THE APPLICATION OF REGULARISATION METHODS TO ANALYSIS OF STRUCTURE DYNAMICS

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Abstract
In the paper there are discussed issues concerning ill-posed problems. Mathematical definition and a method of detecting ill-posed problems as well as a method of improving such problems conditioning by the use of the Tikhonov regularisation are presented. The results of transfer function noise reduction by the use of the Tikhonov regularisation method are shown.

Keywords: ill-posed problem, ill-conditioned matrix, regularisation

ZASTOSOWANIE METOD REGULARYZACJI W ANALIZIE DYNAMIKI KONSTRUKCJI

Streszczenie
W pracy omówiono zagadnienia dotyczące zagadnień źle zdefiniowanych. Przedstawiono definicję matematyczną, metodę wykrywania zagadnień źle zdefiniowanych oraz metodę poprawiania uwarunkowania tych zagadnień przy użyciu regularyzacji Tikhonova. Zaprezentowano również możliwość zastosowania metody regularyzacji Tikhonova do redukcji szumów widmowych funkcji przejścia.

Słowa kluczowe: zagadnienie źle zdefiniowane, macierz źle uwarunkowana, regularyzacja

1. INTRODUCTION

For many years ill-posed problems were treated as a mathematical curiosity. The first mathematical description was proposed by Hadamard in 1915. In 1977 N. Tikhonov and V. Y. Arsenin proved that the class of ill-posed problems includes many classical mathematical problems and, what is more important, that such problems find practical applications.

Nowadays, for the purposes of identification of complex mechanical structures, the methods of inverse identification are frequently used. Determining an inverse problem solution is complicated by the fact that measurement characteristics are always burdened with a variety of errors. In case of ill-posed problems even small errors of measured system responses have a great influence on accuracy of estimated parameters. Estimation of a correct solution is impossible without earlier improvement of problem formulation. Therefore regularisation as a method of ill-defined problems effective solving arouse great interest.

In this paper the application of the Tikhonov regularisation method to transfer function noise reduction is discussed. The noise reduction of transfer functions on the basis of which the modal models are estimated results in increase in the parameters accuracy of these models. It is especially important in case of diagnosing structure state on the basis of changes in the modal parameters such as natural frequencies, modal damping factors and system mode shapes [4].

2. MATHEMATICAL DESCRIPTION OF ILL–POSED PROBLEMS

According to the Hadamard definition, the equation:

$$[A]\{x\}=\{y\} \qquad [A]: X \to Y \tag{1}$$

is well–posed provided:

1. solution existence for each $\forall \{y\} \in Y$, $\{x\} \in X$ such that $[A]\{x\} = \{y\}$,
2. uniqueness: $[A]\{x_1\} = [A]\{x_2\} \Rightarrow \{x_1\} = \{x_2\}$,
3. stability: $[A]^{-1}$ is continuous.

Equation (1) is ill–posed if one of the above conditions is not met.

3. IDENTIFICATION METHODS OF ILL – POSED PROBLEMS

The SVD method is the most popular method that allows for identification of ill–posed problems. Singular values resulting from the SVD decomposition of a system matrix $[A] \in R^{m \times n}$ ($m \geq n$) are described by the equation [1]:

$$[A]=[U][\Sigma][V]^T = \sum_{i=1}^{n}\{u_i\}\sigma_i\{v_i\}^T \tag{2}$$

where: [U], [V]: orthonormal matrixes of singular vectors: $[U]^T[U] = [V]^T[V] = [I]_n$,

$$[\Sigma]=\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \sigma_n\end{bmatrix}: \text{ diagonal matrix}$$

such that: $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$, σ_i: singular value of [A] matrix; $\{v_i\}$, $\{u_i\}$: right and left singular vector of [A] matrix.

A system matrix [A] of a discrete ill-posed problem is always ill-conditioned. In such a case a determinant of the [A] matrix is close to zero, which means that the [A] matrix is almost rank-deficient. The SVD decomposition of such an ill-conditioned matrix has the following properties [3]:

– singular values σ_i gradually decay to zero,
– along with the increase in *i* index, in the $\{v_i\}$, $\{u_i\}$ vectors more changes in signs of elements are observed,
– [A] matrix condition number is high (the highest to smallest singular value ratio > 10^{14}).

4. REGULARISATION

As a regularisation we understand an improvement of a problem posedness or, in a discrete case, of a system matrix [A] conditioning. From the mathematical point of view, the method idea is to estimate approximate inverse operator $[R_\alpha]$ which, under the condition that:

$$\{y_n\}=[A]\{x_{real}\}+\{\eta_n\} \qquad \{\eta_n\}\to 0 \qquad (3)$$

satisfies the equation:

$$\{x_{\alpha n}\}=[R_{\alpha n}]\{y_n\} \quad \to \quad \{x_{true}\} \qquad (4)$$

4.1. Tikhonov regularisation method

Measured response of a real system (1) is described by the equation:

$$[A]\{x\}=\{y_{sz}\} \quad \Leftrightarrow \quad [A]\{x\}=\{y_{ideal}\}+\{\eta\} \qquad (5)$$

where: $\{y_{sz}\}\in R^{n\times 1}$: measured noisy system response; $\{\eta\}\in R^{n\times 1}$: noise; $[A]\in R^{n\times m}$: system matrix; $\{x\}\in R^{m\times 1}$: unknown solution; n, m: integers.

Numerical solution of the least squares method, which is commonly used for solving algebraical equations, is unique and unbiased only when the [A] matrix rank equals m. Therefore an ill-posed problem solution obtained by the use of the least squares method:

$$\{x_{ls}\}=\arg\min_x \left\|\{y_{sz}\}-[A]\{x\}\right\|_2^2 \qquad (5)$$

is unstable – the more noisy is the measurement data the more obtained solution differs from the correct one. Modification of the equation of interest by replacing the [A] matrix with a well-conditioned matrix as well as introducing additional constraints: do not guarantee obtaining correct solutions. Determining a correct solution by the use of an inverse method is usually impossible without earlier improvement of problem formulation (system matrix conditioning). In case of the Tikhonov regularisation method, an unknown solution has a form of [3]:

$$\{x_\alpha\}=\arg\min_x\left\{\left\|\{y_{sz}\}-[A]\{x\}\right\|_2^2+\alpha^2\left\|[L]\{x\}\right\|_2^2\right\} \qquad (6)$$

where: α: regularisation parameter describing a compromise between an accurate fitting and a smoothness of the obtained curve; [L]: usually a unit matrix; [I]: unit matrix.

The L-curve is the most popular method of determining an optimal regularisation parameter α [2].

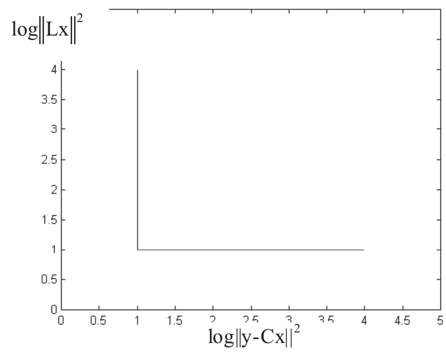

Fig. 1. L-curve method

The L-curve method [2, 3] consists in determining a graphical dependence between $\left\|\{y_{sz}\}-[A]\{x_\alpha\}\right\|_2^2$ and $\left\|[L]\{x_\alpha\}\right\|_2^2$ for all the possible α values in a logarithmic scale (Fig. 1). The optimal value of the regularisation parameter α_{opt} corresponds to the coordinates of the L-curve corner. If $\alpha < \alpha_{opt}$ then the solution is close to a solution obtained by the use of the least squares method. Assumption of $\alpha > \alpha_{opt}$ leads to a solution of an equation that differs significantly from the original one.

4.2. Tikhonov regularisation as a filtration method

On the basis of the equation (2), an inverse operator value R_α can be determined according to the formula:

$$[R_\alpha]=\left([A]^T[A]+\alpha[I]\right)^{-1}[A]^T[U]^T \qquad (7)$$

so:

$$[R_\alpha]=\left([V][\Sigma]^T[U]^T[U][\Sigma][V]^T+\alpha[V][I][V]^T\right)^{-1}[V][\Sigma]^T \qquad (8)$$

therefore:

$$[R_\alpha]=[V]\left([\Sigma]^T[\Sigma]+\alpha[I]\right)^{-1}[\Sigma]^T[U]^T \qquad (10)$$

or:

$$[R_\alpha]=[V]\cdot diag\left(\frac{\sigma_i^2}{\sigma_i^2+\alpha}\cdot\frac{1}{\sigma_i}\right)[U]^T \qquad (11)$$

Expression: $w_\alpha(s_i^2)=\dfrac{\sigma_i^2}{\sigma_i^2+\alpha}$ *for* $[L]=[I]_n$ is called a Tikhonov filter function. If $\alpha \to 0$ then $w_\alpha(\sigma_i^2)\to 1$ so $[R_\alpha]\to [V]\cdot diag\left((s_i)^{-1}\right)[U]^T$.

The Tikhonov filter function performance consists in filtering out small singular values ($\sigma_i<\alpha$).

4.3. Application of Tikhonov regularisation to noise reduction

Seven-degree-of-freedom discrete system was considered (Fig. 3).

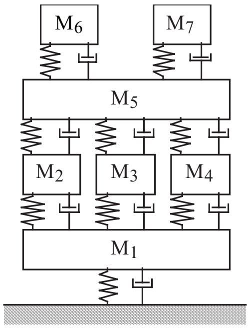

Fig.3. Scheme of a considered system

Physical system parameters are as follows: $M_1 = 5$ [kg], $M_2 = M_3 = M_4 = 1$ [kg], $M_5 = 4$, $M_6 = M_7 = 2$ [kg], $c_{01} = 12$ [Ns/m], $c_{12} = c_{13} = c_{14} = c_{25} = c_{35} = c_{45} = 5$ [Ns/m], $c_{56} = c_{57} = 9$ [Ns/m], $k_{01} = 80000$ [N/m], $k_{12} = k_{13} = k_{14} = k_{35} = k_{45} = 15000$ [N/m], $k_{25} = 14800$ [N/m], $k_{56} = k_{57} = 28000$ [N/m]. The following notation was assumed: k_{ij}, c_{ij} – values of stiffness and damping between masses M_i and M_j.
Dynamic system motion equation has the for of:

$$[M]\cdot\{\ddot{x}\}+[C]\cdot\{\dot{x}\}+[K]\cdot\{x\}=\{f\} \qquad (9)$$

where: [M], [C], [K]: mass, damping and stiffness matrixes.
For a system modal model determined analytically the transfer functions were estimated. On this basis the matrix of transfer functions [H(s)] was formed:

$$[H(s)] = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \dots & H_{1n}(s) \\ H_{21}(s) & H_{22}(s) & \dots & H_{2n}(s) \\ \vdots & \vdots & \dots & \vdots \\ H_{n1}(s) & H_{2n}(s) & \dots & H_{nn}(s) \end{bmatrix} \qquad (10)$$

where: $s = j\omega$: Laplace variable, $n = 1, \ldots, 7$.
Transfer functions determined with respect to the mass M_3 ($\{H_{31}\}, \ldots, \{H_{37}\}$) were burdened with an additional zero-mean random noise of values from the range of $\pm$ 10 % maximal amplitude value of transfer function determined analytically. Matrix [H(s)] of noisy elements $\{H_{31\ sz}\}, \ldots, \{H_{37\ sz}\}$ was marked as $[H_{sz}]$.
Noisy system response $\{X_{sz}\}$ was determined on the basis of the equation [4]:

$$[H_{sz}]_{i\times j}\cdot\{F_{sz}\}_{i\times 1}=\{X_{sz}\}_{i\times 1} \qquad (11)$$

under assumption that the force $\{F_{sz}\}$ applied to the considered system has the form of: $\{F_{sz}\}_{7\times 1}=[1+j\cdot 1,\ldots,1+j\cdot 1]^{T}$.
Taking into account (11), by the use of the Tikhonov regularisation method, the $\{F_{reg}\}$ force value was determined. On the basis of the equations:

$$[H_{sz}]\cdot\{F_{reg}\}=\{X_{reg}\} \qquad (12)$$

the matrix of transfer functions $[H_{reg}]$, obtained as a result of $\{F_{sz}\}$ vector regularisation, was computed:

$$[H_{reg}]=\{X_{reg}\}\cdot\{F_{reg}\}^{-1} \qquad (13)$$

The comparison of an example transfer function obtained as a result of Tikhonov regularisation and a transfer function determined analytically is presented in the Fig. 4a and Fig 4b.

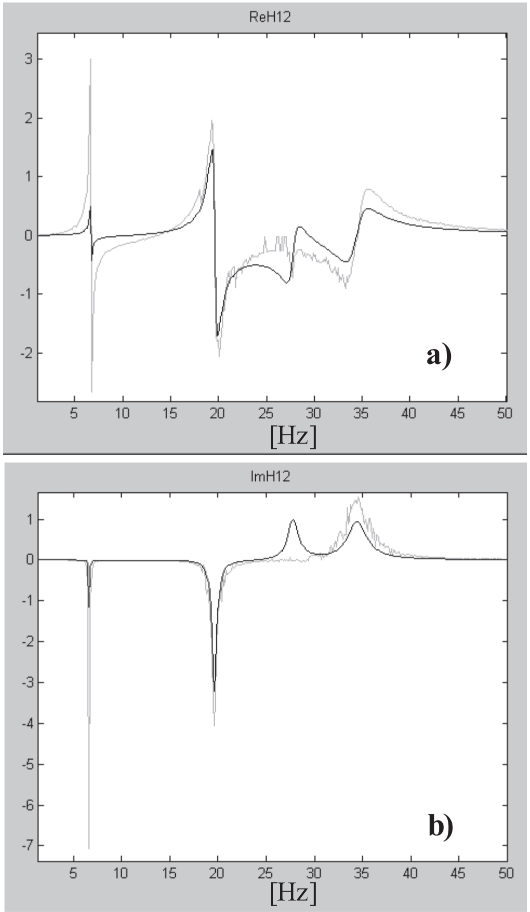

Fig. 4. Real (a) and imaginary (b) parts of a transfer functions without noise (H_{12}: black) and burdened with 10% noise and regularised ($H_{12\ reg}$: grey).

For the purposes of the further analysis, from the $[H_{reg}]$ matrix the transfer functions ($[H_{31\ reg}], \ldots, [H_{37\ reg}]$) were chosen. For such a set of characteristics the estimation of modal parameters was carried out by the use of the ERA method implemented in the VIOMA toolbox.

In the Table 1 there are gathered natural frequencies and modal damping factors corresponding to poles values estimated by the use of the ERA method for the set of chosen transfer functions without noise, burdened with 10% noise, burdened with 10% noise and regularised.
Table 2 and Table 3 contain the percentage relative errors of estimation of natural frequencies e_f and modal damping coefficients e_ζ for unsmoothed characteristics burdened with 10% random noise as well as characteristics burdened with 10% random noise and smoothed by the use of curve smoothing

methods[1] (MA: moving avarage, LOESS: locally weigthed scatter plot smooth, SG: Savitzky - Golay) and Tikhonov regularisation

Table 1

No	noise 0% (ERA)		noise 10% (ERA)		noise 10% regularisation (ERA)	
	f [Hz]	ζ [%]	f [Hz]	ζ [%]	f [Hz]	ζ [%]
1.	6,641	0,59	-	-	6,661	0,64
2.	19,619	1,51	19,645	1,59	19,720	1,41
3.	27,493	2,89	-	-	-	-
4.	27,555	2,89	27,529	2,89	-	-
5.	27,731	2,53	-	-	-	-
6.	34,414	3,50	34,407	3,43	34,421	3,40

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Table 2

No	noise 10% (ERA)		noise 10%, MA, (ERA)		noise 10%, LOESS, (ERA)	
	e_f [%]	E_ζ	e_f [%]	e_ζ [%]	e_f [%]	e_ζ [%]
1.	-	-	0,994	698,00	1,024	418,64
2.	0,132	5,298	0,005	64,238	0,025	33,775
3.	-	-	-	-	0,036	9,688
4.	0,094	0,000	0,036	12,802	0,058	6,574
5.	-	-	-	-	-	-
6.	0,020	2,000	0,017	8,571	0,020	2,857

Table 3

No	noise 10% (ERA)		noise 10%, SG, (ERA)		noise 10%, regularisation, (ERA)	
	e_f [%]	E_ζ	e_f [%]	e_ζ [%]	e_f [%]	e_ζ [%]
1.	-	-	0,994	698,00	0,300	8,475
2.	0,132	5,298	0,005	64,238	0,515	6,622
3.	-	-	0,236	12,803	-	-
4.	0,094	0,000	0,127	25,605	-	-
5.	-	-	-	-	-	-
6.	0,020	2,000	0,017	8,571	0,020	2,857

In the considered case percentage relative errors e_f for the Tikhonov regularisation method are comparable to the errors determined for the other curve smoothing (MA, LOESS, SG) methods. Percentage relative errors e_ζ are the smallest for the Tikhonov regularisation method in the whole estimation band; the most noticeable differences are observed for the low frequency band.
The Table 4 contains MAC coefficients for the mode shapes corresponding to the system poles from the Table 1. The MAC values for the mode shapes Ψ_1, Ψ_2, Ψ_6 estimated for the noisy characteristics smoothed by
the use of the MA, LOESS, SG and Tikhonova regularisation methods approach unity while for the Ψ_3, Ψ_5 are low.

[1] The most popular application of curve smoothing methods is a data noise reduction.

Table 4

No	noise 10%	noise 10%, MA	noise 10%, LOES	noise 10% SG	noise 10% regulari sation
1.	-	0,9936	0,9910	0,9945	0,9477
2.	0,9949	0,9962	0,9965	0,9958	0,8331
3.	-	-	0,2593	0,2681	-
4.	0,7431	0,7403	0,7348	0,7384	-
5.	-	-	-	-	-
6.	0,9989	0,9990	0,9989	0,9984	0,8506

4.4. Conclusions

Application of the Tikhonova regularization to the noisy transfer functions resulted in improvement of estimated poles quality. Percentage relative errors of estimated natural frequencies and modal damping coefficients are significantly lower (especially in the low frequency band) than in case of noise reduction by the use of curve smoothing methods.

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