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Run-Up of Linear Long Waves on Uniform Slope in Lagrangian Description

Włodzimierz Chybicki

Institute of Hydro-Engineering of the Polish Academy of Sciences,
ul. Kościerska 7, 80-328 Gdańsk, Poland, e-mail: wchyb@ibwpan.gda.pl

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Abstract

The case of linear, two-dimensional long waves on a uniform slope is considered. It is assumed that the fluid is nonviscous and incompressible. In the present paper the description of the long wave proposed by Wilde (Wilde, Chybicki 2004) is based on the fundamental assumption that the vertical material lines of fluid remain vertical during the entire motion. The equations of motion are derived with the help of a variational formulation of the problem. The Lagrangian is the difference between the kinetic and potential energy of the fluid. In the paper a correction followed from dispersion to the results obtained by Shuto is presented.

Key words: Wave run-up, Lagrangian description, Hamiltonian principle, hypergeometric functions

1. Introduction

Usually the wave run-up is defined as the maximum vertical height above still water level reached by the wave. Only a few models of water wave propagation are available for waves on a sloping bottom. In the Eulerian description usually used, the solution of this problem is writen in terms of the first kind Bessel functions of zero order. In the Lagrangian description the solution of this problem was found by Shuto (Shuto 1967).

Its application to this problem has several advantages. The boundary condition on the bottom in this system is more easily satisfied than in the Eulerian description. Secondly the maximum run-up height is the maximum of vertical displacement of the particle of water which at rest was on the shore line.

Another solution of this problem is presented. The equations of motion are derived with the help of a variational formulation of the problem. The Lagrangian is the difference between the kinetic and potential energy of the fluid. The basic assumption of the theory of long water waves simplifies the geometry of the displacement field. The correction follows from taking into account the additional term of kinetic energy.

2. Theory and Results

Let us consider a periodic two-dimensional motion of an infinite layer of inviscid fluid. It is assumed that the water for a time $t \leq 0$ is at rest and the corresponding particle co-ordinates are named a, b, $(-h(a) \leq b \leq 0)$, where h denotes depth of water. Let us assume that the bottom is described as

$$b = -\beta a, \tag{1}$$

hence the depth of the water is $h(a) = \beta a$. The still water surface is taken as the a-axis. The motion of the fluid is described by the mapping of the names into the positions occupied by the points at the time t. Let us assume that horizontal displacement is independent of the vertical co-ordinate. Thus, the mapping can be given as:

$$\begin{aligned} x &= x(a,b,t) = a + u(a,t), \\ y &= y(a,b,t) = b + w(a,b,t)\ . \end{aligned} \tag{2}$$

The incompressibility condition

$$\frac{\partial(x,y)}{\partial(a,b)} = \frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a} = 1, \tag{3}$$

leads to the following relation:

$$\frac{\partial w}{\partial b} = \frac{-u_a(a,t)}{1+u_a(a,t)} = -\frac{u_a(a,t)}{1+u_a(a,t)}\ . \tag{4}$$

The solution of the equation reads

$$w(a,b,t) = -\frac{(b+\beta a)\,u_a(a,t)}{1+u_a(a,t)} + f(a,t) \tag{5}$$

and the continuity equation is satisfied for any function $f(a,t)$. This function can be specified from the boundary condition on the bottom, which in the Lagrangian description can be formulated as follows: the water particle originally rested on the bottom should stay here. It is easy to verify that function

$$\begin{aligned} f(a,t) &= -h\,(a + u\,(a,t)) + h\,(a) \approx \\ &\approx -\frac{\partial h}{\partial a}u\,(a,t) - \frac{1}{2}\frac{\partial^2 h}{\partial a^2}u\,(a,t)^2 - \frac{1}{2}\frac{\partial^3 h}{\partial a^3}u\,(a,t)^3 \ldots \end{aligned} \tag{6}$$

guarantees, that this condition is fulfilled.

The equations of motion are derived from the hamiltonian principle (Herivel 1954, Szmidt 1988):

$$\delta \int_t \int_a L da dt = 0, \tag{7}$$

with the Lagrangian function

$$L = \int_{-h}^{0} \left[\rho \frac{1}{2} \left(x_t^2 + y_t^2 \right) + p \left(\frac{\partial (x, y)}{\partial (a, b)} - 1 \right) - \rho g y \right] db. \tag{8}$$

In the considered case the determinant of matrix of Jacobi is always equal to unity and the Lagrangian function presents the density of the difference between the kinetic and potential energy, given as:

$$L = \int_{-h}^{0} \left[\rho \frac{1}{2} \left(x_t^2 + y_t^2 \right) - \rho g y \right] db. \tag{9}$$

Using relations (1) and (5) the Lagrangian function can be expressed in terms of horizontal displacements. This function can be integrated analytically with respect to vertical co-ordinate b from $-\beta a$ to 0. In order to obtain the linear equation of motion, let us consider the two lowest orders of the Lagrangian function:

$$L = L_1 + L_2, \tag{10}$$

where

$$L_1 = \rho \frac{\partial}{\partial a} \left(\frac{hu}{2} \right) \text{ and}$$

$$\begin{aligned} L_2 = \rho & \left[\frac{h}{2} \left(\frac{\partial u}{\partial t} \right)^2 + \frac{h^3}{6} \left(\frac{\partial^2 u}{\partial t \partial a} \right)^2 - \frac{g h^2}{2} \left(\frac{\partial u}{\partial a} \right)^2 \right] + \\ + \rho & \left[h \left(\frac{\partial h}{\partial a} \right)^2 \left(\frac{\partial u}{\partial t} \right)^2 + \frac{gh}{2} \frac{\partial^2 h}{\partial a^2} u^2 + \frac{h^3}{2} \frac{\partial h}{\partial a} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t \partial a} \right]. \end{aligned} \tag{11}$$

It is proper to notice that for constant depth the last bracket vanishes. The Euler equation reads

$$\left(1 + \frac{h}{2} \frac{\partial^2 h}{\partial a^2} \right) \frac{\partial^2 u}{\partial t^2} - g \frac{\partial^2}{\partial a^2} (hu) - \frac{h^2}{3} \frac{\partial^4 u}{\partial t^2 \partial a^2} - h \frac{\partial h}{\partial a} \frac{\partial^3 u}{\partial t^2 \partial a} = 0. \tag{12}$$

This is an equation of linear long waves with weak dispersion for the case of uneven bottom. For constant depth equation (12) takes the form:

$$\frac{\partial^2 u}{\partial t^2} - gh \frac{\partial^2 u}{\partial a^2} - \frac{h^2}{3} \frac{\partial^4 u}{\partial t^2 \partial a^2} = 0 \tag{13}$$

and determines dispersion relation the same as in the Boussinesq equations:

$$\omega^2 = \frac{ghk^2}{1 + k^2h^2/3}. \tag{14}$$

For very long waves the terms followed from kinetic energy of vertical displacement are usually neglected as negligible quantities. Thus, for long waves equation (12) reads:

$$\frac{\partial^2 u}{\partial t^2} - g\frac{\partial^2}{\partial a^2}(hu) = 0. \tag{15}$$

The above equation was obtained by Shuto (Shuto 1967).

Using (1) equation (12) is rewritten as:

$$\frac{\partial^2 u}{\partial t^2} - g\frac{\partial^2}{\partial a^2}(\beta a u) - \frac{\beta^2 a^2}{3}\frac{\partial^4 u}{\partial t^2 \partial a^2} - \beta^2 a\frac{\partial^3 u}{\partial a\,\partial t^2} = 0. \tag{16}$$

Let us consider the case in which the movement is harmonic in time, i.e.

$$\eta = \bar{u}(a)e^{i\omega t} \tag{17}$$

and equation (16) reduces to ordinary differential equation

$$\frac{\beta a}{3}\left(\beta a\omega^2 - 3g\right)\frac{d^2\bar{u}}{da^2} + \beta\left(\beta a\omega^2 - 2g\right)\frac{d\bar{u}}{da} - \omega^2\bar{u} = 0. \tag{18}$$

Introducing a new variable $y = \frac{\beta\omega^2}{3g}a$ equation (18) takes the form:

$$\beta^2 y\,(y-1)\frac{d^2\bar{u}}{dy^2} + \beta^2(3y-2)\frac{d\bar{u}}{dy} - 3\bar{u} = 0, \tag{19}$$

or after some manipulation

$$\begin{aligned} &-\beta^2\left(y\,(1-y)\frac{d^2\bar{u}}{dy^2} + (2-3y)\frac{d\bar{u}}{dy} + \frac{3}{\beta^2}\bar{u}\right) = \\ &= -\beta^2\left(y\,(1-y)\frac{d^2\bar{u}}{dy^2} + [C - (A+B+1)\,y]\frac{d\bar{u}}{dy} - AB\bar{u}\right) = 0, \end{aligned} \tag{20}$$

where $A = 1 - \frac{\sqrt{3+\beta^2}}{\beta}$, $B = 1 + \frac{\sqrt{3+\beta^2}}{\beta}$, $C = 2$. The equation in the last brackets is a so called hypergeometric differential equation (Morse, Feshbach 1953). The solution, which has a regular singular point at the origin is a hypergeometric function

$$\bar{u} = {}_2F_1(A, B, C, y) = {}_2F_1\left(1 - \frac{\sqrt{3+\beta^2}}{\beta}, 1 + \frac{\sqrt{3+\beta^2}}{\beta}, 2, y\right) = \\ = {}_2F_1\left(1 - \frac{\sqrt{3+\beta^2}}{\beta}, 1 + \frac{\sqrt{3+\beta^2}}{\beta}, 2, \frac{\beta\omega^2}{3g}a\right). \tag{21}$$

The free surface is determined by the equality $b = 0$. Setting $b = 0$ to equation (5) and using (6) we obtain the formula of the vertical displacement:

$$w(a,t) = -\frac{\beta a u_a(a,t)}{1 + u_a(a,t)} - \beta u(a,t). \tag{22}$$

Hence, a linear part of Eq. (22) is

$$w(a,t) = -\beta a u_a(a,t) - \beta u(a,t). \tag{23}$$

Using formula

$$\frac{d\left({}_2F_1(a,b,c,z)\right)}{dz} = \frac{ab}{c}{}_2F_1(a+1, b+1, c+1, z), \tag{24}$$

we can write free surface elevation for standing waves in parametric form:

$$u(a,t) = A \cdot {}_2F_1\left(1 - \frac{\sqrt{3+\beta^2}}{\beta}, 1 + \frac{\sqrt{3+\beta^2}}{\beta}, 2, \frac{\beta\omega^2}{3g}a\right)\cos(\omega t), \tag{25}$$

$$w(a,t) = -A\beta \cdot {}_2F_1\left(1 - \frac{\sqrt{3+\beta^2}}{\beta}, 1 + \frac{\sqrt{3+\beta^2}}{\beta}, 2, \frac{\beta\omega^2}{3g}a\right)\cos(\omega t) + \\ + A\frac{a\omega^2}{2g} \cdot {}_2F_1\left(2 - \frac{\sqrt{3+\beta^2}}{\beta}, 2 + \frac{\sqrt{3+\beta^2}}{\beta}, 3, \frac{\beta\omega^2}{3g}a\right)\cos(\omega t). \tag{26}$$

The maximum horizontal and vertical displacement of the particle which rests on the shore-line at the initial instant is A and $A\beta$ respectively. Relations (25–26) represent the solution of equation (16) when complete reflection is assumed. Hypergeometric functions (25) and (26) have their singular point at $\frac{\beta\omega^2 a}{3g} = 1$. For long waves however, this argument can be approximated as

$$\frac{\beta\omega^2 a}{3g} = \frac{h}{3g}\omega^2 \approx \frac{h}{3g}\left(\frac{ghk^2}{1 + k^2h^2/3}\right) = \frac{k^2h^2}{3 + k^2h^2} \tag{27}$$

and for long waves, for $L > 8h$ this value is always less than 0.2.

The solution of equation (14) in case of non-dispersive waves for horizontal and vertical displacements was obtained by Shuto (Shuto 1967):

$$u(a,t) = A_1 \frac{\sqrt{\beta g}}{\omega} J_1\left(2\omega\sqrt{\frac{a}{g\beta}}\right)\cos(\omega t), \tag{28}$$

$$w(a,t) = A_1 \beta J_0\left(2\omega\sqrt{\frac{a}{g\beta}}\right)\cos(\omega t). \tag{29}$$

Relationships (25–26) and (28–29) describe the surface elevation on uniform slopes in parametric form respectively with and without taking into consideration the influence of dispersion. Figure 1 shows the comparison of free surface elevation calculated with the help of formulas (25–26) and (28–29).

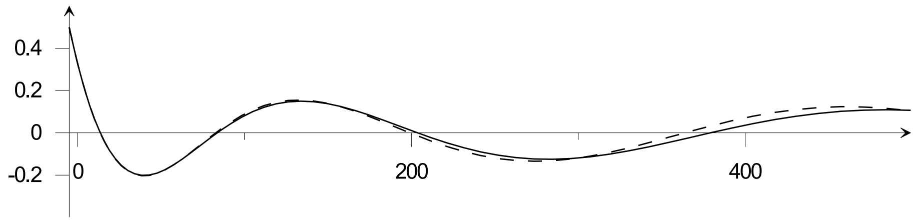

Fig. 1. Free surface elevation with dispersion – dashed line and without dispersion – solid line. Wave parameters, $\omega = 0.3\ \mathrm{s}^{-1}$, $\beta = 0.1$

The influence of dispersion causes that increase of wave amplitude is a bit smaller and waves for great depth are shorter. This effect is more visible when the waves are shorter and the slope is smaller. For standing waves with period of about 10 s and slope 1 : 25 the same comparison is shown in Fig. 2.

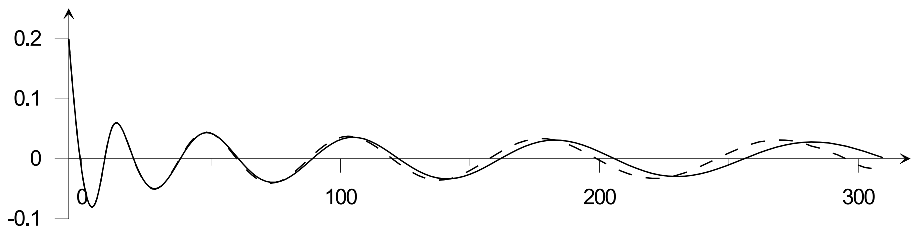

Fig. 2. Free surface elevation with dispersion – dashed line and without dispersion – solid line. Wave parameters, $\omega = 0.6\ \mathrm{s}^{-1}$, $\beta = 0.04$

The maximum run-up amplitude on uniform slope was obtained by Carrier and Greenspan (Carrier, Greenspan 1958, Voltzinger et al 1989):

$$R_{\max} = A_0 2\sqrt{\pi}\left(\frac{\omega^2 h_0}{g\beta^2}\right)^{\frac{1}{4}}, \tag{30}$$

where A_0 is the amplitude of the incident wave with frequency ω on depth h_0. The ratio $R_c/R_{\max}$ calculated run-up amplitude R_c to the theoretical results, given by Eq. (30), as a function of dimensionless frequency $\omega\sqrt{\frac{h_0}{g\beta^2}}$ is displayed in Fig. 3.

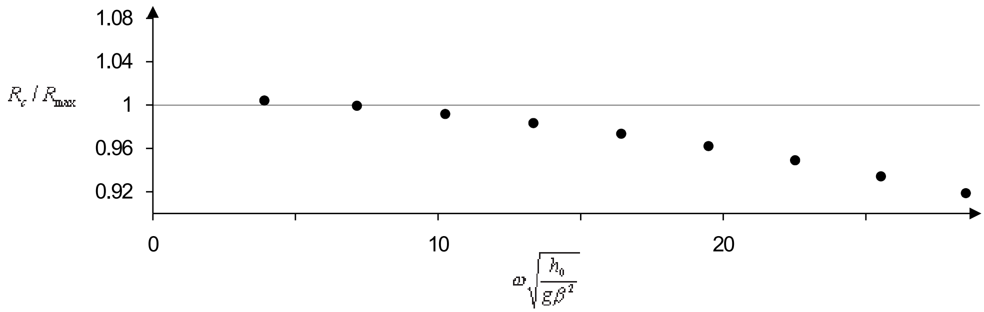

Fig. 3. The ratio calculated run-up amplitude to the theoretical results as a function of dimensionless frequency

The influence of dispersion is significantly stronger for dimensionless frequency parameter.

3. Conclusions

1. For the long waves, the assumption that horizontal displacements do not depend on vertical co-ordinates leads to the expression for vertical displacement.
2. The part of kinetic energy following vertical displacement introduces dispersion to the equation obtained.
3. When the motion is harmonic in time, the equation of motion can be simplified to a hypergeometric differential equation. The solution for horizontal and vertical displacement is expressed in terms of hypergeometric functions.
4. The comparison of the obtained solution with result described by Shuto was shown. If the dispersion of the waves is taken into account, the resulting waves are shorter. The increase of wave amplitude is less than in the non-dispersive case.

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