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Stefan problems in non-cylindrical domains arising in Czochralski process of crystal growth

by

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Abstract: In this paper we discuss a two-phase Stefan problem with convection in a non-cylindrical (time-dependent) domain. This work is motivated by phase change phenomenon arising in the Czochralski process of crystal growth. The time-dependence of domain is a mathematical description of the situation in which the material domain changes its shape with time by crystal growth. We consider the so-called enthalpy formulation for it and give its solvability, assuming that the time-dependence of the material domain is prescribed and smooth enough in time, and the convective vector is prescribed, too. Our main idea is to apply the theory of quasi-linear equations of parabolic type.

Keywords: Stefan problem, non-cylindrical domain, prescribed convection.

1. Introduction

The Czochralski process is widely used for the production of a column of simple crystal from the melt. But its theoretical analysis seems still incomplete, though many interesting phenomena are observed in this process from the mathematical point of view. Recently, models of the Czochralski process were discussed in Pawłow (2000) in a more general setting, and some special cases of those model

In the original model of crystal growth the shape of material (crystal and melt) is determined by three (unknown) interfaces of solid-liquid, liquid-gas and solid-gas. But, in this paper, supposing that the material domain is prescribed we consider the solid-liquid phase transition in the material domain.

We use the following notation: For $0 < T < +\infty$ and $t \in [0,T]$,

$\Omega_\ell(t)$: liquid (melt) region, $\Omega_s(t)$: solid (crystal) region,

$S(t)$: solid-liquid interface,

$\Omega_m(t) := \Omega_\ell(t) \cup \Omega_s(t) \cup S(t)$: material domain,

$\Gamma(t) := \partial\Omega_m(t) = \Gamma_\ell(t) \cup \Gamma_s(t)$: material boundary,

$\nu = \nu(t,x)$: 3-dimensional unit vector outward normal to $\Gamma(t)$ at $x \in \Gamma(t)$,

$\mathrm{n} = \mathrm{n}(t,x)$: 3-dimensional unit vector normal to $S(t)$ at $x \in S(t)$ pointing to $\Omega_\ell(t)$,

$Q_i := \bigcup_{t\in(0,T)}\{t\} \times \Omega_i(t)$, $i = m, \ell, s$, $\Sigma := \bigcup_{t\in(0,T)}\{t\} \times \Gamma(t)$,

$\Sigma_i := \bigcup_{t\in(0,T)}\{t\} \times \Gamma_i(t)$, $i = \ell, s$, $S := \bigcup_{t\in(0,T)}\{t\} \times S(t)$.

Note that $\Gamma_\ell(t)$ is the union of the the liquid-gas interface and the liquid boundary attached to the crucible, and $\Gamma_s(t)$ is the solid-gas interface.

Next we denote by $v_\Sigma := v_\Sigma(t,x)$ and $v_S := v_S(t,x)$ the normal speed of $\Gamma_m(t)$ at $(t,x) \in \Sigma_m$ and of $S(t)$ at $(t,x) \in S$, respectively. Then, the 4-dimensional (with respect to (t,x)-space) unit vectors $\vec{\nu}$ outward normal to Σ and $\vec{\mathrm{n}}$ normal to S pointing to the liquid region Q_ℓ are given by

$$\vec{\nu} = \frac{1}{(|v_\Sigma|^2+1)^{\frac{1}{2}}}(-v_\Sigma, \nu), \quad \vec{\mathrm{n}} = \frac{1}{(|v_S|^2+1)^{\frac{1}{2}}}(-v_S, \mathrm{n}).$$

These notations will be used in the derivation of our weak formulation.

It is easily understood that by the crystal growth the shape of material domain $\Omega_m(t)$ changes with time and hence a 3-dimensional convective vector field $\mathbf{v} := \mathbf{v}(t,x)$ is caused in Q_m. The determination of $\mathbf{v}$ is also one of the important questions in the mathematical modeling of the Czochralski crystal growth process. It is reasonable to postulate that $\mathbf{v}$ is equal to the pulling velocity v_p in the crystal and is a solution of the incompressible Navier-Stokes (or simply Stokes) equation in the melt (see Crowley, 1983, DiBenedetto and O'Leary, 1993). Nevertheless, in this paper, we suppose that the convective field $\mathbf{v}$ is prescribed, too, assumed to be sufficiently smooth and satisfying

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_m, \tag{1.1}$$

$$\mathbf{v}\cdot\nu = v_\Sigma \quad \text{on } \Sigma. \tag{1.2}$$

Now, from the usual energy balance laws we derive the following system to determine the temperature field $\theta := \theta(t,x)$ and the interface $S(t)$; note that $\theta(t,x)$ together with $S(t)$ is a solution of the two-phase Stefan problem (SPC):= $\{(1.3)\text{-}(1.6)\}$ with prescribed convection $\mathbf{v}$ formulated in the non-cylindrical domain Q_m,

$$\theta_\ell = \theta_s = 0, \quad \left(k_\ell \frac{\partial \theta_\ell}{\partial \mathrm{n}} - k_s \frac{\partial \theta_s}{\partial \mathrm{n}} \right) = L(\mathbf{v} \cdot \mathrm{n} - v_S) \quad \text{on } S, \tag{1.4}$$

$$k_i \frac{\partial \theta_i}{\partial \nu} + n_0 k_i \theta_i = p \quad \text{on } \Sigma_i, \quad i = \ell, s, \tag{1.5}$$

$$\theta(0, \cdot) = \theta_0 \quad \text{on } \Omega(0), \quad S(0) = S_0, \tag{1.6}$$

where $\theta_{i,t} := \partial\theta_i/\partial t$, θ_ℓ and θ_s denote the temperature in the liquid and solid region, respectively, and the phase change temperature is supposed to be 0 for simplicity; k_ℓ, k_s and L are positive constants which are the heat conductivities and latent heat, respectively; f is a given heat source on Q_m, p is a boundary datum prescribed on Σ and n_0 is a positive constant; θ_0 is the initial temperature on $\Omega_m(0)$ and S_0 is the initial location of the solid-liquid interface, satisfying that

$$\theta_0 > 0 \quad \text{on } \Omega_\ell(0), \quad \theta_0 < 0 \quad \text{on } \Omega_s(0), \quad \theta_0 = 0 \quad \text{on } S_0. \tag{1.7}$$

When the material domain does not change in time, the Stefan problem without convection was skillfully treated by Damlamian (1977) in the time-dependent subdifferential operator theory and the problem with convection was discussed in Rodrigues and Yi (1990), Rodrigues (1994), in connection with models of the continuous casting process of steel. On the other hand, the case of non-cylindrical domains was treated by Kenmochi and Pawłow (1986) and only the existence result was there obtained, but the uniqueness question has been left open. The main difficulty apparently comes from the time-dependence of the material domain and the analysis is much harder, for instance, in getting uniform estimates for approximate solutions. Another point of our approach is the use of properties (1.1) and (1.2) required for the convection vector $\mathbf{v}$. The main result of this paper says that these properties of convection vector $\mathbf{v}$ are significant especially for our weak variational formulation.

This paper is organized as follows. In Section 2 we derive a weak variational formulation, which is called the enthalpy formulation, from the system (1.3)–(1.6). In Sections 3 and 4 we propose regular approximate problems for it and give various uniform estimates for approximate solutions. In the final section we discuss the convergence of approximate solutions and construct a weak solution of our problem as a limit, and the uniqueness is also proved.

2. Weak formulation

The enthalpy u and a function $\beta : \mathbf{R} \to \mathbf{R}$ are defined as follows:

$$u := \begin{cases} \theta + L & \text{if } \theta > 0, \\ [0, L] & \text{if } \theta = 0, \\ \theta & \text{if } \theta < 0, \end{cases} \qquad \beta(r) := \begin{cases} k_s r & \text{if } r < 0, \\ 0 & \text{if } 0 \le r \le L, \\ k_\ell (r - L) & \text{if } r > L. \end{cases}$$

Then, β is a non-decreasing Lipschitz continuous function on $\mathbf{R}$, and its Lipschitz constant is $L_\beta := \max\{k_\ell, k_s\}$

By using the enthalpy function u our problem (SPC) is reformulated as an initial-boundary value problem for a degenerate parabolic equation in the non-cylindrical domain Q_m of the following form

$$\text{(E)} \quad \begin{cases} u_t - \Delta\beta(u) + \mathbf{v}\cdot\nabla u = f \quad \text{in } Q_m, \\ \frac{\partial\beta(u)}{\partial\nu} + n_0\beta(u) = p \quad \text{on } \Sigma, \\ u(0) = u_0 \quad \text{on } \Omega_m(0), \end{cases}$$

where $u_0 := \theta_0 + L\chi_{\Omega_\ell(0)}$ with the characteristic function $\chi_{\Omega_\ell(0)}$ of $\Omega_\ell(0)$. In fact, we multiply equations (1.3) by any test function $\eta \in C^2(\overline{Q_m})$ with $\eta = 0$ on $\Omega_m(T)$, and then integrate them over Q_ℓ and Q_s, respectively, and add these two resultants. Then, with the help of the Green-Stokes' formula and the relations $d\Sigma_i = (|v_\Sigma|^2+1)^{1/2}d\Gamma_i(t)dt$, $i = \ell, s$, utilizing the first condition in (1.4), we have

$$\begin{aligned}
&\int_{Q_\ell}\theta_{\ell,t}\eta dxdt + \int_{Q_s}\theta_{s,t}\eta dxdt \\
&= -\int_{Q_\ell}\theta_\ell\eta_t dxdt + \int_S \theta_\ell\eta(-(\vec{\mathrm{n}})^t)dS + \int_{\Sigma_\ell}\theta_\ell\eta(\vec{\nu})^t d\Sigma_\ell - \int_{\Omega_\ell(0)}\theta_0\eta(0)dx \\
&\quad - \int_{Q_s}\theta_s\eta_t dxdt + \int_S \theta_s\eta(\vec{\mathrm{n}})^t dS + \int_{\Sigma_s}\theta_s\eta(\vec{\nu})^t d\Sigma_s - \int_{\Omega_s(0)}\theta_0\eta(0)dx \\
&= -\int_{Q_m} u\eta_t dxdt - \int_{\Omega_m(0)} u_0\eta(0)dx \\
&\quad + \int_{Q_\ell} L\eta_t dxdt + \int_{\Omega_\ell(0)} L\eta(0)dx - \int_{\Sigma_\ell}\theta_\ell\eta v_\Sigma d\Gamma_\ell(t)dt - \int_{\Sigma_s}\theta_s\eta v_\Sigma d\Gamma_s(t)dt,
\end{aligned}$$

where $(\vec{\nu})^t$ and $(\vec{\mathrm{n}})^t$ denote the time-axis component of vectors $\vec{\nu}$ and $\vec{\mathrm{n}}$, respectively. Next, by (1.4) and (1.5) we have

$$\begin{aligned}
&-\int_{Q_\ell} k_\ell\Delta\theta_\ell\eta dxdt - \int_{Q_s} k_s\Delta\theta_s\eta dxdt \\
&= \int_{Q_m}\nabla\beta(u)\cdot\nabla\eta dxdt + \int_\Sigma (n_0\beta(u) - p)\eta d\Gamma(t)dt \\
&\quad - \int_S L(\mathbf{v}\cdot\mathbf{n} - v_S)\eta dS(t)dt.
\end{aligned}$$

Moreover, recalling (1.1) and (1.2), by the first condition in (1.4) and the continuity of $\mathbf{v}\cdot\mathbf{n}$ on $S(t)$, we see that

$$\begin{aligned}
&\int_{Q_\ell}(\mathbf{v}\cdot\nabla\theta_\ell)\eta dxdt + \int_{Q_s}(\mathbf{v}\cdot\nabla\theta_s)\eta dxdt \\
&= -\int_{Q_m} u(\mathbf{v}\cdot\nabla\eta)dxdt + \int_{Q_\ell} L(\mathbf{v}\cdot\nabla\eta)dxdt + \int_{\Sigma_\ell}\theta_\ell v_\Sigma\eta d\Gamma_\ell(t)dt \\
&\quad + \int \theta_s v_\Sigma \eta d\Gamma_s(t)dt
\end{aligned}$$

Here, with the help of the Green-Stokes' formula, we see from conditions (1.1)–(1.2) again and the relation $dS = (|v_S|^2 + 1)^{1/2} dS(t)dt$ that

$$\int_{Q_\ell} L\eta_t dxdt + \int_{\Omega_\ell(0)} L\eta(0)dx + \int_{Q_\ell} L(\mathbf{v}\cdot\nabla\eta)dxdt$$
$$= \int_S L(\mathbf{v}\cdot\mathbf{n} - v_S)\eta dS(t)dt. \tag{2.1}$$

Summing up these equalities, we obtain that

$$-\int_{Q_m} u\eta_t dxdt + \int_{Q_m} \nabla\beta(u)\cdot\nabla\eta dxdt + n_0\int_\Sigma \beta(u)\eta d\Gamma(t)dt$$
$$-\int_{Q_m} u(\mathbf{v}\cdot\nabla\eta)dxdt$$
$$= \int_{Q_m} f\eta dxdt + \int_\Sigma p\eta d\Gamma(t)dt + \int_{\Omega_m(0)} u_0\eta(0)dx \tag{2.2}$$

for all $\eta \in C^2(\overline{Q_m})$ with $\eta = 0$ on $\Omega_m(T)$. As usual, this is regarded as a variational form of (E).

Now, we define a weak solution of our problem.

DEFINITION 2.1 *A function u is called a weak solution of* (SPC), *if u, $\beta(u) \in L^2(Q_m)$ and $\beta(u(t,\cdot)) \in H^1(\Omega_m(t))$ for a.e. $t \in [0,T]$ with*

$$\int_0^T |\beta(u(t))|^2_{H^1(\Omega_m(t))} dt < \infty,$$

$u(t,\cdot) \in L^2(\Omega_m(t))$ for each $t \in [0,T]$, the function

$$t \longmapsto \int_{\Omega_m(t)} u(t,x)\xi(x)dx \quad \text{is continuous on } [0,T] \text{ for all } \xi \in L^2(\mathbf{R}^3),$$

and u satisfies the variational identity (2.2).

We suppose that the material domain $\Omega_m(t)$ depends smoothly on time t in the sense that there is a transformation $y = X(t,x) := (X_1(t,x),\ X_2(t,x),\ X_3(t,x))$ of C^2-class from $\overline{Q_m}$ into $\mathbf{R}^3$, satisfying that

$$(\star)\quad \begin{cases} X(t,\cdot) \text{ maps } \overline{\Omega_m(t)} \text{ onto } \overline{\Omega_m(0)} \text{ and for all } t \in [0,T], \\ X(0,\cdot) = I \text{ (identity) on } \overline{\Omega_m(0)}. \end{cases}$$

Now, fix the following notation:

$$\Omega_0 := \Omega_m(0), \quad \Gamma_0 := \Gamma(0),$$
$$Q_0 := (0,T)\times\Omega_0, \quad \Sigma_0 := (0,T)\times\Gamma_0, \quad y = (y_1,y_2,y_3) \in \overline{\Omega_0};$$

and denote the inverse of X by $x = Y(t,y) := (Y_1(t,y),\ Y_2(t,y),\ Y_3(t,y))$.

THEOREM 2.1 *Assume that* $f \in L^2(Q_m)$, $p \in C^1(\overline{\Sigma})$, $u_0 \in L^2(\Omega_0)$ *and* $\beta(u_0) \in H^1(\Omega_0)$. *Also, assume that* $\mathbf{v} \in C^1(\overline{Q_m})^3$ *and* (1.1), (1.2) *are satisfied. Then, there is one and only one weak solution* u *of* (SPC).

The proof of our theorem is given in the Sections 3 through 5.

As will be understood from our proof given in Section 5, the presence of the convection term $\mathbf{v}$ plays an important role for the uniqueness of weak solutions of Stefan problems formulated in non-cylindrical domains. This is one of interesting aspects of Theorem 2.1.

3. Regular approximation to (SPC)

In this section, let us consider an approximate problem $(\mathrm{SPC})_\delta := \{(3.1)–(3.3)\}$ in the non-cylindrical domain Q_m, with parameter $\delta \in (0,1]$, to (SPC):

$$u_{\delta,t} - \Delta\beta_\delta(u_\delta) + \mathbf{v}\cdot\nabla u_\delta = f_\delta \quad \text{in } Q_m, \tag{3.1}$$

$$\frac{\partial\beta_\delta(u_\delta)}{\partial\nu} + n_0\beta_\delta(u_\delta) = p_\delta \quad \text{on } \Sigma, \tag{3.2}$$

$$u_\delta(0) = u_{0\delta} \quad \text{on } \Omega_0, \tag{3.3}$$

where β_δ, f_δ, p_δ and $u_{0\delta}$ are regular approximations of β, f, p and u_0, respectively, as follows:

(1) β_δ is a smooth, increasing and Lipschitz continuous function on $\mathbf{R}$ such that

$$\delta \le \beta_\delta'(r) \left(= \frac{d}{dr}\beta_\delta(r)\right) \le C_0 \quad \text{for all } r \in \mathbf{R},$$

for a positive constant C_0 independent of δ, and such that

$$\beta_\delta \to \beta \quad \text{uniformly on } \mathbf{R} \text{ as } \delta \to 0;$$

we put $\hat\beta_\delta(r) := \int_0^r \beta_\delta(s)ds$ as well as $\hat\beta(r) := \int_0^r \beta(s)ds$ for all $r \in \mathbf{R}$.

(2) f_δ is a smooth function on $\overline{Q_m}$ such that

$$f_\delta \to f \quad \text{in } L^2(Q_m) \text{ as } \delta \to 0.$$

(3) p_δ is a smooth function on $\overline{\Sigma}$ such that

$$p_\delta \to p \quad \text{in } C^1(\overline{\Sigma}) \text{ as } \delta \to 0.$$

(4) $u_{0\delta}$ is a smooth function on $\overline{\Omega_0}$ such that $u_{0\delta} \to u_0$ in $L^2(\Omega_0)$, $\beta_\delta(u_{0\delta}) \to \beta(u_0)$ in $H^1(\Omega_0)$ as $\delta \to 0$ and the compatibility condition

$$\frac{\partial\beta_\delta(u_{0\delta})}{\partial\nu} + n_0\beta_\delta(u_{0\delta}) = p_\delta \quad \text{on } \Gamma_0, \tag{3.4}$$

holds.

We give first an existence-uniqueness result for the approximate problem $(\mathrm{SPC})_\delta$.

LEMMA 3.1 $(\mathrm{SPC})_\delta$ *has one and only one solution* u_δ *such that* u_δ *and* <u>*all*</u> *the*

Proof. Using $y = X(t,x)$ we transform $(\mathrm{SPC})_\delta$ to a problem $(\overline{\mathrm{SPC}})_\delta :=$ $\{(3.5)\text{–}(3.7)\}$ formulated in the cylindrical domain Q_0:

$$\bar{u}_{\delta,t} - \sum_{i,j=1}^{3} \frac{\partial}{\partial y_i}\left\{a_{ij}\frac{\partial \beta_\delta(\bar{u}_\delta)}{\partial y_j}\right\} + \mathbf{w}_1 \cdot \nabla\beta_\delta(\bar{u}_\delta) + \mathbf{w}_2 \cdot \nabla \bar{u}_\delta$$

$$= \bar{f}_\delta \quad \text{in } Q_0, \tag{3.5}$$

$$\frac{\partial \beta_\delta(\bar{u}_\delta)}{\partial \nu_A} + \bar{n}_0 \beta_\delta(\bar{u}_\delta) = \bar{p}_\delta \quad \text{on } \Sigma_0, \tag{3.6}$$

$$\bar{u}_\delta(0) = u_{0\delta} \quad \text{on } \Omega_0. \tag{3.7}$$

Here

$$\bar{u}_\delta(t,y) := u_\delta(t, Y(t,y)), \quad \bar{f}_\delta(t,y) := f_\delta(t, Y(t,y)),$$
$$\bar{n}_0(t,y) := (\|\widetilde{J}_Y(t,y)\| / \|J_Y(t,y)\|) n_0,$$
$$\bar{p}_\delta(t,y) := (\|\widetilde{J}_Y(t,y)\| / \|J_Y(t,y)\|) p_\delta(t, Y(t,y)),$$

where $J_Y(t,\cdot)$ denotes the Jacobian of $x = Y(t,\cdot)$ with its determinant $\|J_Y(t,\cdot)\|$, and $\|\widetilde{J}_Y(t,\cdot)\|$ denotes the ratio between the surface elements $d\Gamma(t)$ and $d\Gamma_0$, which is determined by the restriction of $x = Y(t,\cdot)$ on Γ_0; hence

$$dx = \|J_Y\| dy \quad \text{on } \Omega_0, \quad d\Gamma(t) = \|\widetilde{J}_Y\| d\Gamma_0 \quad \text{on } \Gamma_0.$$

Moreover

$$a_{ij}(t,y) := \sum_{k=1}^{3} \frac{\partial X_i}{\partial x_k}(t, Y(t,y)) \frac{\partial X_j}{\partial x_k}(t, Y(t,y)), \quad i,j = 1,2,3,$$

$\mathbf{w}_1 := (w_{11}, w_{12}, w_{13})$, $\mathbf{w}_2 := (w_{21}, w_{22}, w_{23})$ with

$$w_{1j}(t,y) := \sum_{k,\ell=1}^{3} \frac{\partial}{\partial y_\ell}\left(\frac{\partial X_\ell}{\partial x_k}(t, Y(t,y))\right) \frac{\partial X_j}{\partial x_k}(t, Y(t,y)), \quad j = 1,2,3,$$

$$w_{2j}(t,y) := \sum_{k=1}^{3} \frac{\partial X_j}{\partial x_k}(t, Y(t,y)) \left(v_k(t, Y(t,y)) - \frac{\partial X_k}{\partial t}(t, Y(t,y))\right),$$
$$j = 1,2,3,$$

$$\frac{\partial(\cdot)}{\partial \nu_A} := \sum_{i,j=1}^{3} a_{ij} \frac{\partial(\cdot)}{\partial y_i} \bar{\nu}_j = \frac{\|\widetilde{J}_Y\|}{\|J_Y\|} \frac{\partial(\cdot)}{\partial \nu} \quad \text{on } \Sigma_0,$$

where $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}_3)$ is the unit vector outward normal to Γ_0.

Since $X(0,\cdot) = I$ on $\overline{\Omega_0}$, the matrix $\{a_{ij}(0,y)\}$ is the unit on $\overline{\Omega_0}$ and hence $\{a_{ij}(t,y)\}$ is strictly positive definite on $\overline{\Omega_0}$ for $t \in [0, T']$ with a certain positive $T'(\le T)$. Therefore, $(\overline{\mathrm{SPC}})_\delta$ is uniformly parabolic quasi-linear equation with condition for initial and boundary data is satisfied. Now, apply the general existence and uniqueness theorem due to Chapter 5, Section 7 of Ladyzhenskaya, Solonnikov and Ural'tseva (1968) to $(\overline{\mathrm{SPC}})_\delta$. Then we see that $(\overline{\mathrm{SPC}})_\delta$ has a unique solution $\bar{u}_\delta$ in the Hölder space $H^{2+\alpha,1+\alpha/2}(\overline{Q_0(T')})$ for a certain exponent $\alpha \in (0,1)$. It is also easy to check that $u_\delta(t,x) := \bar{u}_\delta(t, X(t,x))$ is a solution of $(\mathrm{SPC})_\delta$ on $Q_m(T') := \bigcup_{t\in(0,T')}\{t\} \times \Omega_m(t)$, satisfying the required regularities. If $T' < T$, then the solution u_δ can be extended beyond time T' by repeating the same argument as above with initial time T'. Finally we can construct a unique solution u_δ of $(\mathrm{SPC})_\delta$ on Q_m in the Hölder class. ■

Next we prepare two lemmas about uniform estimates of approximate solutions.

LEMMA 3.2 *There exists a positive constant M_1, independent of parameter $\delta \in (0,1]$, such that*

$$\sup_{t\in[0,T]} |u_\delta(t)|^2_{L^2(\Omega_m(t))} + \int_0^T |\beta_\delta(u_\delta(t))|^2_{H^1(\Omega_m(t))} dt \le M_1$$
$$\text{for all } \delta \in (0,1]. \tag{3.8}$$

Proof. We use essentially the conditions (1.1) and (1.2) in order to get the uniform estimates (3.8). For each $t \in [0,T]$, put $Q_m(t) := \bigcup_{\tau\in(0,t)}\{\tau\} \times \Omega_m(\tau)$. First, by multiplying (3.1) by $\beta_\delta(u_\delta)$ and integrating the resultant over $Q_m(t)$, we get

$$\int_{Q_m(t)} \frac{\partial u_\delta}{\partial \tau}\beta_\delta(u_\delta)dxd\tau - \int_{Q_m(t)} \Delta\beta_\delta(u_\delta)\beta_\delta(u_\delta)dxd\tau$$
$$+ \int_{Q_m(t)} (\mathbf{v}\cdot\nabla u_\delta)\beta_\delta(u_\delta)dxd\tau = \int_{Q_m(t)} f_\delta\beta_\delta(u_\delta)dxd\tau. \tag{3.9}$$

Here, by the Stokes' formula,

$$\int_{Q_m(t)} \frac{\partial u_\delta}{\partial \tau}\beta_\delta(u_\delta)dxd\tau = \int_{Q_m(t)} \frac{\partial}{\partial \tau}\hat{\beta}_\delta(u_\delta)dxd\tau$$
$$= \int_0^t \int_{\Gamma(\tau)} \hat{\beta}_\delta(u_\delta)\frac{-v_\Sigma}{(|v_\Sigma|^2+1)^{\frac{1}{2}}}d\Sigma + \int_{\Omega_m(t)} \hat{\beta}_\delta(u_\delta(t))dx - \int_{\Omega_0} \hat{\beta}_\delta(u_{0\delta})dx$$
$$= -\int_0^t \int_{\Gamma(\tau)} \hat{\beta}_\delta(u_\delta)v_\Sigma d\Gamma(\tau)d\tau + \int_{\Omega_m(t)} \hat{\beta}_\delta(u_\delta(t))dx - \int_{\Omega_0} \hat{\beta}_\delta(u_{0\delta})dx,$$

and by the boundary condition (3.2),

$$-\int_{Q_m(t)} \Delta\beta_\delta(u_\delta)\beta_\delta(u_\delta)dxd\tau$$
$$\int |\nabla\beta_\delta(u_\delta)|^2 dxd\tau \quad \int^t \int (n_\delta - n_\delta\beta_\delta(u_\delta))\beta_\delta(u_\delta)d\Gamma(\tau)d\tau$$

Moreover, we have, by (1.1) and (1.2)

$$\int_{Q_m(t)} (\mathbf{v}\cdot\nabla u_\delta)\beta_\delta(u_\delta)dxd\tau = \int_{Q_m(t)} \mathbf{v}\cdot\nabla\hat{\beta}_\delta(u_\delta)dxd\tau$$
$$= \int_0^t \int_{\Gamma(\tau)} \hat{\beta}_\delta(u_\delta)(\mathbf{v}\cdot\nu)d\Gamma(\tau)d\tau$$
$$= \int_0^t \int_{\Gamma(\tau)} \hat{\beta}_\delta(u_\delta)v_\Sigma d\Gamma(\tau)d\tau.$$

Now, substituting the above expressions in (3.9), we obtain with the help of Schwarz's inequality that for each $\varepsilon > 0$

$$\int_{\Omega_m(t)} \hat{\beta}_\delta(u_\delta(t))dx + \int_0^t \int_{\Omega_m(\tau)} |\nabla\beta_\delta(u_\delta)|^2 dxd\tau$$
$$+ (n_0 - \varepsilon)\int_0^t \int_{\Gamma(\tau)} |\beta_\delta(u_\delta)|^2 d\Gamma(\tau)d\tau$$
$$\leq \frac{1}{4\varepsilon}\int_{Q_m} |f_\delta|^2 dxd\tau + \varepsilon\int_0^t \int_{\Omega_m(\tau)} |\beta_\delta(u_\delta)|^2 dxd\tau + \frac{1}{4\varepsilon}\int_\Sigma |p_\delta|^2 d\Gamma(\tau)d\tau$$
$$+ \int_{\Omega_0} \hat{\beta}_\delta(u_{0\delta})dx \quad \text{for all } t \in [0,T]. \tag{3.10}$$

From the definitions of β_δ and $\hat{\beta}_\delta$ it follows that there exist positive constants c_β, c'_β and c''_β, independent of parameter $\delta \in (0,1]$, such that

$$\hat{\beta}_\delta(r) \geq c_\beta|r|^2 - c'_\beta \quad \text{and} \quad |\beta_\delta(r)|^2 \leq c''_\beta(|r|^2+1) \quad \text{for all } r \in \mathbf{R}. \tag{3.11}$$

Therefore, by choosing $\varepsilon > 0$ small enough in (3.10) and using Gronwall's inequality we obtain a uniform estimate of the form (3.8) for a positive constant M_1 independent of $\delta \in (0,1]$. ■

LEMMA 3.3 *There exists a positive constant M_2, independent of $\delta \in (0,1]$, such that*

$$\int_{Q_m} \beta'_\delta(u_\delta)|\nabla u_\delta|^2 dxdt \leq M_2 \quad \textit{for all } \delta \in (0,1]. \tag{3.12}$$

Proof. Just as in the proof of Lemma 3.2, by multiplying (3.1) by u_δ and integrating over Q_m, and noting that $(\mathbf{v}\cdot\nabla u_\delta)u_\delta = 1/2(\mathrm{div}(u_\delta^2\mathbf{v}))$, we get

$$\int_{Q_m} \frac{1}{2}\frac{\partial}{\partial t}|u_\delta|^2 dxdt + \int_{Q_m} \nabla\beta_\delta(u_\delta)\cdot\nabla u_\delta dxdt$$
$$-\int^T \int \frac{\partial\beta_\delta(u_\delta)}{} u_\delta d\Gamma(t)dt = -\int \frac{1}{}\mathrm{div}(u^2\mathbf{v})dxdt + \int f u dxdt$$

Now, by using (1.2), (3.8) and Young's inequality,

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_m(T)}|u_\delta(T)|^2dx+\int_{Q_m}\beta_\delta'(u_\delta)|\nabla u_\delta|^2dxdt\\
&\le -\frac{1}{2}\int_0^T\int_{\Gamma(t)}|u_\delta|^2\frac{-v_\Sigma}{(|v_\Sigma|^2+1)^{\frac{1}{2}}}d\Sigma-\frac{1}{2}\int_0^T\int_{\Gamma(t)}|u_\delta|^2(\mathbf{v}\cdot\nu)d\Gamma(t)dt\\
&+\int_0^T\int_{\Gamma(t)}(p_\delta-n_0\beta_\delta(u_\delta))u_\delta d\Gamma(t)dt+\frac{1}{2}\int_{Q_m}|f_\delta|^2dxdt\\
&+\frac{1}{2}\int_{Q_m}|u_\delta|^2dxdt+\frac{1}{2}\int_{\Omega_0}|u_{0\delta}|^2dx\\
&\le \frac{1}{2}\int_0^T\int_{\Gamma(t)}|p_\delta|^2d\Gamma(t)dt+\frac{n_0^2}{2}\int_0^T\int_{\Gamma(t)}|\beta_\delta(u_\delta)|^2d\Gamma(t)dt\\
&+\int_0^T\int_{\Gamma(t)}|u_\delta|^2d\Gamma(t)dt+\frac{1}{2}|f_\delta|^2_{L^2(Q_m)}+\left(\frac{1}{2}+\frac{T}{2}\right)M_1.
\end{aligned}$$

By (3.11) again we have

$$|u_\delta(t,x)|^2\le\frac{1}{c_\beta}|\beta_\delta(u_\delta(t,x))|^2+\frac{c_\beta'}{c_\beta}\quad\text{for all } (t,x)\in Q_m,$$

so that there exists a positive constant M_2', independent of $\delta\in(0,1]$, such that

$$\begin{aligned}
&\int_{Q_m}\beta_\delta'(u_\delta)|\nabla u_\delta|^2dxdt\\
&\le M_2'\int_0^T|\beta_\delta(u_\delta(t))|^2_{H^1(\Omega_m(t))}dt+\frac{1}{2}|p_\delta|^2_{L^2(\Sigma)}+\frac{1}{2}|f_\delta|^2_{L^2(Q_m)}+M_1M_2'.
\end{aligned}$$

This, together with (3.8) gives a uniform estimate of the form (3.12) for a constant M_2, which is independent of $\delta\in(0,1]$. ■

4. Estimates of regular approximate solutions

In this section we prove some uniform estimates of the time derivative of $\beta_\delta(u_\delta)$ and the H^1-norm of $\beta_\delta(u_\delta)$. These estimates seem to be more complicated in the non-cylindrical case than in the cylindrical one.

LEMMA 4.1 *There exists a positive constant M_3, independent of parameter $\delta\in(0,1]$, such that*

$$\int_{Q_m}\left|\frac{\partial}{\partial t}\beta_\delta(u_\delta)\right|^2dxdt+\sup_{t\in[0,T]}|\beta_\delta(u_\delta(t))|^2_{H^1(\Omega_m(t))}\le M_3 \tag{4.1}$$

Proof. For each $\delta \in (0,1]$ and $t \in (0,T]$ we consider the time-dependent convex functional $\Phi_\delta(t,\cdot)$ on $L^2(\Omega_0)$ defined by

$$\Phi_\delta(t,z) := \begin{cases} \frac{1}{2}\sum_{i,j=1}^{3}\int_{\Omega_0} a_{ij}(t)\frac{\partial z}{\partial y_i}\frac{\partial z}{\partial y_j}dy \\ +\frac{1}{2}\int_{\Gamma_0}\bar{n}_0(t)z^2 d\Gamma_0 - \int_{\Gamma_0}\bar{p}_\delta(t)z d\Gamma_0 & \text{if } z \in H^1(\Omega_0), \\ \qquad +\infty & \text{if } z \in L^2(\Omega_0)\setminus H^1(\Omega_0). \end{cases}$$

Then, it is easy to see that $\Phi_\delta(t,\cdot)$ is proper and lower semi-continuous on $L^2(\Omega_0)$ and $\Phi_\delta(\cdot,z)$ is Lipschitz continuous on $[0,T]$ for each $z \in H^1(\Omega_0)$; actually, it holds that

$$\frac{d}{dt}\Phi_\delta(t,z) \le K_0(K_0' + \Phi_\delta(t,z)) \\ \text{for a.e. } t \in [0,T] \text{ and all } z \in H^1(\Omega_0), \tag{4.2}$$

where K_0 and K_0' are positive constants determined only by the Lipschitz constants of a_{ij}, $\bar{n}_0$ and $\bar{p}_\delta$; they can be chosen so as to be independent of δ, too. It is derived from this property in the same way as Lemma 1.2.5 in Kenmochi (1981) (or Lemma 2.3 in Kenmochi and Pawłow, 1986), namely that if $v \in W^{1,2}(0,T;L^2(\Omega_0))$, $\partial\Phi_\delta(\cdot,v(\cdot)) \in L^2(0,T;L^2(\Omega))$ and $v(0,\cdot) \in H^1(\Omega_0)$, then $\Phi_\delta(\cdot,v(\cdot))$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\Phi_\delta(t,v(t)) - (v_t(t),\partial\Phi_\delta(t,v(t)))_{L^2(\Omega_0)} \le K_0(K_0' + \Phi_\delta(t,v(t))), \tag{4.3}$$

for a.e. $t \in [0,T]$, where $\partial\Phi_\delta(t,\cdot)$ is the subdifferential of $\Phi_\delta(t,\cdot)$. In fact, for each $s,t \in [0,T]$ with $s < t$ by the definition of the subdifferential and (4.2) we get

$$\frac{1}{t-s}\{\Phi_\delta(t,v(t)) - \Phi_\delta(s,v(s))\} \\ \le \left(\partial\Phi_\delta(t,v(t)), \frac{v(t)-v(s)}{t-s}\right)_{L^2(\Omega_0)} + \frac{1}{t-s}\int_s^t K_0(K_0' + \Phi_\delta(\tau,v(s)))d\tau,$$

where $(\cdot,\cdot)_{L^2(\Omega_0)}$ stands for the standard inner product in $L^2(\Omega_0)$. For a.e. $t \in [0,T]$ at which $\Phi_\delta(\cdot,v)$ and v are differentiable, we have (4.3) by letting $s \nearrow t$. Moreover, $\partial\Phi_\delta(t,v(t))$ is characterized by

$$(\partial\Phi_\delta(t,v(t)),w)_{L^2(\Omega_0)} \\ = \sum_{i,j=1}^{3}\int_{\Omega_0} a_{ij}(t)\frac{\partial v(t)}{\partial y_i}\frac{\partial w}{\partial y_j}dy + \int_{\Gamma_0}\bar{n}_0(t)v(t)w d\Gamma_0 - \int_{\Gamma_0}\bar{p}_\delta(t)w d\Gamma_0,$$

for all $w \in H^1(\Omega_0)$ and hence $\partial\Phi_\delta(t,v(t)) = -\sum_{i,j=1}^{3}\partial/\partial y_j\{a_{ij}(t)(\partial v(t)/\partial y_i)\}$

$\mathbf{w}_1 \cdot \nabla\beta_\delta(\bar{u}_\delta) + \mathbf{w}_2 \cdot \nabla\bar{u}_\delta - \bar{f}_\delta$ (cf. (3.5)), it follows from (4.3) by taking $\beta_\delta(\bar{u}_\delta)$ as v that

$$\begin{aligned}
&\frac{d}{d\tau}\Phi_\delta(\tau,\beta_\delta(\bar{u}_\delta(\tau))) + \int_{\Omega_0} \beta_\delta'(\bar{u}_\delta(\tau))|\bar{u}_{\delta,\tau}(\tau)|^2 dy \\
&\le K_0(K_0' + \Phi_\delta(\tau,\beta_\delta(\bar{u}_\delta(\tau))) - \int_{\Omega_0} \{\mathbf{w}_1(\tau)\cdot\nabla\beta_\delta(\bar{u}_\delta(\tau))\}\frac{\partial}{\partial\tau}\beta_\delta(\bar{u}_\delta(\tau))dy \\
&- \int_{\Omega_0} \{\mathbf{w}_2(\tau)\cdot\nabla\bar{u}_\delta(\tau)\}\beta_\delta'(\bar{u}_\delta(\tau))\bar{u}_{\delta,\tau}(\tau)dy \\
&+ \int_{\Omega_0} \bar{f}_\delta(\tau)\frac{\partial}{\partial\tau}\beta_\delta(\bar{u}_\delta(\tau))dy, \qquad (4.4)
\end{aligned}$$

for a.e. $\tau \in [0,T]$. Here, integrating (4.4) over $[0,t]$ with respect to τ and using Lemmas 3.2 and 3.3, we obtain for an arbitrary small positive number ε and with notation $Q_0(t) := (0,t)\times\Omega_0$ that

$$\begin{aligned}
&\Phi_\delta(t,\beta_\delta(\bar{u}_\delta(t))) + \int_{Q_0(t)} \beta_\delta'(\bar{u}_\delta)|\bar{u}_{\delta,\tau}|^2 dy d\tau \\
&\le -\int_{Q_0(t)} (\mathbf{w}_1\cdot\nabla\beta_\delta(\bar{u}_\delta))\frac{\partial}{\partial\tau}\beta_\delta(\bar{u}_\delta)dyd\tau - \int_{Q_0(t)} (\mathbf{w}_2\cdot\nabla\bar{u}_\delta)\beta_\delta'(\bar{u}_\delta)\bar{u}_{\delta,\tau}dyd\tau \\
&+ \int_{Q_0(t)} \bar{f}_\delta\frac{\partial}{\partial\tau}\beta_\delta(\bar{u}_\delta)dyd\tau + K_0\int_0^t \{K_0' + \Phi_\delta(\tau,\beta_\delta(\bar{u}_\delta(\tau)))\}d\tau \\
&+ \Phi_\delta(0,\beta_\delta(\bar{u}_{0\delta})) \\
&\le \frac{|\mathbf{w}_1|_{C(\overline{Q_0})^3}}{4\varepsilon}\int_{Q_0(t)} |\nabla\beta_\delta(\bar{u}_\delta)|^2 dyd\tau \\
&+ \varepsilon(|\mathbf{w}_1|_{C(\overline{Q_0})^3}+1)\int_{Q_0(t)} \left|\frac{\partial}{\partial\tau}\beta_\delta(\bar{u}_\delta)\right|^2 dyd\tau \\
&+ \frac{|\mathbf{w}_2|_{C(\overline{Q_0})^3}}{4\varepsilon}\int_{Q_0(t)} |\nabla\bar{u}_\delta|^2\beta_\delta'(\bar{u}_\delta)dyd\tau \\
&+ \varepsilon|\mathbf{w}_2|_{C(\overline{Q_0})^3}\int_{Q_0(t)} \beta_\delta'(\bar{u}_\delta)|\bar{u}_{\delta,\tau}|^2 dyd\tau \\
&+ \frac{1}{4\varepsilon}|\bar{f}_\delta|^2_{L^2(Q_0)} + K_0\int_0^t \{K_0' + \Phi_\delta(\tau,\beta_\delta(\bar{u}_\delta(\tau)))\}d\tau + \Phi_\delta(0,\beta_\delta(\bar{u}_{0\delta})) \\
&\le \varepsilon(|\mathbf{w}_1|_{C(\overline{Q_0})^3}+1)\int_{Q_0(t)} \left|\frac{\partial}{\partial\tau}\beta_\delta(\bar{u}_\delta)\right|^2 dyd\tau + C_\varepsilon|\mathbf{w}_1|_{C(\overline{Q_0})^3}M_1 \\
&+ \varepsilon|\mathbf{w}_2|_{C(\overline{Q_0})^3}\int_{Q_0(t)} \beta_\delta'(\bar{u}_\delta)|\bar{u}_{\delta,\tau}|^2 dyd\tau + C_\varepsilon|\mathbf{w}_2|_{C(\overline{Q_0})^3}M_2 + C_\varepsilon|\bar{f}_\delta|^2_{L^2(Q_0)} \\
&+ K_0K_0'T + K_0\int_0^T \Phi_\delta(\tau,\beta_\delta(\bar{u}_\delta(\tau)))d\tau + \Phi_\delta(0,\beta_\delta(\bar{u}_{0\delta})). \qquad (4.5)
\end{aligned}$$

where C_ε is a positive constant depending only on ε, and M_1, M_2 are the same constants as in Lemmas 3.2 and 3.3. Since $|\partial(\beta_\delta(\bar{u}_\delta))/\partial t| \le C_0|\bar{u}_{\delta,t}|$, it follows that

$$\int_{Q_0(t)} \beta_\delta'(\bar{u}_\delta)|\bar{u}_{\delta,\tau}|^2 dyd\tau \ge \frac{1}{C_0}\int_{Q_0(t)} \left|\frac{\partial}{\partial\tau}\beta_\delta(\bar{u}_\delta)\right|^2 dyd\tau.$$

Therefore it follows from (4.5) with a small $\varepsilon > 0$ and Lemma 3.2 that

$$\Phi_\delta(t, \beta_\delta(\bar{u}_\delta(t))) + \int_{Q_0(t)} \left|\frac{\partial}{\partial\tau}\beta_\delta(\bar{u}_\delta)\right|^2 dyd\tau \le M_4 \quad \text{for all } t \in [0,T], \quad (4.6)$$

where M_4 is a positive constant independent of $\delta \in (0,1]$. From the definition of Φ_δ and (4.6) it follows immediately that

$$\sup_{t\in[0,T]} |\beta_\delta(\bar{u}_\delta(t))|^2_{H^1(\Omega_0)} + \int_{Q_0} \left|\frac{\partial}{\partial t}\beta_\delta(\bar{u}_\delta)\right|^2 dydt \le M_5, \quad (4.7)$$

for a positive constant M_5 independent of $\delta \in (0,1]$. Finally, describe the quantities of the left hand side of (4.7) in the (t,x)-coordinate of the non-cylindrical domain Q_m. Then we obtain a uniform estimate of the form (4.1).■

5. Proof of the Theorem

EXISTENCE: Let $\{u_\delta\}_{\delta\in(0,1]}$ be the family of approximate solutions of $(\mathrm{SPC})_\delta$. By Lemmas 3.2, 3.3 and 4.1 with the standard compactness argument we can find a sequence $\{\delta_n\}$ with $\delta_n \to 0$ as $n \to +\infty$ and functions u, ζ such that

$$u_n := u_{\delta_n} \to u \quad \text{weakly in } L^2(Q_m),$$
$$\beta_n(u_n) := \beta_{\delta_n}(u_n) \to \zeta \quad \text{in } L^2(Q_m) \text{ and weakly in } H^1(Q_m).$$

The monotonicity argument implies that $\zeta = \beta(u)$ in $L^2(Q_m)$. We now show that u is a weak solution of (SPC). To do so, multiply (3.1) by any test function $\eta \in C^2(\overline{Q_m})$ with $\eta(T,\cdot) = 0$ and integrate it over Q_m. Then, just as in the derivation of our weak formulation, we see by the Green-Stokes' formula that the approximate solution u_n satisfies

$$-\int_{Q_m} u_n\eta_t dxdt - \int_\Sigma u_n\eta v_\Sigma d\Gamma(t)dt + \int_{Q_m} \nabla\beta_{\delta_n}(u_n)\cdot\nabla\eta dxdt$$
$$+ n_0\int_\Sigma \beta_{\delta_n}(u_n)\eta d\Gamma(t)dt - \int_{Q_m} u_n(\mathrm{v}\cdot\nabla\eta)dxdt + \int_\Sigma u_n\eta(\mathrm{v}\cdot\nu)d\Gamma(t)dt$$
$$= \int f_\delta \eta dxdt + \int n_\delta \eta d\Gamma(t)dt + \int u_{0\delta}\eta(0)dx$$

Here, noting condition (1.2) again and passing to the limit in n we get

$$
\begin{aligned}
&-\int_{Q_m} u\eta_t dxdt + \int_{Q_m} \nabla\beta(u)\cdot\nabla\eta dxdt + n_0\int_\Sigma \beta(u)\eta d\Gamma(t)dt \\
&-\int_{Q_m} u(\mathbf{v}\cdot\nabla\eta)dxdt \\
&= \int_{Q_m} f\eta dxdt + \int_\Sigma p\eta d\Gamma(t)dt + \int_{\Omega_0} u_0\eta(0,\cdot)dx,
\end{aligned}
$$

which is the required variational identity. Moreover, on account of the uniform estimates obtained in Sections 3 and 4, we see that $u, \beta(u) \in L^2(Q_m)$ and

$$
\int_0^T |\beta(u(t))|^2_{H^1(\Omega_m(t))}dt \le M_1.
$$

Finally, let us check the continuity property of u in time. To do so, we use the weak continuity of the function $\bar{u}(t) := u(t, Y(t,\cdot))$ in $L^2(\Omega_0)$, which is easily seen from the fact that $\{\bar{u}_{\delta,t}\}$ is bounded in $L^2(0,T;H^{-1}(\Omega_0))$ (see (3.5)). For each function $\xi \in L^2(\mathbf{R}^3)$, we observe

$$
\begin{aligned}
&\int_{\Omega_m(t+\Delta t)} u(t+\Delta t, x)\xi(x)dx - \int_{\Omega_m(t)} u(t,x)\xi(x)dx \\
&= \int_{\Omega_0} \bar{u}(t+\Delta t, y)\xi(y)\|J_Y(t+\Delta t, y)\|dy - \int_{\Omega_0} \bar{u}(t,y)\xi(y)\|J_Y(t,y)\|dy \\
&= \int_{\Omega_0} \{\bar{u}(t+\Delta t, y) - \bar{u}(t,y)\}\xi(y)\|J_Y(t+\Delta t, y)\|dy \\
&+ \int_{\Omega_0} \bar{u}(t,y)\{\xi(y)\|J_Y(t+\Delta t, y)\| - \xi(y)\|J_Y(t,y)\|\}dy,
\end{aligned}
$$

where $\|J_Y(t,\cdot)\|$ is the Jacobian determinant of the transformation $x = Y(t,y)$ (see the proof of Lemma 3.1). Clearly, as $\Delta t \to 0$, the right hand side of the above equalities goes to 0, so that the integral $\int_{\Omega_m(t)} u(t,x)\xi(x)dx$ is continuous in t. This completes the existence proof. ■

UNIQUENESS: The idea of our uniqueness proof is due to Chapter 3, Section 3 of Ladyzhenskaya, Solonnikov and Ural'tseva (1968), and this was also extensively used in Niezgódka and Pawłow (1983), Rodrigues and Yi (1990), Rodrigues (1994), as well as Fukao, Kenmochi and Pawłow (2002).

Let u_1 and u_2 be two weak solutions and take their difference. Then

$$
\begin{aligned}
&-\int_{Q_m} (u_1-u_2)\eta_t dxdt - \int_{Q_m} (\beta(u_1)-\beta(u_2))\Delta\eta dxdt \\
&+ \int_\Sigma (\beta(u_1)-\beta(u_2))\frac{\partial\eta}{\partial\nu}d\Gamma(t)dt \\
&+ n_0\int_\Sigma (\beta(u_1)-\beta(u_2))\eta d\Gamma(t)dt - \int_{Q_m} (u_1-u_2)(\mathbf{v}\cdot\nabla\eta)dxdt = 0 \qquad (5.1)
\end{aligned}
$$

As usual, consider the function

$$b(t,x) := \begin{cases} \frac{\beta(u_1(t,x))-\beta(u_2(t,x))}{u_1(t,x)-u_2(t,x)} & \text{if } u_1(t,x) \neq u_2(t,x), \\ 0 & \text{if } u_1(t,x) = u_2(t,x), \end{cases}$$

which is non-negative and bounded on Q_m. Then, by using (5.1),

$$\begin{aligned} &-\int_{Q_m} (u_1 - u_2)\{\eta_t + b\Delta\eta + \mathrm{v}\cdot\nabla\eta\}dxdt \\ &+\int_{\Sigma} (\beta(u_1) - \beta(u_2))\left\{\frac{\partial\eta}{\partial\nu} + n_0\eta\right\} d\Gamma(t)dt \\ &= 0 \quad \text{for all } \eta \in C^2(\overline{Q_m}) \quad \text{with } \eta(T,\cdot) = 0; \end{aligned} \tag{5.2}$$

it is easy to see that (5.2) holds for any function $\eta \in W^{1,2}(Q_m)$ with $\Delta\eta \in L^2(Q_m)$ and $\eta(T,\cdot) = 0$. Now take a smooth and strictly positive approximation b_ε of b such that

$$\begin{aligned} &\varepsilon \le b_\varepsilon \le C_1 \quad \text{a.e. on } Q_m, \\ &b_\varepsilon \to b \quad \text{a.e. on } Q_m \text{ as } \varepsilon \to 0, \end{aligned}$$

where C_1 is a positive constant independent of the approximation parameter $\varepsilon \in (0,1]$, and consider the following auxiliary linear parabolic problem formulated in the non-cylindrical domain Q_m for any given $\ell \in C_0^\infty(Q_m)$:

$$(\mathrm{P})_\varepsilon \quad \begin{cases} \eta_{\varepsilon,t} + b_\varepsilon\Delta\eta_\varepsilon + \mathrm{v}\cdot\nabla\eta_\varepsilon = \ell \quad \text{in } Q_m, \\ \dfrac{\partial\eta_\varepsilon}{\partial\nu} + n_0\eta_\varepsilon = 0 \quad \text{on } \Sigma, \\ \eta_\varepsilon(T,\cdot) = 0 \quad \text{on } \Omega_m(T). \end{cases}$$

This problem has a unique Hölder continuous solution η_ε such that η_ε, $\eta_{\varepsilon,t}$, η_{ε,x_i} and $\eta_{\varepsilon,x_ix_j}$, $i,j = 1,2,3$, are Hölder continuous on $\overline{Q_m}$. In fact, this is reformulated as the following backward problem $(\overline{\mathrm{P}})_\varepsilon$ formulated in the cylindrical domain Q_0:

$$(\overline{\mathrm{P}})_\varepsilon \quad \begin{cases} \bar\eta_{\varepsilon,t} + \displaystyle\sum_{i,j=1}^{3} \bar b_\varepsilon \frac{\partial}{\partial y_i}\left\{a_{ij}\frac{\partial\bar\eta_\varepsilon}{\partial y_j}\right\} + (-\bar b_\varepsilon \mathrm{w}_1 + \mathrm{w}_2)\cdot\nabla\bar\eta_\varepsilon = \bar\ell \quad \text{in } Q_0, \\ \dfrac{\partial\bar\eta_\varepsilon}{\partial\nu_A} + \bar n_0\bar\eta_\varepsilon = 0 \quad \text{on } \Sigma_0, \\ \bar\eta_\varepsilon(T,\cdot) = 0 \quad \text{on } \Omega_0, \end{cases}$$

where w_i, $i = 1,2$, and $\bar n_0$ are the same as in section 3, $\bar\eta_\varepsilon(t,y) := \eta_\varepsilon(t, Y(t,y))$, $\bar b_\varepsilon(t,y) := b_\varepsilon(t, Y(t,y))$ and $\bar\ell(t,y) := \ell(t, Y(t,y))$. We can solve $(\overline{\mathrm{P}})_\varepsilon$ by applying the general theory of quasi-linear parabolic equations in Ladyzhenskaya, Solonnikov and Ural'tseva (1968) and see that it has a unique solution that $\eta_\varepsilon(t,x) := \bar{\eta}_\varepsilon(t, X(t,x))$ is a solution of $(\mathrm{P})_\varepsilon$ on Q_m, satisfying the required regularities.

Here we are going to show some uniform estimates for η_ε with respect to ε.

LEMMA 5.1 *There exists a positive constant M_6, which depends on ℓ and is independent of parameter $\varepsilon \in (0,1]$, such that*

$$
\begin{aligned}
&|\nabla \eta_\varepsilon(s)|^2_{L^2(\Omega_m(s))} + \int_s^T \int_{\Omega_m(t)} b_\varepsilon |\Delta \eta_\varepsilon|^2 dx dt \\
&\le M_6 \int_s^T |\nabla \eta_\varepsilon(t)|^2_{L^2(\Omega_m(t))} dt + n_0 \int_s^T \int_{\Gamma(t)} \frac{\partial}{\partial t} \eta_\varepsilon^2 d\Gamma(t) dt \\
&+ n_0 \int_s^T \int_{\Gamma(t)} \mathbf{v} \cdot \nabla(\eta_\varepsilon^2) d\Gamma(t) dt + M_6 \textit{ for all } s \in [0,T] \textit{ and } \varepsilon \in (0,1]. \qquad (5.3)
\end{aligned}
$$

Proof. Multiplying the first equation in $(\mathrm{P})_\varepsilon$ by $\Delta\eta_\varepsilon$ and integrating it over $\Omega_m(t)$ with respect to x, we get for each t

$$
\begin{aligned}
&\int_{\Omega_m(t)} \eta_{\varepsilon,t} \Delta \eta_\varepsilon dx + \int_{\Omega_m(t)} b_\varepsilon |\Delta \eta_\varepsilon|^2 dx + \int_{\Omega_m(t)} (\mathbf{v} \cdot \nabla \eta_\varepsilon) \Delta \eta_\varepsilon dx \\
&= - \int_{\Omega_m(t)} \nabla \ell \cdot \nabla \eta_\varepsilon dx. \qquad (5.4)
\end{aligned}
$$

Here we observe that

$$
\begin{aligned}
&\int_{\Omega_m(t)} \eta_{\varepsilon,t} \Delta \eta_\varepsilon dx \\
&= - \int_{\Omega_m(t)} (\nabla \eta_{\varepsilon,t} \cdot \nabla \eta_\varepsilon)\, dx + \int_{\Gamma(t)} \eta_{\varepsilon,t} \frac{\partial \eta_\varepsilon}{\partial \boldsymbol{\nu}} d\Gamma(t) \\
&= -\frac{1}{2} \int_{\Omega_m(t)} \frac{\partial}{\partial t} |\nabla \eta_\varepsilon|^2 dx - \frac{n_0}{2} \int_{\Gamma(t)} \frac{\partial}{\partial t} \eta_\varepsilon^2 d\Gamma(t) \\
&= -\frac{1}{2} \frac{d}{dt} \int_{\Omega_m(t)} |\nabla \eta_\varepsilon|^2 dx + \frac{1}{2} \int_{\Gamma(t)} |\nabla \eta_\varepsilon|^2 v_\Sigma d\Gamma(t) \\
&- \frac{n_0}{2} \int_{\Gamma(t)} \frac{\partial}{\partial t} \eta_\varepsilon^2 d\Gamma(t). \qquad (5.5)
\end{aligned}
$$

Also we have by (1.1)

$$
\begin{aligned}
&- \int_{\Omega_m(t)} (\mathbf{v} \cdot \nabla \eta_\varepsilon) \Delta \eta_\varepsilon dx \\
&= \int_{\Omega_m(t)} \nabla(\mathbf{v} \cdot \nabla \eta_\varepsilon) \cdot \nabla \eta_\varepsilon dx - \int_{\Gamma(t)} (\mathbf{v} \cdot \nabla \eta_\varepsilon) \frac{\partial \eta_\varepsilon}{\partial \boldsymbol{\nu}} d\Gamma(t) \\
&= \int \sum^{3} \left\{ \frac{\partial v_j}{\partial x_i} \frac{\partial \eta_\varepsilon}{\partial x_j} \frac{\partial \eta_\varepsilon}{\partial x_i} + v_j \frac{\partial^2 \eta_\varepsilon}{\partial x_i \partial x_j} \frac{\partial \eta_\varepsilon}{\partial x_i} \right\} dx + n_0 \int (\mathbf{v} \cdot \nabla \eta_\varepsilon) \eta_\varepsilon d\Gamma(t)
\end{aligned}
$$

$$\le 3|\mathbf{v}|_{C^1(\overline{Q_m})^3} \int_{\Omega_m(t)} |\nabla \eta_\varepsilon|^2 dx + \frac{1}{2}\int_{\Omega_m(t)} \mathrm{div}\left(|\nabla\eta_\varepsilon|^2 \mathbf{v}\right) dx$$
$$+ \frac{n_0}{2}\int_{\Gamma(t)} \mathbf{v}\cdot\nabla(\eta_\varepsilon^2) d\Gamma(t), \tag{5.6}$$

and by (1.2)

$$\frac{1}{2}\int_{\Omega_m(t)} \mathrm{div}\left(|\nabla\eta_\varepsilon|^2 \mathbf{v}\right) dx = \frac{1}{2}\int_{\Gamma(t)} |\nabla\eta_\varepsilon|^2(\mathbf{v}\cdot\boldsymbol{\nu}) d\Gamma(t)$$
$$= \frac{1}{2}\int_{\Gamma(t)} |\nabla\eta_\varepsilon|^2 v_\Sigma d\Gamma(t). \tag{5.7}$$

Integrating (5.4) in time over $[s,T]$ and using (5.5)–(5.7), we get

$$\int_{\Omega_m(s)} |\nabla\eta_\varepsilon(s)|^2 dx + \int_s^T \int_{\Omega_m(t)} b_\varepsilon |\Delta\eta_\varepsilon|^2 dx dt$$
$$\le (6|\mathbf{v}|_{C^1(\overline{Q_m})^3} + 1)\int_s^T\int_{\Omega_m(t)} |\nabla\eta_\varepsilon|^2 dxdt + \int_s^T\int_{\Omega_m(t)} |\nabla\ell|^2 dxdt$$
$$+ n_0\int_s^T\int_{\Gamma(t)} \frac{\partial}{\partial t}\eta_\varepsilon^2 d\Gamma(t)dt + n_0\int_s^T\int_{\Gamma(t)} \mathbf{v}\cdot\nabla(\eta_\varepsilon^2) d\Gamma(t)dt.$$

Thus, a uniform estimate of the form (5.3) is derived. ■

LEMMA 5.2 *There exists a positive constant M_7, which depends on ℓ and is independent of parameter $\varepsilon \in (0,1]$, such that*

$$\int_{\Gamma(t)} \frac{\partial}{\partial t}\eta_\varepsilon(t)^2 d\Gamma(t) \le M_7 \frac{d}{dt}\int_{\Gamma(t)} \eta_\varepsilon(t)^2 d\Gamma(t) + M_7|\eta_\varepsilon(t)|^2_{L^2(\Gamma(t))} \tag{5.8}$$

for all $t\in[0,T]$ and $\varepsilon\in(0,1]$.

Proof. Our geometric condition $(\star)$ ensures that there exists a finite open covering $\{U_k(t)\}_{k=1}^N$ of $\Gamma(t)$ and a local coordinate system $y=(y_1,y_2,y_3)=\widetilde{X}_k(t,x)$ $:= (\widetilde{X}_{k1}(t,x),\ \widetilde{X}_{k2}(t,x),\ \widetilde{X}_{k3}(t,x))$ from $U_k(t)$ onto an open subset $\widetilde{U}_k$ of the y-space for all $t\in[0,T]$ such that

- $\widetilde{X}_k(t, U_k(t)\cap\Omega_m(t)) = \widetilde{U}_k \cap \{y; y_3<0\}$ and $\widetilde{X}_k(t,U_k(t)\cap\Gamma(t)) = \widetilde{U}_k\cap \{y;y_3=0\}$ $(\subset \mathbf{R}^2)$ for all $k=1,2,\dots,N$ and all $t\in[0,T]$, that is, every point (t,x) with $x\in U_k(t)\cap\Gamma(t)$ is mapped to $(t,y)=(t,\widetilde{X}_{k1}(t,x),\widetilde{X}_{k2}(t,x),0)$ for all $k=1,2,\dots,N$ and all $t\in[0,T]$;
- $\frac{\partial(\cdot)}{\partial\boldsymbol{\nu}} = \widetilde{a}_k(t,y')\frac{\partial(\cdot)}{\partial y_3}$ on $\widetilde{U}_k\cap\{y;y_3=0\}$, where $y' := (y_1,y_2,0)$ and $\widetilde{a}_k(\cdot,\cdot)$ is positive and of C^2-class on $[0,T]\times(\widetilde{U}_k\cap\{y;y_3=0\})$ for all $k=1,2,\dots,N$ and all $t\in[0,T]$;
- $d\Gamma(t) := S_k(t,y')dy'$ on $\widetilde{U}_k\cap\{y;y_3=0\}$ for $k=1,2,\cdots,N$, where $S_k(\cdot,\cdot)$

Moreover, take a partition of unity $\{\phi_k(t,\cdot)\}$ on $\Gamma(t)$, namely

$$\phi_k \in C_0^\infty(\mathbf{R}_t \times \mathbf{R}_x^3), \quad \mathrm{supp}(\phi_k(t,\cdot)) \subset U_k(t),$$

$$\sum_{k=1}^N \phi_k(t,\cdot) = 1 \quad \text{on } \Gamma(t), \quad t \in [0,T],\ 0 \le \phi_k \le 1, \quad k = 1,2,\ldots,N,$$

and put $\widetilde{\eta}_\varepsilon(t,y) := \eta_\varepsilon(t, \widetilde{Y}_k(t,y))$ and $\widetilde{\phi}_k(t,y) := \phi(t, \widetilde{Y}_k(t,y))$, where $\widetilde{Y}_k(t,\cdot) := \widetilde{X}_k^{-1}(t,\cdot) : \widetilde{U}_k \to U_k(t)$ for all $k = 1,2,\ldots,N$ and all $t \in [0,T]$. Since

$$\frac{\partial \eta_\varepsilon^2}{\partial t} = \frac{\partial \widetilde{\eta}_\varepsilon^2}{\partial t} + \sum_{i=1}^3 \frac{\partial \widetilde{\eta}_\varepsilon^2}{\partial y_i} \frac{\partial \widetilde{X}_{ki}}{\partial t},$$

it follows that for each $t \in [0,T]$

$$\begin{aligned}
&\int_{\Gamma(t)} \frac{\partial}{\partial t} \eta_\varepsilon(t)^2 d\Gamma(t) \\
&= \sum_{k=1}^N \int_{\Gamma(t)\cap U_k(t)} \phi_k(t) \frac{\partial \eta_\varepsilon(t)^2}{\partial t} d\Gamma(t) \\
&= \sum_{k=1}^N \int_{\mathbf{R}^2} \widetilde{\phi}_k(t) \frac{\partial \widetilde{\eta}_\varepsilon(t)^2}{\partial t} S_k(t) dy' \\
&+ \sum_{k=1}^N \sum_{i=1}^3 \int_{\mathbf{R}^2} \widetilde{\phi}_k(t) \frac{\partial \widetilde{\eta}_\varepsilon(t)^2}{\partial y_i} \frac{\partial \widetilde{X}_{ki}(t)}{\partial t} S_k(t) dy'.
\end{aligned} \tag{5.9}$$

The first term of the last equality in (5.9) is estimated as follows:

$$\begin{aligned}
&\sum_{k=1}^N \int_{\mathbf{R}^2} \widetilde{\phi}_k(t) \frac{\partial \widetilde{\eta}_\varepsilon(t)^2}{\partial t} S_k(t) dy' \\
&= \frac{d}{dt} \left\{ \sum_{k=1}^N \int_{\mathbf{R}^2} \widetilde{\eta}_\varepsilon(t)^2 \widetilde{\phi}_k(t) S_k(t) dy' \right\} - \sum_{k=1}^N \int_{\mathbf{R}^2} \widetilde{\eta}_\varepsilon(t)^2 \frac{\partial}{\partial t} \left(\widetilde{\phi}_k(t) S_k(t) \right) dy' \\
&\le \frac{d}{dt} \int_{\Gamma(t)} \eta_\varepsilon(t)^2 d\Gamma(t) + \int_{\Gamma(t)} \eta_\varepsilon(t)^2 \left\{ \sum_{k=1}^N \left| \frac{\partial}{\partial t} \left(\widetilde{\phi}_k(t) S_k(t) \right) \right| \frac{1}{S_k(t)} \right\} d\Gamma(t).
\end{aligned}$$

Now, note that

$$\frac{\partial \widetilde{\eta}_\varepsilon^2}{\partial y_3} = \frac{1}{a_k} \frac{\partial \eta_\varepsilon^2}{\partial \nu} = -\frac{2n_0}{a_k} \eta_\varepsilon^2,$$

where $a_k(t,x) := \widetilde{a}_k(t, X(t,x))$. Then, the second term of the last equality in

$$\begin{aligned}
&\sum_{k=1}^{N}\sum_{i=1}^{3}\int_{\mathbf{R}^2}\widetilde{\phi}_k(t)\frac{\partial\widetilde{\eta}_\varepsilon(t)^2}{\partial y_i}\frac{\partial\widetilde{X}_{ki}(t)}{\partial t}S_k(t)dy' \\
&=\sum_{k=1}^{N}\sum_{i=1}^{2}\int_{\mathbf{R}^2}\left\{-\frac{\partial}{\partial y_i}\left(\widetilde{\phi}_k(t)\frac{\partial\widetilde{X}_{ki}(t)}{\partial t}S_k(t)\right)\widetilde{\eta}_\varepsilon^2\right. \\
&\left.+\widetilde{\phi}_k(t)\frac{\partial\widetilde{\eta}_\varepsilon(t)^2}{\partial y_3}\frac{\partial\widetilde{X}_{k3}(t)}{\partial t}S_k(t)\right\}dy' \\
&\le\int_{\Gamma(t)}\eta_\varepsilon(t)^2\sum_{k=1}^{N}\left\{\left|\frac{\partial}{\partial y_1}\left(\widetilde{\phi}_k(t)\frac{\partial\widetilde{X}_{k1}(t)}{\partial t}S_k(t)\right)\right.\right. \\
&\left.\left.+\ \frac{\partial}{\partial y_2}\left(\widetilde{\phi}_k(t)\frac{\partial\widetilde{X}_{k2}(t)}{\partial t}S_k(t)\right)\right|\frac{1}{S_k(t)}\right\}d\Gamma(t) \\
&+\int_{\Gamma(t)}\eta_\varepsilon(t)^2\sum_{k=1}^{N}\left|\frac{2n_0}{a_k(t)}\frac{\partial\widetilde{X}_{k3}(t)}{\partial t}\right|d\Gamma(t);
\end{aligned}$$

in the first integral of the last inequality we consider the integrals as functions of (t,x) by the inverse transformation of $y=\widetilde{X}_k(t,x)$. Therefore (5.8) holds for a constant $M_7>0$ having the required properties. ■

LEMMA 5.3 *There exists a positive constant M_8, which depends on ℓ and is independent of parameter $\varepsilon\in(0,1]$, such that*

$$\int_{\Gamma(t)}\mathbf{v}(t)\cdot\nabla(\eta_\varepsilon(t)^2)d\Gamma(t)\le M_8|\eta_\varepsilon(t)|^2_{L^2(\Gamma(t))} \tag{5.10}$$

for all $t\in[0,T]$ and $\varepsilon\in(0,1]$.

Proof. We can obtain a uniform estimate of the form (5.10) in the same way as (5.9) in the proof of Lemma 5.2. ■

Now, by Lemmas 5.1-5.3 and utilizing that $n_0>0$ we see that there exists a positive constant M_9, which depends on ℓ and is independent of parameter $\varepsilon\in(0,1]$, such that

$$\begin{aligned}
&|\eta_\varepsilon(s)|^2_{H_1(\Omega_m(s))}+\int_s^T\int_{\Omega_m(t)}b_\varepsilon|\Delta\eta_\varepsilon|^2dxdt \\
&\le M_9\left\{\int_s^T|\eta_\varepsilon(t)|^2_{H^1(\Omega_m(t))}dt+1\right\}
\end{aligned} \tag{5.11}$$

for all $s\in[0,T]$. Accordingly, applying the Gronwall's inequality to (5.11), we finally have

$$\sup\ |\eta_\varepsilon(t)|^2_{H^1(\Omega_m(t))}+\int\ b_\varepsilon|\Delta\eta_\varepsilon|^2dxdt\le M_{10}, \tag{5.12}$$

where M_{10} is a positive constant, which depends only on ℓ (it is independent of $\varepsilon \in (0,1]$). Taking η_ε as a test function η in (5.2), we have

$$\begin{aligned}
0 &= -\int_{Q_m} (u_1 - u_2)\{\eta_{\varepsilon,t} + b\Delta\eta_\varepsilon + \mathbf{v}\cdot\nabla\eta_\varepsilon\}dxdt \\
&= -\int_{Q_m} (u_1 - u_2)\{\eta_{\varepsilon,t} + b_\varepsilon\Delta\eta_\varepsilon + \mathbf{v}\cdot\nabla\eta_\varepsilon\}dxdt \\
&+ \int_{Q_m} (u_1 - u_2)(b_\varepsilon - b)\Delta\eta_\varepsilon dxdt \\
&= -\int_{Q_m} (u_1 - u_2)\ell dxdt + \int_{Q_m} (u_1 - u_2)(b_\varepsilon - b)\Delta\eta_\varepsilon dxdt.
\end{aligned}$$

Thanks to (5.12) and $\varepsilon \to 0$, we have

$$\left|\int_{Q_m} (u_1 - u_2)(b_\varepsilon - b)\Delta\eta_\varepsilon dxdt\right| \le \left\{\int_{Q_m} |u_1 - u_2|^2 |b_\varepsilon - b| dxdt\right\}^{\frac{1}{2}} (2M_{10})^{\frac{1}{2}} \\ \to 0.$$

Therefore

$$\int_{Q_m} (u_1 - u_2)\ell dxdt = 0 \quad \text{for all } \ell \in C_0^\infty(Q_m),$$

which implies that $u_1 = u_2$ a.e. on Q_m. ■

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