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An Approximate Method for Estimation of Fluid Motion in a Lake Containing Islands

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Abstract

The physical phenomenon considered in the paper deals with flow in a lake, which contains one or several islands. The flow is generated by rivers, their inlets and outlets being distributed on the shoreline of the lake. The problem to be solved consists in determination of the velocity field – in the domain bounded by the shorelines of the lake and the islands.

In fact, it is attempted to arrive solely at an *estimation* of the field. Consequently, a rather simple physical model of the phenomenon, as well as its simple mathematical description has been applied. In particular, plane, irrotational and steady flow of ideal liquid has been introduced, the inlets and outlets of the rivers being simulated by sources and sinks.

Hence, the problem reduces to determination of a complex function, representing the complex velocity field, which satisfies the impermeability condition on *all* contours representing the shorelines, the field being generated by the singularities already mentioned.

Unfortunately, the so formulated problem is "overconditioned" or "too stiff", what means that the impermeability condition on the outer contour cannot be satisfied. Nevertheless, we arrived at a simple method for circumventing this obstacle, the payoff consisting in some *modification* of this contour. We had this particular modification in mind, applying the word "approximate" in the title of the paper.

The paper contains results – in the form of streamline patterns – for lakes containing from 1 to 3 islands. In the relevant figures the distances between the given and the modified exterior contours can be seen distinctly – allowing the reader to draw conclusions, whether the errors due to the modifications are admissible or not. Of course, it depends anyway on the point of view of the user of the results.

Key words: conformal mapping, complex potential, streamline pattern, multiply connected domain, circular domain

1. Introduction

Some fundamental concepts and formulae are recalled in this Chapter – in order to provide background for further considerations.

1.1. Conformal Mapping

The fundamental mathematical operation used in the present paper, consists in conformal mapping of a class of multiply connected domains. An exemplary member of this class is shown in Fig. 1, and it has the following characteristic features:

- it is bounded by a given (exterior) contour;
- it contains a number of inner contours, every one being separate with respect to the remaining ones.

The domain bounded by the contours is multiply connected and finite.

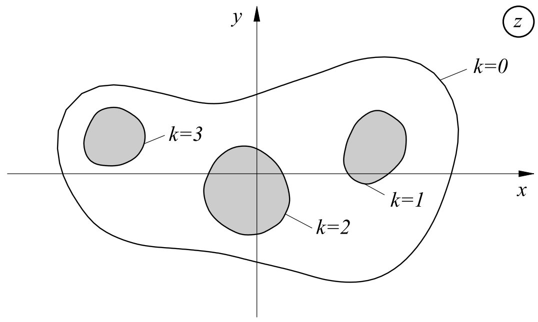

Fig. 1. A given, finite, multiply connected domain (Prosnak, Klonowska 1996)

The so-called *Koebe theorem* (Gaier 1964, Prosnak 1987) ensures existence and uniqueness of a mapping function, transforming the given domain onto a circular one, or *vice versa*. We assume this function in the following *standard* form (Prosnak, Klonowska 1996):

$$z = \sum_{n=0}^{\infty} G_{0n}\zeta^n + \sum_{k=1}^{K}\sum_{n=1}^{\infty} G_{kn}\left(\frac{a_k}{\zeta - \zeta_k}\right)^n, \tag{1}$$

the relevant circular domain being shown in Fig. 2.

The following symbols applied in the series (1):

$$a_k, \quad \zeta_k; \quad k = 0, 1, \ldots, K; \quad a_0 = 1; \quad \zeta_0 = 0, \tag{2}$$

denote radii and complex centers of the circles; the further ones:

$$G_{kn}; \quad k = 0, 1, \ldots, K; \quad n = 0, 1, 2, \ldots \tag{3}$$

– complex coefficients of the series. Of course, the series have to be truncated, a rather large number of terms to be retained at the first one.

Determination of the mapping function (1) reduces to determination of the unknown constants (2) and (3). It is rather a serious, nonlinear problem, which

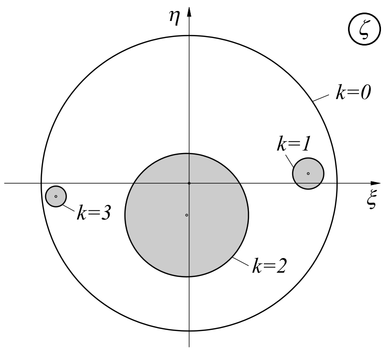

Fig. 2. Conformal mapping of the given domain onto a circular one (Prosnak, Klonowska 1996)

can be solved only by means of an iterative process. Fortunately, ready algorithms and computer programs for this task can be found in (Prosnak, Klonowska 1996) and (Klonowska-Prosnak 2004), so that in further considerations we will treat the mapping function (1) as known.

The standard form (1) of the mapping function is consistent with the so-called *theorem of Runge* (Leja 1973, Prosnak 1987), which ensures – roughly speaking – that any complex function defined in a finite multiply connected domain can be represented by power series and series of rational fractions.

1.2. Complex Potential and Complex Velocity

Any plane, steady, irrotational flow of ideal liquid can be described by means of the so-called *complex potential*:

$$w = w(z) = \varphi(x, y) + i\psi(x, y), \tag{4}$$

where:

$$z = x + iy$$

denotes domain of flow (see Fig. 1), the symbols φ and ψ standing for the velocity potential, and for the streamfunction, respectively. The derivative of (4) has the following property:

$$\frac{dw}{dz} = u(x, y) - i\, v(x, y), \tag{5}$$

the right-hand side containing rectangular velocity components. It is referred to as the *complex velocity*, although – in fact – it is *conjugate* of this value.

Assuming that the mapping function (1) is known in a case under consideration, we can use it – looking for the complex potential (4) and the complex velocity (5) not in the original complex plane z, but in the auxiliary, circular one ζ – see Fig. 2. Namely, denoting the mapping function (1) as

$$z = z(\zeta),$$

where

$$\zeta = \xi + i\eta,$$

we can perform transformation of (4) and (5) onto the ζ-domain, as follows:

$$w(z) = w(z(\zeta)) = W(\xi, \eta) = \Phi(\xi, \eta) + i\,\Psi(\xi, \eta);$$

$$\frac{dw}{dz} = \frac{dw(z(\zeta))}{d\zeta} \cdot \frac{d\zeta}{dz} = \frac{dW}{d\zeta} \cdot \left(\frac{dz}{d\zeta}\right)^{-1}.$$

In the last formula, the derivative

$$\frac{dW}{d\zeta} = U(\xi, \eta) - i\,V(\xi, \eta) \tag{6}$$

contains *rectangular* velocity components in the ζ-plane.

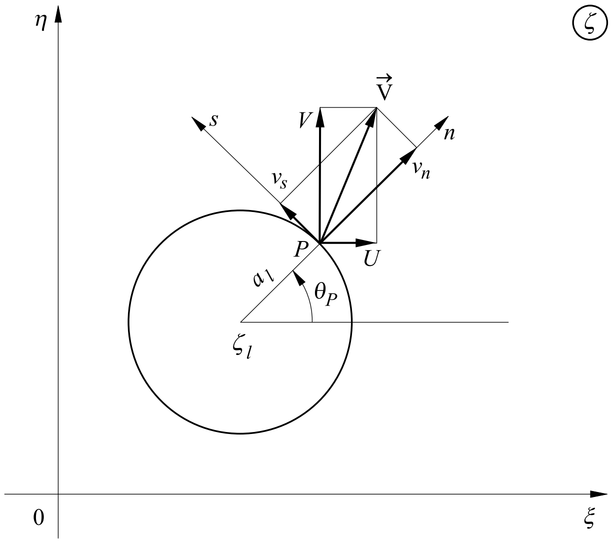

Fig. 3. Relations between velocity components in the Cartesian and the polar coordinate system

In order to be able to formulate boundary conditions for velocity on circles bounding the circular domain, formulae for the normal and for the tangent velocity components on every circle – should be derived.

Let us consider an arbitrary point P on an arbitrarily selected circle No. l – see Fig. 3 – and denote its complex coordinate as follows:

$$\zeta_P = \zeta_l + a_l e^{i\theta_P}.$$

Let us introduce also the symbols:

$$\vec{V}, \quad U, \quad V, \quad v_n, \quad v_s, \quad \theta_P \ ,$$

denoting:

$\vec{V}$	–	velocity vector in point P;
U, V	–	its rectangular components;
$v_n, \ v_s$	–	its components normal and tangent to the circle;
θ_P	–	an angle, defining position of point P on the circle No. l; see Fig. 3.

In further considerations the index $_P$ will be omitted at the angle θ_P – for the sake of simplicity.

The relations between $v_n, \ v_s$ and U, V following from Fig. 3 can be formulated as:

$$v_n = U\cos\theta + V\sin\theta;$$

$$v_s = V\cos\theta - U\sin\theta,$$

wherefrom:

$$v_n - i\, v_s = (U - i\, V)\, e^{i\theta}. \tag{7}$$

Returning now to formula (6) we will assume the complex velocity also in the standard form, containing the power series, as well as series of rational fractions – compare (1). Moreover, a function $f(\zeta)$, generating the flow will be introduced as well as the relation (7). Finally, the following formula for the complex velocity will be assumed:

$$(v_n - i\, v_s)_P = e^{i\theta}\left\{ f(\zeta_P) + \left[\sum_{n=0}^{N_0} C_{0n}\,(\zeta_P)^n\right] + \sum_{k=1}^{K}\sum_{n=1}^{N_k} C_{kn}\left(\frac{a_k}{\zeta_P - \zeta_k}\right)^n \right\}, \tag{8}$$

referring already to the particular point ζ_P on the l-th circle.

It should be kept in mind, that the function generating the flow has not been specified so far. Moreover, the coefficients of the series are unknown – contrary to the geometrical parameters a_k, ζ_k, defining the circular domain, the parameters following from the known mapping function (1).

1.3. Function Generating the Flow

The only function generating flow in the original domain – see Fig. 1 – consists of a set of sources and sinks, simulating inlets and outlets of rivers. As such, they are distributed on the *outer contour* of this domain. In consequence, all their images following from mapping of the original domain onto the circular one (Fig. 2) – have to be located on the outer circle. An exemplary system of such configuration of five singularities is shown in Fig. 4, the numbers referring to consecutive singularities.

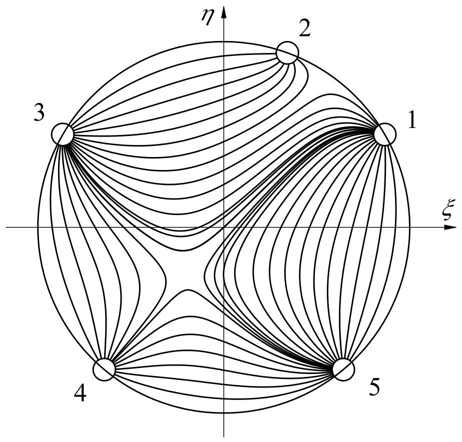

Fig. 4. Streamline pattern in a disc, generated by three sources and two sinks distributed on its boundary

The complex velocity for such a system of sources and sinks is represented by the following general formula:

$$f(\zeta) = \frac{dW_Q}{d\zeta} = \frac{1}{2\pi} \sum_{k=1}^{N_Q} \frac{Q_k}{\zeta - \zeta_{Q_k}} = U_Q(\xi, \eta) - i V_Q(\xi, \eta), \tag{9}$$

where – in the particular case presented in Fig. 4 – the following values have been assumed:

$$\left|\zeta_{Q_k}\right| = 1; \quad k = 1, 2, \ldots, 5;$$

$$Q_1 = 0.2; \quad Q_2 = 0.1; \quad Q_3 = -0.2; \quad Q_4 = 0.1; \quad Q_5 = -0.2.$$

It can be proved, that any function (9) has the interesting property, that the circle is identical with a streamline only if the sum of the intensities is equal to zero:

$$\sum_{k=1}^{N_Q} Q_k = 0. \tag{10}$$

This is undoubtly true in the case presented in Fig. 4.

The complex velocity (9) can be split – exactly – into two rectangular components:

$$U(\xi, \eta) = \frac{1}{2\pi} \sum_{k=1}^{N_Q} \mathrm{Re}\left\{ \frac{Q_k}{(\xi - \xi_{Q_k}) + i\,(\eta - \eta_{Q_k})} \right\};$$
$$V(\xi, \eta) = -\frac{1}{2\pi} \sum_{k=1}^{N_Q} \mathrm{Im}\left\{ \frac{Q_k}{(\xi - \xi_{Q_k}) + i\,(\eta - \eta_{Q_k})} \right\}, \tag{11a}$$

the symbols Re and Im denoting the real, and the imaginary part, respectively. Therefore, the ordinary differential problem for determination of a streamline can be formulated as follows:

$$\frac{d\xi}{dt} = U(\xi, \eta); \quad \frac{d\eta}{dt} = V(\xi, \eta); \tag{11b}$$

$$\xi = \xi(t); \quad \eta = \eta(t); \tag{11c}$$

$$\xi(t_0) = \xi_0; \quad \eta(t_0) = \eta_0, \tag{11d}$$

where (11b) denotes the system of two differential equations and (11c) – the parametric equations of the streamline, with ξ_0, η_0, t_0 defining its initial point. A standard Runge-Kutta method (Prosnak 1993) has been applied for solution of the differential problem.

2. Determination of the Velocity Field

2.1. Formulation of the Problem

As already mentioned, the problem under consideration consists in determination of velocity field in a lake which contains one or several islands. The field is generated by rivers, flowing in – or out – of the lake.

The physical model of this phenomenon is defined by the following simplifying assumptions:

- the domain of the solution is plane, finite and multiply connected;
- the fluid in the domain is simulated by ideal liquid;
- the flow is plane, irrotational, steady and generated by a given set of sources and sinks distributed on the outer contour of the domain.

Furthermore, it is assumed, that the domain of the solution can be transformed conformally onto a circular domain, which is also finite, multiply connected and bounded by a set of separate circles. The corresponding mapping function is regarded as known.

2.2. The Problem Considered in the Circular Domain. The Unavoidable Modification

It is well known, that the solution to the problem – to be understood as the velocity field – must be represented by a proper *holomorphic function* – in the original, as well as in the circular domain. In the last mentioned one, the complex velocity may be assumed in the standard form of (1), which leads to the formula (8), wherein the right hand side of (9) should be substituted instead of $f(\zeta)$.

Now, in order to determine the unknown coefficients

$$C_{kn} = \mu_{kn} + i\nu_{kn}; \quad k = 0, 1, \ldots, K; \quad n = 0, 1, \ldots, N_k$$

the proper system of algebraic equations must be developed, stemming from the requirement of impermeability in all the points ζ_P, distributed on the circles bounding the domain.

First of all, the fundamental question arises, as to whether the requirement may concern all the circles. The answer is no, and this follows from introduction of the singularities on the outer circle, which has to be a streamline.

In order to see the situation clearly, let us imagine that we formulate the problem in an *infinite* domain, "forgetting" the outer circle. The streamline pattern will cover in this case the whole infinite plane, all the inner circles will be impermeable, and an infinite set of regular quasi-circular streamlines will connect all the singular points.

What we can do now, is to look for such a particular streamline, which is as close as possible to the outer circle, and to accept it as the outer boundary line of the domain – instead of the given outer circle, which becomes permeable. The obvious *approximation* of this operation manifests itself by means of a gap between the circle, and the selected streamline. The gap represents the *error* of the approximation.

In consequence of replacing the original problem formulated in the finite domain by the modified one, concerning the infinite domain – the power series contained in the brackets of (8) must vanish, and the condition of impermeability will be assumed as:

$$v_n = Re\left\{ e^{i\theta}\left[\frac{1}{2\pi}\sum_{k=1}^{N_Q} \frac{Q_k}{\zeta_P - \zeta_{Q_k}} + \sum_{k=1}^{K}\sum_{n=1}^{N_k} C_{kn}\left(\frac{a_k}{\zeta_P - \zeta_k}\right)^n \right]\right\} = 0, \tag{12a}$$

with the only unknowns:

$$C_{kn} = \mu_{kn} + i\nu_{kn}; \quad k = 1, 2, \ldots, K; \quad n = 1, 2, \ldots, N_k. \tag{12b}$$

They are determined in this paper by means of the so called pseudo-spectral method (Klonowska-Prosnak 2004). It consists in the distribution of a sufficient number of nodes on the circles, every one being defined by the number of the circle, and by the value of the angular variable θ, so that:

$$\zeta_P = \zeta_k + a_k \exp(i\theta_{kn}); \quad k = 1, 2, \ldots, K, \tag{12c}$$

where

$$\theta_{kn} = n\frac{2\pi}{M_k}; \quad n = 1, 2, \ldots, M_k \tag{12d}$$

denotes the position of a node on the k-th circle, which contains M_k nodes in total.

Algorithms and computer programmes for generation of the system of equations for the real and imaginary parts of (12b), and for solving the system – can be found in (Klonowska-Prosnak et al 2005).

3. Selected Numerical Examples

3.1. Case No. 1: the Lake Containing One Island

The set of four diagrams, illustrating determination of the streamline pattern in the case described in the title of the present Section, is collected in Fig. 5.

In part (a) of this Figure the doubly connected, given domain is shown in the z-plane, the contours being drawn doubly: as solid and as dashed lines. One of them denotes the *given* contours, the other one is calculated by means of the suitable mapping function.

In part (b) the circular domain is shown in the ζ-plane, representing the counter-image of the former one.

The constants of the mapping function (1) are read in by the mentioned programme from a proper file, and some parts of the output are reproduced in Table 1 (Klonowska-Prosnak et al 2005, p. 45).

Some remarks concerning accuracy of the mapping can be also found in the cited Report.

Part (c) is the most interesting one. In order to obtain it, the coefficients given by (12b) had to be determined. The respective output is shown in Table 2.

Next, the previously mentioned ordinary, differential problems for determination of the selected streamlines had to be solved. It can be seen, that one of the streamlines is rather close to the circle, indicated by the dashed line.

Part (d) contains final results of the calculations, consisting of images of the streamlines appearing in the $\widetilde{\zeta}$-domain.

Table 1. Constants of the mapping function for case No. 1

Name of file, containing ready coefficients of the mapping function:
N,a,eps,kap,ksi,eta,mi,ni: S1_mn.res
. . . .
CONSTANTS OF THE MAPPING FUNCTION:
Radii of the discs:
a[0] = 1.000000000
a[1] = 0.198296832
Coordinates of the centres:
ksi[0] = 0.000000000 eta[0] = 0.000000000
ksi[1] = 0.178844305 eta[1] = -0.202094392
Coefficients of the series:
Contour number: 0

k	mi[0,k]	ni[0,k]
0	-0.018961842	0.010338522
1	0.290253971	-0.023057852
2	0.038354764	0.037805058
. . . .		
300	0.000000559	0.000000255

Contour number: 1

k	mi[1,k]	ni[1,k]
1	-0.001853660	-0.002387020
2	0.000228548	-0.001653475
3	0.000613304	0.000251658
. . . .		
50	-0.000000001	-0.000000001

Table 2. Coefficients of the complex velocity for case No. 1

ELEMENTS OF THE FUNCTION FOR GENERATING THE FLOW.
Number of nodes on circle No 1: Nw[1] = 98
NGaus = 98
Nk[1] = 50
Number of singularities on the outer circle: NQ = 2
Intensities and positions of the singularities:

k	Q[k]	thQ[k]
1	-2.000000	5.000000
2	2.000000	187.000000

CONSTANTS OF THE COMPLEX VELOCITY:
Coefficients of the series:
Contour number: 1

k	mi[1,k]	ni[1,k]
1	0.000000000	0.000000000
2	-0.616565363	-0.103955930
3	-0.019917612	-0.059057157
4	-0.017614645	-0.014792721
5	0.000340695	-0.004648183
6	-0.000077697	-0.000922120
7	0.000138209	-0.000219691
. . . .		
49	0.000000000	0.000000000
50	0.000000000	0.000000000

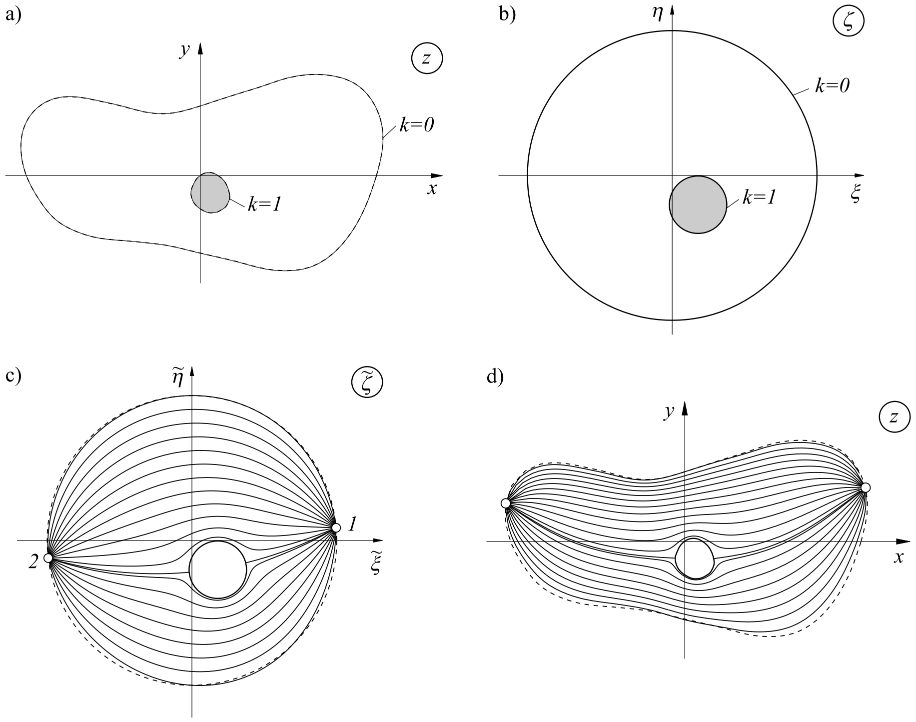

Fig. 5. A set of three conformal mappings necessary for determination of streamline pattern in a given doubly connected domain

The images are determined by means of the mapping function (1), for the proper values of the constants.

The replica of the contour given in (a) is shown in (d) – by the dashed line. According to the introduced approximation it coincides with the closest streamline only partly. It must be recalled, that the distance between the two lines represents the error of the approximation.

It should be emphasized, that the accuracy of the velocity field is important for the success of the whole project. Hence, it must be carefully checked. The obvious way of performing this operation consists in evaluation of the normal velocity component on the inner circles. In the case under consideration there is only one inner circle in the domain of solution, and values of the velocity components appear in the Table 3.

Checking the normal velocity component solely in the 98 nodes – is insufficient. It means in fact confirmation, whether the consecutive equations of the system are satisfied or not. The real validation must concern points situated between the nodes – as can be seen in the second part of Table 3.

Table 3. Validation of results in case No. 1

VALIDATION OF THE RESULTS.
Checking the consecutive equations.
det = 1.28743097491510E+0067
Velocity components vn, vs in the nodes.
l = 1

m	vn	vs
1	-0.000000000000	0.367268886684
2	0.000000000000	0.274324234736
3	-0.000000000000	0.177317465671
...	...	...
96	-0.000000000000	0.615479814168
97	-0.000000000000	0.538323435822
98	0.000000000000	0.455452530410

Velocity components vn, vs in points situated between original nodes.
l = 1

m	vn	vs
1	-0.000000000000	0.367268886684
2	0.000000000000	0.322280881084
3	-0.000000000000	0.276225499692
4	0.000000000000	0.229189029682
5	0.000000000000	0.181265668562
...	...	...
196	0.000000000000	0.573769597053
197	0.000000000000	0.535116128331
198	0.000000000000	0.495091556397
199	-0.000000000000	0.453739332180
200	0.000000000000	0.411111520094

3.2. Case No. 2: the Lake Containing Two Islands

The consecutive parts of the entire operation, consisting of a definition of the given domain a), determination of the circular domain (b), generation of the streamline pattern in the circular domain (c), and – finally – returning to the given domain (d) by the use of the mapping function (1) – are collected in Fig. 6.

It is worthy of noting, that in Fig. 6c the (dashed) circle is rather close to the corresponding streamline: they even coincide along rather long intervals, and intersect several times. However, after mapping the plane $\widetilde{\zeta}$ onto z by means of the mapping function (1), this pattern seems to change in this respect: if not qualitatively, then – undoubtly – quantitatively. The distance between the given contour (the dashed one), and its closest streamline – seems to grow relatively. This follows from the property of conformal mapping known as *dilatation of the mapping* (see e.g. Prosnak, Klonowska 1996, p. 45).

We do not think that introduction of some *numerical* results – as in the former case – is very profitable. Hence, they are abandoned. The same will be done in the next case.

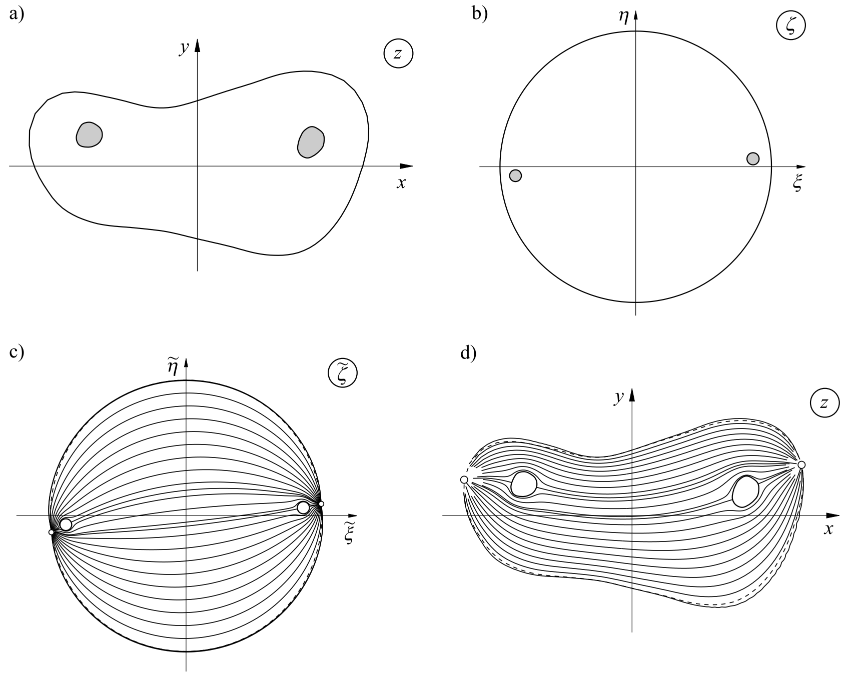

Fig. 6. A set of three conformal mappings necessary for determination of streamline pattern in a given trebly connected domain

3.3. Case No. 3: the Lake Containing Three Islands

As formerly, all the operations necessary to determine the streamline pattern are collected in just one Fig. 7.

The given domain shown in Fig. 7a resembles the one presented in Fig. 1. They differ with respect to dimensions and positions of the islands.

4. Conclusions and Comments

The phenomenon of flow in a lake containing islands, the flow being generated by in- and outcoming rivers, has been dealt with in the paper, and development of an approximate method for evaluation of velocity field in the lake has been attempted. The model of the phenomenon had been defined by some simplifying assumptions, concerning properties of the domain of solution, of the fluid, and of the independent variables governing the flow. The model can be described as plane, steady, irrotational motion of ideal liquid, in a finite multiply connected domain.

The domain has been transformed by a holomorphic function onto a circular one, wherein the solution has been sought in the form of a "standard" function, representing complex velocity, and satisfying the impermeability condition on the circles bounding the domain.

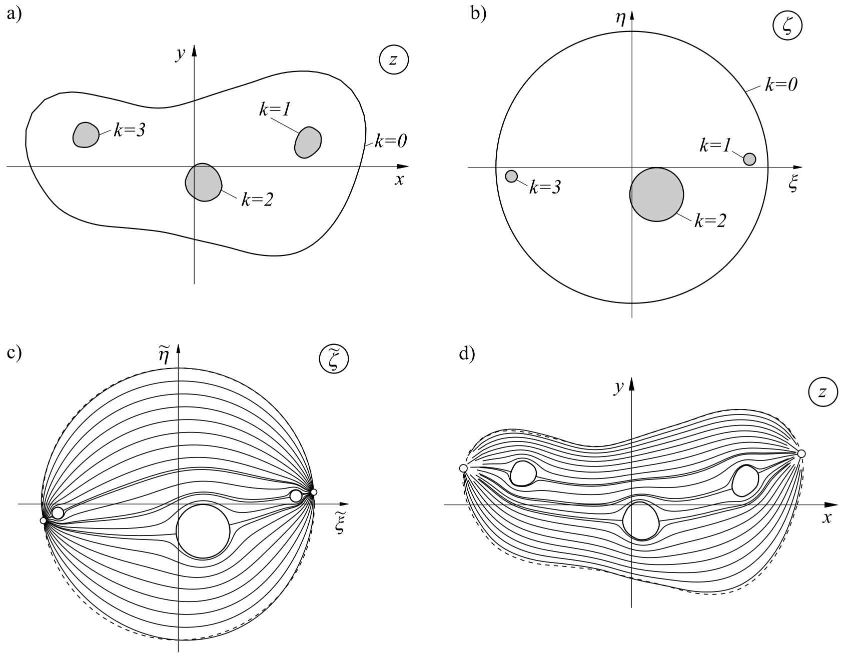

Fig. 7. A set of three conformal mappings necessary for determination of streamline pattern in a given quadri connected domain

The fundamental approximation characterising the method consists in *modification of the outer contour*. In so modified a domain, accuracy of solution, expressed by values of the normal velocity component at all contours bounding the domain – appears to be rather good.

The method has been applied in three cases differing by the number of the inner contours. No difficulties have been noticed. However, it should be kept in mind, that the success as a whole depends to a large extent on an element, which – in fact – does not belong to the method. Viz., accuracy of the mapping function is meant here. Nevertheless, in this paper the function is regarded as known, the reader being referred to (Prosnak, Klonowska 1996; Klonowska-Prosnak 2004), as far as determination of the function is concerned.

On the other hand, arriving at the complex velocity reduces to the solution of a system of algebraic equations of the first order, i.e. to a linear problem.

Some possibility of generalisation of presented results should now be briefly considered.

In all the three examples just one source, and one sink simulate the in- and outcoming river. As follows from Fig. 4, more such singularities can be easily introduced.

True enough, the flow is assumed in all cases as irrotational, but it seems obvious, that a vortex can be inserted into any of the inner contours – similarly as is done in the case of a system of airfoils. The necessity of such generalisation depends on the shape of an island, and existence of the vortex can be explained analogically, i.e. by action of the (neglected) viscosity.

Also – it seems possible to take into account streams, flowing out from the islands. This possibility has not, however, been explored by ourselves.

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