated in (H). Without loss of generality it can be assumed that $\widetilde{u}_k \to u$ a.e. in Ω. Since it is already known that $\chi \in \partial j(u)$, one can apply (3) to obtain $\chi \cdot (-u) \le j^0(u; -u) \le \widehat{k}\,|u|$. Hence

$$\chi \cdot \widetilde{u}_k = (1 - \varepsilon_k)\chi \cdot u \ge -\widehat{k}\,|u|\,. \tag{56}$$

This implies that the sequence $\{\chi \cdot \widetilde{u}_k\}$ is bounded from below by an integrable function and $\chi \cdot \widetilde{u}_k \to \chi \cdot u$ a.e. in Ω. On the other hand, one gets

$$\int_\Omega \chi_n \cdot (\widetilde{u}_k - u_n)\, d\Omega \le \int_\Omega \widetilde{j}_n^0(u_n; \widetilde{u}_k - u_n)\, d\Omega.$$

Thus

$$\int_\Omega \chi \cdot \widetilde{u}_k\, d\Omega - \liminf_{n\to\infty} \int_\Omega \chi_n \cdot u_n\, d\Omega \le \limsup_{n\to\infty} \int_\Omega \widetilde{j}_n^0(u_n; \widetilde{u}_k - u_n)\, d\Omega,$$

and due to (52) we are led to the estimate

$$\begin{aligned}
\int_\Omega \chi \cdot \widetilde{u}_k\, d\Omega &\le \liminf_{n\to\infty} \int_\Omega \chi_n \cdot u_n\, d\Omega + \int_\Omega j^0(u; \widetilde{u}_k - u)\, d\Omega \\
&\le \liminf_{n\to\infty} \int_\Omega \chi_n \cdot u_n\, d\Omega + \int_\Omega j^0(u; -\varepsilon_k u)\, d\Omega \\
&\le \liminf_{n\to\infty} \int_\Omega \chi_n \cdot u_n\, d\Omega + \int_\Omega \varepsilon_k \widehat{k}\,|u|\, d\Omega \le C, \quad C = \text{const}\,.
\end{aligned}$$

Thus, by Fatou's lemma we are allowed to conclude that $\chi \cdot u \in L^1(\Omega)$, i.e. (54) holds. Taking into account that $\varepsilon_k \to 0$ as $k \to \infty$ a.e. in Ω (passing to a subsequence if necessary) we establish (55), as required.

Step 5. In this step it will be shown that

$$\limsup \langle Au_n, u_n - u\rangle_V \le 0. \tag{57}$$

Taking into account (49) one has

$$\langle Au_n - g, u_n - \widetilde{u}_k \rangle_V \le \Phi(\widetilde{u}_k) - \Phi(u_n) + \int_\Omega \widetilde{j}_n^0(u_n; \widetilde{u}_k - u_n)\, d\Omega.$$

Since

$$\limsup_{n\to\infty} \langle Au_n - g, u_n - \widetilde{u}_k \rangle_V = \limsup_{n\to\infty} \langle Au_n, u_n - u \rangle_V + \langle B - g, u - \widetilde{u}_k \rangle_V,$$

we obtain

$$\limsup_{n\to\infty} \langle Au_n, u_n - u \rangle_V \le \Phi(\widetilde{u}_k) - \Phi(u) + \int_\Omega j^0(u; \widetilde{u}_k - u)\, d\Omega + \langle B - g, \widetilde{u}_k - u \rangle_V. \tag{58}$$

Here, because of the boundedness of A, we have supposed without loss of generality that for some $B \in V^\star$, $Au_n \to B$ weakly in $V^\star$. Now observe that $\widetilde{u}_k \to u$ a.e. in Ω, and

$$j^0(u; \widetilde{u}_k - u) = \varepsilon_k j^0(u; -u) \le \varepsilon_k \widehat{k}\, |u| \le \widehat{k}\, |u|,$$

hence, Fatou's lemma ensures

$$\limsup_{k\to\infty} \int_\Omega j^0(u; \widetilde{u}_k - u)\, d\Omega \le 0. \tag{59}$$

On passing to the limit in (58) as $k \to \infty$ and taking into account (H), (52) and (59) we are allowed to conclude (57), as claimed.

Step 6. Now it will be shown that (46) holds for any $v \in V$ with $v \in \operatorname{Dom}\Phi$ and $j^0(u; v - u) \in L^1(\Omega)$.

In view of (57) the pseudo-monotonicity of A yields

$$Au_n \to Au \quad \text{weakly in } V \quad (\text{i.e. } B = Au) \tag{60}$$

$$\langle Au_n, u_n \rangle_V \to \langle Au, u \rangle. \tag{61}$$

Let $v \in V \cap L^\infty(\Omega; \mathbb{R}^s)$ with $\Phi(v) < +\infty$ be chosen arbitrarily. Thus, (49) combined with (52) and lower semicontinuity of Φ yields the assertion. Now let $j^0(u; v - u) \in L^1(\Omega)$ with $v \notin V \cap L^\infty(\Omega; \mathbb{R}^s)$ and $\Phi(v) < +\infty$. According to (H) there exists a sequence $\widetilde{v}_k = (1 - \varepsilon_k)v$ such that $\{\widetilde{v}_k\} \subset V \cap L^\infty(\Omega; \mathbb{R}^s)$, $\widetilde{v}_k \to v$ strongly in V and $\Phi(\widetilde{v}_k) \to \Phi(v)$. Since, as it already has been established,

$$\langle Au - g, \widetilde{v}_k - u \rangle_V + \Phi(\widetilde{v}_k) - \Phi(u) + \int_\Omega j^0(u; \widetilde{v}_k - u)\, d\Omega \ge 0,$$

so in order to show (46) it remains to deduce that

For this purpose let us observe that $\widetilde{v}_k - u = (1-\varepsilon_k)(v-u) + \varepsilon_k(-u)$, which, combined with the convexity of $j^0(u;\cdot)$ yields the estimate

$$j^0(u;\widetilde{v}_k - u) \le (1-\varepsilon_k) j^0(u; v-u) + \varepsilon_k j^0(u;-u) \le \left|j^0(u;v-u)\right| + \widehat{k}\,|u|\,.$$

Thus, Fatou's lemma implies the assertion.

Step 7. In the final step of the proof we are to consider the case in which $v \notin \operatorname{Dom}\Phi$ or $j^0(u;v-u) \notin L^1(\Omega)$. Recall that if $j^0(u;v-u) \notin L^1(\Omega)$ then according to the convention that $+\infty - \infty = +\infty$ we have

$$\int_\Omega j^0(u;v-u)\,d\Omega$$
$$= \begin{cases} +\infty & \text{if } \int_\Omega [j^0(u;v-u)]^+\,d\Omega = +\infty \\ -\infty & \text{if } \int_\Omega [j^0(u;v-u)]^+\,d\Omega < +\infty \text{ and } \int_\Omega [j^0(u;v-u)]^-\,d\Omega = +\infty, \end{cases}$$

where the notation has been used: $r^+ := \max\{r,0\}$ and $r^- := \max\{-r,0\}$ for any $r \in \mathbb{R}$.

Thus, if $v \notin \operatorname{Dom}\Phi$ or $\int_\Omega j^0(u;v-u)\,d\Omega = +\infty$ then $\Phi(v) + \int_\Omega j^0(u;v-u)\,d\Omega = +\infty$ and (46) holds immediately.

Finally, it remains to consider the case of $v \in \operatorname{Dom}\Phi$ and $\int_\Omega j^0(u;v-u)\,d\Omega = -\infty$. It will be proved that in such a case we are led to the contradiction, which means that whenever $v \in \operatorname{Dom}\Phi$, we get either $\int_\Omega j^0(u;v-u)\,d\Omega = +\infty$ or $j^0(u;v-u) \in L^1(\Omega)$ showing in view of the previous results that (46) holds.

According to the hypothesis (H) there exists a sequence $\widetilde{v}_k = (1-\varepsilon_k)v$ such that $\{\widetilde{v}_k\} \subset V \cap L^\infty(\Omega;\mathbb{R}^s)$, $\widetilde{v}_k \to v$ strongly in V and $\Phi(\widetilde{v}_k) \to \Phi(v)$. Since, as already has been established,

$$\langle Au - g, \widetilde{v}_k - u\rangle_V + \Phi(\widetilde{v}_k) - \Phi(u) + \int_\Omega j^0(u;\widetilde{v}_k - u)\,d\Omega \ge 0,$$

we get

$$\int_\Omega j^0(u;\widetilde{v}_k - u)\,d\Omega \ge \langle Au - g, -\widetilde{v}_k + u\rangle_V - \Phi(\widetilde{v}_k) + \Phi(u) \ge -C,$$
$$C = \text{const},$$

and consequently

$$\int_\Omega [j^0(u;\widetilde{v}_k - u)]^+\,d\Omega \ge \int_\Omega [j^0(u;\widetilde{v}_k - u)]^-\,d\Omega - C. \tag{62}$$

By the hypothesis, $\int_\Omega [j^0(u;v-u)]^-\,d\Omega = +\infty$ and $\int_\Omega [j^0(u;v-u)]^+\,d\Omega < +\infty$. Since

$$j^0(u;\widetilde{v}_k - u) \le (1-\varepsilon_k) j^0(u;v-u) + \varepsilon_k j^0(u;-u)$$

so we obtain

$$\int_\Omega [j^0(u;\widetilde{v}_k - u)]^+ \, d\Omega \le \int_\Omega [j^0(u;v-u)]^+ \, d\Omega + \int_\Omega \widehat{k}\, |u| \, d\Omega \le D,$$
$$D = \text{const},$$

which, combined with (62) yields

$$\int_\Omega [j^0(u;\widetilde{v}_k - u)]^- \, d\Omega \le C + D.$$

Thus, the application of Fatou's lemma concludes

$$\int_\Omega [j^0(u;v-u)]^- \, d\Omega \le C + D.$$

contrary to the assumption that $\int_\Omega j^0(u;v-u)\, d\Omega = -\infty$. This contradiction completes the proof of Theorem 5. ■

5. Final remarks and comments

As it is well known, Naniewicz and Panagiotopoulos (1995), in case of the classical growth condition of the form

$$|\partial j(\xi)| \le c(1 + |\xi|^{p-1}), \quad \xi \in \mathbb{R}^s, \tag{63}$$

the problem described by hemivariational inequality (46) admits a solution $u \in V$ and, moreover, there exist $\chi \in L^q(\Omega;\mathbb{R}^s)$, $1/p + 1/q = 1$, and $\psi \in V^\star$ such that

$$\chi \in \partial j(u) \text{ a.e. in } \Omega \quad \text{and} \quad \psi \in \partial\Phi(u),$$
$$g = Au + \psi + l_\chi,$$

where $l_\chi \in V^\star$ is a linear continuous functional defined by

$$\langle l_\chi, v\rangle := \int_\Omega \chi \cdot v \, d\Omega, \quad v \in V. \tag{64}$$

Recall that the subdifferential $\partial\Phi(u) \subset V^\star$ in the sense of Convex Analysis (Ekeland and Temam, 1976) is defined for $u \in \operatorname{Dom}\Phi$ by means of the formula

$$\Phi(v) - \Phi(u) \ge \langle \psi, v - u\rangle, \quad \forall\, v \in V.$$

Thus, in case of (63) a statement that $u \in V$ is a solution of hemivariational inequality (46) is equivalent to

(65)

The situation changes essentially when (63) is replaced by the unilateral growth condition (2). In such a case we have only ensured the $L^1(\Omega)$-regularity of χ and consequently, the corresponding functional l_χ (given by the formula (64)) is linear on its domain $\operatorname{Dom} l_\chi \supset L^\infty(\Omega;\mathbb{R}^s) \cap V$, but not necessarily continuous. It may happen that l_χ does not have the continuous extension onto the whole space V (l_χ is discontinuous). If it is the case, l_χ can be extended onto the whole space V as a function from V into $\mathbb{R} \cup \{+\infty, -\infty\}$ by setting

$$l_\chi(v) := \begin{cases} \int_\Omega \chi \cdot v \, d\Omega & \text{if } \chi \cdot v \in L^1(\Omega) \\ +\infty & \text{if } \int_\Omega [\chi \cdot v]^+ \, d\Omega = +\infty \\ -\infty & \text{if } \int_\Omega [\chi \cdot v]^+ \, d\Omega < +\infty \text{ and } \int_\Omega [\chi \cdot v]^- \, d\Omega = +\infty, \end{cases} \tag{66}$$

for each $v \in V$. Thus we deal with a functional $l_\chi : V \to \mathbb{R} \cup \{+\infty, -\infty\}$ which is discontinuous whenever $l_\chi(v) = +\infty$ or $l_\chi(v) = -\infty$ for at least one point of V. Notice that l_χ can be expressed as a difference of two convex lower semicontinuous proper functions $l_\chi^+(v) := \int_\Omega [\chi \cdot v]^+ \, d\Omega$ and $l_\chi^-(v) := \int_\Omega [\chi \cdot v]^- \, d\Omega$, $v \in V$, i.e.

$$l_\chi(v) = l_\chi^+(v) - l_\chi^-(v), \quad \forall v \in V. \tag{67}$$

Denote by $\mathcal{L}(\mathcal{V})$ the class of all linear densely defined functions $l : V \to \mathbb{R} \cup \{+\infty, -\infty\}$ which can be represented by a difference of two convex lower semicontinuous proper functions $l^+ : V \to \mathbb{R} \cup \{+\infty\}$ and $l^- : V \to \mathbb{R} \cup \{+\infty\}$, i.e. $l = l^+ - l^-$, with the convention that

$$l(v) := \begin{cases} l^+(v) - l^-(v) & \text{if } v \in \operatorname{Dom} l^+ \cap \operatorname{Dom} l^- \\ +\infty & \text{if } v \notin \operatorname{Dom} l^+ \\ -\infty & \text{if } v \in \operatorname{Dom} l^+ \text{ and } v \notin \operatorname{Dom} l^-. \end{cases} \tag{68}$$

For a convex, lower semicontinuous, proper function $\varphi : V \to \mathbb{R} \cup \{+\infty\}$ we introduce $\widehat{\partial}\varphi(u) \subset \mathcal{L}(\mathcal{V})$ as follows: if $u \notin \operatorname{Dom} \varphi$ then $\widehat{\partial}\varphi(u) = \emptyset$ while if $u \in \operatorname{Dom} \varphi$ then we set

$$l \in \widehat{\partial}\varphi(u) \Leftrightarrow l(u) \in \mathbb{R} \text{ and } \varphi(v) - \varphi(u) \geq l(v - u), \quad \forall v \in V. \tag{69}$$

The formal definition of $\widehat{\partial}\varphi(u)$ coincides with that of $\partial\varphi(u) \subset V^*$ in the sense of convex analysis. However, $\widehat{\partial}\varphi(u)$, apart from containing elements of $\partial\varphi(u)$, may contain also some discontinuous linear functionals which will be called here discontinuous subgradients. Notice that if $u \in \operatorname{Int}(\operatorname{Dom} \Phi)$, where "Int" means "the interior", then by the Banach-Steinhaus theorem it follows that $\widehat{\partial}\varphi(u)$ and $\partial\varphi(u)$ coincide.

In the terminology of Pallaschke and Rolewicz (1997) a function $l \in \widehat{\partial}\varphi(u)$

reader to Pallaschke and Rolewicz (1997) for the general abstract subdifferential theory.

As we shall see below, the notion of discontinuous subgradient is specially useful in describing some particular aspects of Theorem 5.

THEOREM 6. *Assume all the hypotheses of Theorem 5. Then there exists* $u \in V$ *such that*

$$\langle Au - g, v - u\rangle_V + \Phi(v) - \Phi(u) + \int_\Omega j^0(u; v-u)\, d\Omega \geq 0,\ \forall v \in V. \quad (70)$$

Moreover, there exists $\chi \in L^1(\Omega; \mathbb{R}^s)$, $\chi \in \partial j(u)$ *a.e. in* Ω, *such that for* l_χ *defined by* (66) *it follows that*

$$g - Au - l_\chi \in \widehat{\partial}\Phi(u). \quad (71)$$

Proof. Following the lines of the proof of Theorem 5 we can deduce that the inequality

$$\langle Au - g, v - u\rangle_V + \Phi(v) - \Phi(u) + \int_\Omega \chi \cdot (v-u)\, d\Omega \geq 0$$

holds for any $v \in V \cap L^\infty(\Omega; \mathbb{R}^s)$. It can be written equivalently as

$$\Phi(v) - \Phi(u) \geq \langle -Au + g, v - u\rangle_V - l_\chi(v-u), \quad \forall v \in V \cap L^\infty(\Omega; \mathbb{R}^s), \quad (72)$$

where $l_\chi(v-u) = \int_\Omega \chi \cdot (v-u)\, d\Omega$. It must be shown that this inequality holds for any $v \in V$. If $v \notin \mathrm{Dom}\,\Phi$ then there is nothing to prove because $\Phi(v) = +\infty$.

Let us consider the case of $v \in \mathrm{Dom}\,\Phi$. If $\chi \cdot v \in L^1(\Omega)$ then $-\chi \cdot \widetilde{v}_k \geq -|\chi \cdot v|$ which by Fatou's lemma yields $\liminf_{k\to\infty} -l_\chi(\widetilde{v}_k) \geq -l_\chi(v)$. Thus, in view of (H) the assertion follows ($\widetilde{v}_k$ has been taken as in the hypothesis (H)). If $\chi \cdot v \notin L^1(\Omega)$ then there is nothing to prove if $l_\chi(v) = +\infty$, while, as it will be shown, the case $l_\chi(v) = -\infty$ cannot happen. Indeed, suppose that $l_\chi(v) = -\infty$, i.e. $\int_\Omega [\chi \cdot v]^+\, d\Omega < +\infty$ and $\int_\Omega [\chi \cdot v]^-\, \Omega = +\infty$. Taking into account (72) we are led to $l_\chi(\widetilde{v}_k) \geq -C$ for a constant C. Hence

$$D \geq \int_\Omega [\chi \cdot \widetilde{v}_k]^+\, d\Omega \geq \int_\Omega [\chi \cdot \widetilde{v}_k]^-\, d\Omega - C$$

for some $D = \text{const}$. But, due to Fatou's lemma this yields

$$\int_\Omega [\chi \cdot v]^-\, d\Omega \leq C + D$$

contrary to $\int_\Omega [\chi \cdot v]^-\, d\Omega = +\infty$. This contradiction completes the proof. ∎

REMARK 7. The hypothesis (H) of Theorem 5 involves the truncation result for vector-valued Sobolev spaces $V = W^{1,r}(\Omega;\mathbb{R}^s)$, formulated as

THEOREM 8 (Naniewicz, 1997). *For each* $v \in W^{1,r}(\Omega;\mathbb{R}^s)$, $r \geq 1$, *there exists a sequence of functions* $\{\varepsilon_n\} \subset L^\infty(\Omega)$ *with* $0 \leq \varepsilon_n \leq 1$ *such that*

$$\begin{aligned} &\{(1-\varepsilon_n)v\} \subset W^{1,r}(\Omega;\mathbb{R}^s) \cap L^\infty(\Omega;\mathbb{R}^s) \\ &(1-\varepsilon_n)v \to v \quad \textit{strongly in} \;\; W^{1,r}(\Omega;\mathbb{R}^s). \end{aligned} \tag{73}$$

In the case of scalar valued Sobolev spaces $W^{m,r}(\Omega)$, $r \geq 1$, $m \geq 1$, such truncation procedure is admissible due to the famous result of Hedberg (1978).

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