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The Kummer confluent hypergeometric function and some of its applications in the theory of azimuthally magnetized circular ferrite waveguides

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Abstract— Examples of the application of the confluent hypergeometric functions in miscellaneous areas of the theoretical physics are presented. It is suggested these functions to be utilized as a universal means for solution of a large number of problems, leading to: cylindrical, incomplete gamma, Coulomb wave, Airy, Kelvin, Bateman, Weber's parabolic cylinder, logarithmic-integral and exponential integral functions, generalized Laguerre, Poisson-Charlier and Hermit polynomials, integral sine and cosine, Fresnel and probability integrals, etc. (whose complete list is given), which are their special cases. The employment of such an approach would permit to develop general methods for integration of these tasks, to generalize results of different directions of physics and to find the common features of various phenomena, governed by equations, pertaining to the same family. Emphasis is placed here on the use of the Kummer function in the field of microwaves: the cases of normal and slow rotationally symmetric *TE* modes propagation in the azimuthally magnetized circular ferrite waveguide are considered. Lemmas on the properties of the argument, real and imaginary parts, and positive purely imaginary (real) zeros of the function mentioned in the complex (real) domain, of importance in the solution of boundary-value problem stated for normal (slow) waves, are substantiated analytically or numerically. A theorem for the identity of positive purely imaginary and real zeros of the complex respectively real Kummer function for certain parameters, is proved numerically. Tables and graphs support the results established. The terms for wave transmission are obtained as four bilaterally open intervals of variation of the quantities, specifying the fields. It turns out that the normal (slow) modes may exist in one (two) region(s). The theoretically predicted phase curves for the first waves of the two *TE* sets examined show that the structure explored is suitable for ferrite control components design.

Keywords— microwave propagation in anisotropic media, microwave guides and components, ferrite phase shifters, switches and isolators, eigenvalue problems, function-theoretic and computational methods in electromagnetic theory, theoretical and numerical analysis of special functions.

1. Introduction

The Kummer confluent hypergeometric function (CHF) belongs to an important class of special functions of the mathematical physics [1–19] with a large number of applications in different branches of the quantum (wave) mechanics [2, 5–7, 9, 10, 12, 17, 20, 21], atomic physics [2, 5, 22, 23], quantum theory [23], nuclear physics [23], quantum electronics [24, 25], elasticity theory [2, 5, 7, 9, 26], acoustics [5, 10, 27, 28], theory of oscillating strings [2, 5, 29], hydrodynamics [5, 10, 30], random walk theory [2, 7], optics [31], wave theory [2, 7], fiber optics [32–34], electromagnetic field theory [5, 7, 35, 36], plasma physics [37–39], the theory of probability and the mathematical statistics [5, 7, 10, 13, 40], the pure [5, 41–43] and applied mathematics [44]. In the microwave physics and in particular in the theory of waveguides, such examples are the problems for rotationally symmetric wave propagation in closed and opened circular guiding structures, containing: radially inhomogeneous isotropic dielectric [45–48] or azimuthally magnetized radially stratified anisotropic media (e.g., ferrite or semiconductor) [48–74]. The possibility to obtain signal phase shifting at microwaves makes the geometries of the second type of filling attractive for the development of nonreciprocal devices for this frequency band and is the reason for their extensive study [48–88].

In this paper some properties of the complex and real Kummer CHF and its positive purely imaginary, respectively real zeros are investigated, which are employed in the analysis of normal and slow rotationally symmetric TE modes in the simplest canonical structure of the aforesaid family of anisotropic transmission lines: the circular waveguide, entirely filled with ferrite. Obtained are the propagation conditions and phase characteristics in both cases, too. It is found that there is one (there are two) area(s) of normal (slow) wave transmission, available for both signs (only for the negative sign) of magnetization. The potentialities of the configuration as phaser, switch or isolator are discussed. Symbols with (without) hats "^" stand for quantities, relevant to the slow (normal) TE modes, respectively to the real (complex) Kummer functions.

Besides, the idea is also expressed to replace in the applications the special cases of the CHFs (that are enumerated) by the functions themselves (to replace the multitudinous schemes, utilized at present by a more universal technique) as much as possible. In this way lots of the common traits of different processes which usually remain hidden, owing to the usage of a rather diverse mathematics, would come into sight.

2. Confluent hypergeometric functions

2.1. Basic concepts

Confluent hypergeometric are called four functions: the Kummer and the connected with it Tricomi function $\Phi(a,c;x)$ and $\Psi(a,c;x)$, respectively, and the Whittaker first, and second ones $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$ [10]. The functions $\Phi(a,c;x)$ and $\Psi(a,c;x)$ are solutions of the confluent hypergeometric equation (CHE), written in the standard form of Kummer [1–14, 16–19, 44, 54, 55, 57–59, 61, 69, 72], whereas $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$ – of the same equation, presented in its modified form, suggested by Whittaker [3, 5, 8, 10, 11, 13, 15–17, 19, 44, 55]. The quantities a and c (κ and μ) are called parameters and x – variable [3]. The CHFs except the Kummer one are multiple-valued for which the zero is a branch point. Their main branch is taken in the complex x – plane with a cut along the negative real axis. Both $\Phi(a,c;x)$ and $M_{\kappa,\mu}(x)$ are regular at zero, whereas $\Psi(a,c;x)$ and $W_{\kappa,\mu}(x)$ tend to infinity for $x \to 0$ [1–19, 55, 57–59, 61, 69, 72]. The greater symmetry with respect to the parameters observed in the formulae, involving Whittaker functions [5, 15], as well as the symmetry in the functions themselves (in their values) [55], is the reason for discussing them in parallel with the Kummer and Tricomi ones. In our opinion however, though not symmetrical, the couple $\Phi(a,c;x)$ – $\Psi(a,c;x)$ is to be preferred in the applications in view of the simpler character of power series, determining them. In addition to above definition, due to L. J. Slater [10], worth mentioning also is the one, given by Tricomi who ascertains that CHF is called any solution of CHE, considered in whichever of its forms [3]. Accordingly, such are for example the $\Phi^*(a,c;x)$, $\mathscr{M}_{\kappa,\mu}(x)$ and $N_{\kappa,\mu}(x)$ functions, too, introduced by Tricomi [2, 3, 7, 9, 61], Buchholz [5] and Erdélyi [10], respectively. Beside the notations, accepted here following F. G. Tricomi [2–4, 7, 9] and our previous works [54, 55, 57–61, 63, 64, 66–74], the symbols $M(a,b,x)$, ${}_1F_1[a;b;x]$, $\overset{\infty}{u}(a,b,x)$, and $F(\alpha,\beta,x)$ are employed also in literature instead of $\Phi(a,c;x)$, the symbols $U(a,b,x)$, $\overset{\infty}{v}(a,b,x)$ and $G(a,b,x)$ – instead of $\Psi(a,c;x)$, and the ones $\sqrt{2x/\pi}\, m_{\kappa}^{(2\rho)}(x)$ and $\sqrt{2x/\pi}\, w_{\kappa}^{(2\rho)}(x)$ – instead of $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$, respectively [1, 5, 10, 12, 13]. The term "confluent" in the name of the functions is used, since the Kummer one might be deduced from the Gauss hypergeometric function ${}_2F_1(a,b;c;x)$ through a limiting process, leading to a confluence of two of its three regular singularities (1 and ∞) into an irregular one (the point ∞) [3, 5, 10]. (The regular singularity 0 remains unchanged.) The word "hypergeometric" is applied, as the expressions for the functions can be obtained by adding factors to the terms of the infinite geometric progression [10].

2.2. Special cases

A lot of special functions can be regarded as special cases of CHFs, or combinations of them:

– the ordinary and modified cylindrical and spherical Bessel functions: $J_\nu(x)$, $I_\nu(x)$, $\sqrt{\pi/(2x)}J_{n+1/2}(x)$ or $\sqrt{\pi/(2x)}J_{-n-1/2}(x)$ and $\sqrt{\pi/(2x)}I_{n+1/2}(x)$, respectively [1–3, 7, 9, 10, 12, 13, 15, 16];

– the Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ [1, 2, 7, 12, 13, 16];

– the Neumann function $N_\nu(x)$ [3, 7];

– the cylindrical and spherical McDonald functions $K_\nu(x)$ and $\sqrt{\pi/(2x)}K_{n+1/2}(x)$ [7, 13, 15, 16];

– the Coulomb wave functions: the two pairs $P_L(a,x)$ and $Q_L(a,x)$, and $U_L(a,x)$ and $V_L(a,x)$, considered by Curtis [17], the couples $G_L(\sigma)$ and $H_L(\sigma)$, defined by Hartree [17], and $U(\alpha,\gamma,Z)$ and $V(\alpha,\gamma,Z)$, introduced by Jeffreys and Jeffreys [17] and the most preferable in the applications standard pair $F_L(\eta,\rho)$ and $G_L(\eta,\rho)$, discussed by Abramowitz and Stegun [5, 10, 13, 17];

– the function $H(m,n,x)$, named Coulomb wave function and function of the paraboloid of revolution by Tricomi [2, 7, 9] or confluent hypergeometric function by Miller [13];

– the Laguerre functions $L_\nu^{(\mu)}(x)$ and $U_\nu^{(\mu)}(x)$ [3, 5, 10, 16], denoted also as $S_\nu^{\mu}(x)$ and $V_\nu^{\mu}(x)$ by Mirimanov [5, 10, 35];

– the Airy functions $Ai(x)$ [13, 16, 44] and $Bi(x)$ [13, 16];

– the incomplete $\gamma(a,x)$, the complementary $\Gamma(a,x)$, the modified $\gamma^*(a,x)$ and the fourth incomplete $\gamma_1(a,x)$ gamma functions [1–3, 7, 9, 10, 13, 15, 16, 44], as well as the derivative of them $g(a,x)$, $g_1(a,x)$, $G(a,x)$ and $k(a,x)$ ones, treated by Tricomi [2, 7, 9];

– the Kelvin (Thomson) functions $bei_n(x)$, $ber_n(x)$, $kei_n(x)$, $ker_n(x)$ [13, 16], $hei_n(x)$ and $her_n(x)$ [3, 11], met also like $bei_n x$, $ber_n x$, etc. [13];

– the Bateman function $k_\nu(x)$ [3, 5, 7, 10, 13, 16];

– the Weber's parabolic cylinder functions $D_\nu(x)$ in the Whittaker's notation [2, 3, 5, 7, 9, 10, 13, 15, 16, 44], $E_\nu^{(0)}(x)$ and $E_\nu^{(1)}(x)$ in the Buchholz's one [5, 7, 10, 13, 16], $D_\nu^+(x)$ and $D_\nu^-(x)$, proposed by Tricomi [7], $U(a,x)$, $V(a,x)$ and $W(a,x)$ in the Miller form [13], or $\delta(\xi,\nu)$ and $\rho(\xi,\nu)$ in the symbols by Magnus [5, 10] and $\varphi_n(x)$ and $\Psi_n(x)$, suggested by Janke, Emde and Lösch [11];

– the Cunningham function $\omega_{m,n}(x)$ [5, 10, 13], known as Pearson-Cunningham function, too [15];

– the Heatly Toronto function $T(m,n,r)$ [3, 5, 10, 13, 16];

– the Meixner's functions $F_1(\alpha,\beta,x)$ [3, 5, 10, 16] and $F_2(\alpha,\beta,x)$ [10];

– the MacRobert's function $E(\alpha,\beta::x)$ [3, 5, 16];
– the Erdélyi function ${}_2F_0(\alpha,\beta;x)$ [3, 10, 16];
– the Poiseuille functions $pe(r,w)$ and $qe(r,w)$ [10];
– the Krupp functions ${}_1R(v,l;x)$ and ${}_2R(v,l;x)$ [5, 10];
– the Schlömilch function $S(v,x)$ [5, 15];
– the Chappell function $C(x,k)$ [5, 10];
– the logarithmic-integral function $li(x)$ or lix [1, 3, 7, 9, 10, 12, 13, 15, 16];
– the exponential integral functions Eix or $Ei(x)$ and $E_1(x)$ [3, 7, 9, 10, 13, 16], the generalized exponential integral function $E_n(x)$ [13, 16], marked also as $\mathscr{E}_n(x)$ [16] and the modified exponential integral one $Ein(x)$, used by Tricomi [2, 7, 13];
– the error $erfx$ or $Erf(x)$ and $Erfi(x)$, and complementary error $erfcx$ or $Erfc(x)$ functions (the error and probability integrals) [2, 3, 7, 9, 10, 13, 15, 44], as well as the ones $\Phi(x)$ and $F(x)$ [1, 5, 11–13], $\phi(x)$ and $L(x)$ [10], $\Theta(x)$, $H(x)$ and $\alpha(x)$ [3, 11], connected with them, the multiple probability integral $i^n\ erfcx$ [13] and the Hh – probability function $Hh_n(x)$ [13];
– the normal (Gauss) $P(x)$ and $Z(x)$ [13], and the χ^2-distribution $P(\chi^2|v)$ and $Q(\chi^2|v)$ functions [40], and the F-distribution $P(F|v_1,v_2)$ one [13];
– the Lagrange-Abel function $\phi_m(x)$ [15];
– some elementary (exponential e^x, power x^n, circular $\sin x$ and hyperbolic shx) functions [1, 3, 7, 13, 16];
– the reduced to $n+1$th degree exponential series $e_n(x)$ [7, 9];
– the Laguerre and generalized Laguerre polynomials $L_n(x)$ and $L_n^{(\alpha)}(x)$ [1–3, 7, 9, 10, 12, 13, 16, 44];
– the Sonine polynomials $T_\mu^{(n)}(x)$ [5, 15];
– the Poisson-Charlier polynomials $\rho_n(v,x)$ [10, 13] or $p_n(x)$ in the Tricomi's notation [3];
– the Hermit and modified Hermit polynomials $He_n(x)$ and $H_n(x)$ [1–3, 7, 10, 12, 13, 16, 44];
– some polynomials (in general incomplete) in $1/x$ of nth degree [7];
– the integral sine $Si(x)$ and cosine $Ci(x)$ [1–3, 5, 10, 12, 13, 15, 16]; and the modified cosine $Cin(x)$, employed by Tricomi [2, 7, 13];
– the Fresnel integrals $C(x)$ and $S(x)$ [3, 7, 10, 12, 13, 16], the related to them $C^*(x)$ and $S^*(x)$, and the generalized Fresnel ones $C(\alpha,x)$ and $S(\alpha,x)$ [2, 7].

2.3. Examples of application

The CHFs play an exceptional role in many branches of physics and mathematics. Several examples of their applications are:

– the solution of Schrödinger equation for charged particle motion (e.g., electron motion) in Coulombian field in the quantum mechanics, atomic physics and quantum theory [2, 5–7, 9, 10, 12, 17, 20–23];
– the energy spectrum specification of the isotropic (spherically symmetric) harmonic oscillator in nuclear physics and other related areas [5, 12, 23];
– the quantum mechanical treatment of the operation of the masers and lasers [24, 25];
– the elasticity problem for the flexion of circular or annular plates of lenticular form (resembling to a concave or convex lens), resting on, or rabbeted along its contour, subjected to a normal load whose value at certain point depends on its radial elongation from the center of the plate [2, 5, 7, 9, 26];
– the theory of the reflection of sound waves by a paraboloid [5, 10, 27];
– the consideration of sound waves propagation in parabolic horn, excited by a point source in its focus, and in the space between two co-focal paraboloids of revolution and the construction of the three-dimensional Green function for the homogeneous boundary-value problem of the first kind (Dirichlet problem) and of the second one (Neumann problem) for the wave equation in both cases [5, 28];
– the inquiry of the natural oscillations of a tight stretched string whose mass is distributed symmetrically with respect to its middle, following a parabolic law [5, 29];
– the investigation of a heat generation in a laminary Poiseuille flow through (in a viscous incompressible liquid, flowing through) a thin cylindrical capillary tube of circular cross-section [5, 10, 30];
– the determination of the length of the resultant of a large number of accidentally directed vectors (a special case, connected with the problems of random walk) [2, 7];
– the task for cylindrical-parabolic mirrors [31];
– the description of sea waves motion against a sheer coast [2, 7];
– the analysis of guided modes along a cladded optical fiber of parabolic-index core and homogeneous cladding [32–34];
– the portrayal of electromagnetic waves transmission in parabolic pipes [5];
– the study of the reflection of electromagnetic waves by a parabolic cylinder [2, 5, 7];
– the solution of the diffraction problem for a plane and a spherical electromagnetic wave in a paraboloid of revolution of infinite dinemsions [5, 35];

- the exploration of radiation electromagnetic field in a hollow paraboloid of revolution, launched by an axially oriented electric or magnetic dipole, placed at or before its focus, and between two co-focal paraboloids [5];
- the electrodynamic characterization of the field in an excited by a loop cavity resonator, consisting of two co-focal caps of the form of paraboloids of revolution [5];
- the finding of the normal (Gauss), the χ^2- and the F-distribution for arbitrary quantities in the theory of probability and mathematical statistics [13, 40];
- the development of a mathematical model of the electrical oscillations in a free ending wire [5];
- the assessment of the noise voltages transfer over a linear rectifier [5];
- the explanation of radiation of magnetized dipole in a stratified medium of spherical symmetry (in a globular layered atmosphere) [5];
- the case of electromagnetic waves in plasma with electron density changing linearly along one of the co-ordinate axes, if an infinitely large constant magnetic field is applied along the latter [37];
- the problem for electromagnetic waves in an inhomogeneous plasma whose collision frequency is a constant and the electron density varies in one direction only as a second-degree polynomial of the last-mentioned (or following a parabolic profile) [38];
- the examination of the radiation field from a uniform magnetic ring current around a cylindrical body of infinite length covered by a plasma sheath in the presence of a uniform azimuthal static magnetic field which is of practical application to improve radio communications during the blackout period in the re-entry of a conical space vehicle in the earth's atmosphere at hypersonic speed [39];
- the Tricomi euristic approximate evaluation of the distribution of the positive integers which can be presented as sums of two *k*th powers of possible value in the theory of probability [7];
- the finding of the normal (Gauss) and χ^2 – and F – distribution for arbitrary quantities in the theory of probability and mathematical statistics [13, 40];
- the series expansion of an arbitrary function in terms of eigenfunctions, of significance in the theory of hydrogen atom to describe the point (discrete) and continuous energy spectrum [5, 41];
- some continued fractions expressions of analytic functions in the complex plane, employable in the computational methods [13, 42];
- the realization of irreducible (simple) representations of a group of third order triangular matrixes, in which integral operators whose kernels are written through Whittaker functions, correspond to certain of its elements [43];
- the inspection of TE_{0n} and TM_{0n} modes, sustained in radially inhomogeneous circular dielectric waveguides (plasma columns or optical fibers) whose permittivity alters in radial direction following certain profiles [45–48];
- the theory of normal and slow surface TM_{0n} waves in the azimuthally magnetized millimeter-wave semiconductor (solid-plasma) coaxial waveguides, using *n*-type InSb and GaAs cooled to 77K as a plasma material [49, 53, 56, 62, 65];
- the problems for normal and slow TE_{0n} modes in the azimuthally-magnetized ferrite and ferrite-dielectric circular and co-axial waveguides and for slow waves, propagating along cylindrical helices, closely wound around (or surrounded by) an azimuthally magnetized ferrite rod (toroid) [49–52, 54, 55, 57–61, 63, 64, 66–74];
- the study of microwave radiation from a magnetic dipole in an azimuthally magnetized ferrite cylinder [89] which may also be explored by means of the functions considered.

3. The confluent hypergeometric functions – a universal means for solution of problems of mathematical physics

The above analysis shows that: a lot of tasks from different areas of mathematical physics lead to various representations of CHFs and a large number of functions are special cases of the latter and can be expressed in terms of them. In view of this one might expect to meet the CHFs throughout the literature. In fact, as Lauwerier wrote, "they are only sparingly used" [30]. Even one of the problems from the class examined was categorized as "not a particularly fortunate one" in the words of Suhl and Walker [49]. An attempt to substantiate these inferences is the following assertion (standing nowadays in plenty of fields): "The reason may be that these functions are still too little known, and are therefore evaded as much as possible." [30].

Indeed, the CHFs are more complicated than many other special functions, since they possess two parameters and an independent variable. The lack of numerical tables, or the insufficient tabulation of the functional values and their zeros were a grave obstacle in their applications [30, 49, 75, 80]. Serious computational predicaments arise, if the parameters and variable get large and especially, provided they are complex. The relations between these three quantities also influence the speed of convergence of power series, determining the functions. Due to this, coming upon them, some authors gave only formal analytical results [2, 5, 7, 9, 12, 21, 23, 24, 27, 30, 36–39, 49, 51], whereas others tried to avoid them through:

- reducing the CHFs to their special cases (if possible) [5, 10, 12];
- defining new functions which replace them [75, 79, 80] or harnessing such ones [83, 89];
- elaborating various numerical methods [48, 76, 82, 86, 87].

In our opinion the usage of so many very diverse artificially devised approaches hampers tracing out the connections among the different phenomena explored (which obviously exist, since the latter could be described by the same mathematical language), and impedes the establishment of their common characteristics. It is our conviction that in spite of the drawbacks pointed out, or the difficulties, appearing as a result of their complexity, the CHFs have indisputable advantages: generality and well developed theory together with valuable properties, such as for example symmetry in case of Whittaker functions. Therefore, a way out of this complicated situation, is to find means to overcome the computational challenges, instead of inventing contrivances to obviate the CHFs.

In essence the employment of the special cases, debated in Subsection 2.2, has a similar effect on the process of investigation of the phenomena and their properties, as the just discussed one, when the CHFs have been excluded from the solutions. Utilizing such a great number of functions entails as well a fragmentation of the analysis methods of corresponding tasks. However, unlike before, this state of affairs has sprung up in a natural way, when different problems have been attacked by different schemes.

As a set-off to that, it is suggested to replace the functions in question (the special cases) everywhere, where they attend by the having more universal character CHFs. To this end, the following statement is formulated:

Statement for universality**: The confluent hypergeometric functions, considered in any of their forms, could be used as a universal means instead of any of the functions, being their special cases and the related to them, such as: the cylindrical, incomplete gamma, Coulomb wave, Weber's parabolic cylinder functions, etc. (whose complete list is given above), in the tasks in which they are met.**

Corrolary: Moving from a fragmentation to a generalization would permit:

- to solve enormous number of problems by the same universal mathematical technique;
- to develop general methods for their solution;
- to generalize results of different branches of physics;
- to find common features in different phenomena, governed by equations from the same family.

An undoubted benefit could be derived even from the partial realization of the programme proposed (when the computational hardships are surmountable).

4. Kummer confluent hypergeometric function

4.1. Definition

The Kummer CHF is defined by the absolutely convergent infinite power series [1–14, 16–19, 54, 55, 57–59, 61, 69, 72]:

$$\Phi(a,c;x) = \sum_0^{\infty} \frac{(a)_\nu}{(c)_\nu} \frac{x^\nu}{\nu!}. \tag{1}$$

It is analytic, regular at zero entire single-valued transcendental function of all a, c, x, (real or complex) except $c = 0, -1, -2, -3, \ldots$, for which it has simple poles. $\Phi(a,c;x)$ is a notation, introduced by Humbert, $(\lambda)_\nu = \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+\nu-1) = \Gamma(\lambda+\nu)/\Gamma(\lambda)$, $(\lambda)_0 = 1$, $(1)_\nu = \nu!$, where λ stands for any number (real or complex) and ν for any positive integer or zero, is the Pochhammer's symbol and $\Gamma(\lambda)$ is the Euler gamma function. The series (1) is a solution of the Kummer CHE that is a second order ordinary differential equation [1–14, 16–19, 54, 55, 57–59, 61, 69, 72]:

$$x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay = 0, \tag{2}$$

having regular and irregular singularities at 0 and at ∞, respectively.

4.2. Asymptotic expansion

The asymptotic expansion of $\Phi(a,c;x)$ for large values of variable $x = |x|e^{j\varphi}$, $0 < \varphi < \pi$, is [6, 54, 57–59]:

$$\Phi(a,c;x) \approx \frac{\Gamma(c)}{\Gamma(a)}|x|^{a-c}e^{j(a-c)\varphi}e^{|x|e^{j\varphi}} + \frac{\Gamma(c)}{\Gamma(c-a)}|x|^{-a}e^{ja(\pi-\varphi)}. \tag{3}$$

If $x = jz$ ($z = |x|$ – real, positive), i.e., $\varphi = \pi/2$, both terms in the expression are approximately equally large and should be taken into account. Provided x is real, positive ($\varphi = 0$), the first term in formula (3) is considered only, since the second one becomes less than the unavoidable error, inherent to the asymptotic expansions. When x is real, negative ($\varphi = \pi$), the second term is used solely for the same reason [6, 54, 57–59].

5. Some properties of the complex Kummer function

5.1. Properties due to the analytical study

The case a – complex ($a = \mathrm{Re}\,a + \mathrm{j}\mathrm{Im}\,a$), $c = 2\mathrm{Re}\,a$ – positive integer, $\mathrm{Im}\,a = -k$, k – real [$a = c/2 - \mathrm{j}k$, $k = \mathrm{j}(a - c/2)$], $x = \mathrm{j}z$ – positive purely imaginary ($x = \mathrm{Re}\,x + \mathrm{j}\mathrm{Im}\,x$, $\mathrm{Re}\,x = 0$, $\mathrm{Im}\,x = z$, $|x| = z$, z – real, positive, $\varphi = \arg x = \mathrm{Im}\,x/\mathrm{Re}\,x$, $\varphi = \pi/2$), is discussed. Under these assumptions an application of the first Kummer theorem [1–3, 5–7, 9–13, 16] facilitates to prove the statement [57–59]:

Lemma 1: If $c = 2\mathrm{Re}\,a$, $\mathrm{Re}\,x = 0$ ($x = \mathrm{j}z$ – purely imaginary), then

$$\arg\Phi(a, 2\mathrm{Re}\,a; \mathrm{j}z) = z/2\,, \qquad (4)$$

where $\arg\Phi$ stands for the argument of the Kummer function.

In addition, a new modulus-argument representation of the asymptotic expansion (3) is obtained [57, 58]:

$$\Phi(a, 2\mathrm{Re}\,a; \mathrm{j}z) \approx 2F(\cos v)\mathrm{e}^{\mathrm{j}(z/2)} = 2F|\cos v|\mathrm{e}^{\mathrm{j}(z/2+n\pi)}, \quad (5)$$

where $F = [\Gamma(2\mathrm{Re}\,a)/|\Gamma(a)|]\mathrm{e}^{-(\pi/2)\mathrm{Im}\,a}z^{-\mathrm{Re}\,a}$, $v = (z/2) + \mathrm{Im}\,a \ln z - \arg\Gamma(a) - \mathrm{Re}\,a(\pi/2)$ and $n = 1, 2, 3\ldots$ denotes the number of corresponding zero of cosine, $\arg\Gamma(a)$ is the argument of gamma function. An inspection of expression (5) permits to formulate further to Lemma 1.

Lemma 2: If $c = 2\mathrm{Re}\,a$, $\mathrm{Re}\,x = 0$ ($x = \mathrm{j}z$ – positive purely imaginary), the function $\Phi(a, 2\mathrm{Re}\,a; \mathrm{j}z)$ has an infinite number of simple zeros $\zeta^{(c)}_{k,n}$ in z both for $k > 0$ and $k < 0$ ($k = -\mathrm{Im}\,a$, $n = 1, 2, 3\ldots$), at which $\mathrm{Re}\,\Phi = \mathrm{Im}\,\Phi = |\Phi| = 0$ [57, 58, 69].

Lemma 3: If $c = 2\mathrm{Re}\,a$, $\mathrm{Re}\,x = 0$ ($x = \mathrm{j}z$ – positive purely imaginary) and z exceeds the nth zero $\zeta^{(c)}_{k,n}$ of Kummer CHF $\Phi(a, 2\mathrm{Re}\,a; \mathrm{j}z)$ in z ($k = -\mathrm{Im}\,a$, $n = 1, 2, 3\ldots$) then its argument

$$\arg\Phi(a, 2\mathrm{Re}\,a; \mathrm{j}z) = (z/2) + n\pi \qquad (6)$$

is a linear function of z with finite increase by π at each consecutive zero of the function [57, 58, 69].

Lemma 4: If $c = 2\mathrm{Re}\,a$, $\mathrm{Re}\,x = 0$, ($x = \mathrm{j}z$ – positive purely imaginary), then for the real and imaginary parts of Kummer CHF it holds $\mathrm{Re}\,\Phi(a, 2\mathrm{Re}\,a; \mathrm{j}z) = 0$ for $z = (2m+1)\pi$, whereas $\mathrm{Im}\,\Phi(a, 2\mathrm{Re}\,a; \mathrm{j}z) = 0$ for $z = 2m\pi$, $m = 0, 1, 2, 3, \ldots$, irrespective of the value of $\mathrm{Im}\,a$, (k) [57, 58].

Corollary: An infinite decreasing (if $\mathrm{Re}\,a > 0$) or increasing (if $\mathrm{Re}\,a < 0$ and $\mathrm{Re}\,a \neq t/2$, $t = 0, -1, -2, -3, \ldots$) sequence of maxima of $|\Phi(a, 2\mathrm{Re}\,a; \mathrm{j}z)|$ and a sequence of its zeros alternate with each other when z grows in case $c = 2\mathrm{Re}\,a$, $\mathrm{Re}\,x = 0$ ($x = \mathrm{j}z$ – positive purely imaginary) [57].

5.2. Properties due to the numerical study

The statements of Lemmas 1–4 are confirmed by the numerical evaluation of the function $\Phi(1.5 - \mathrm{j}k, 3; \mathrm{j}z)$ made, using series (1). Figure 1 is a plot of the loci curves of Φ in the complex plane for $k = +0.5$, 0 and -0.5 (solid, dotted and dashed lines, respectively), Fig. 2 visualises

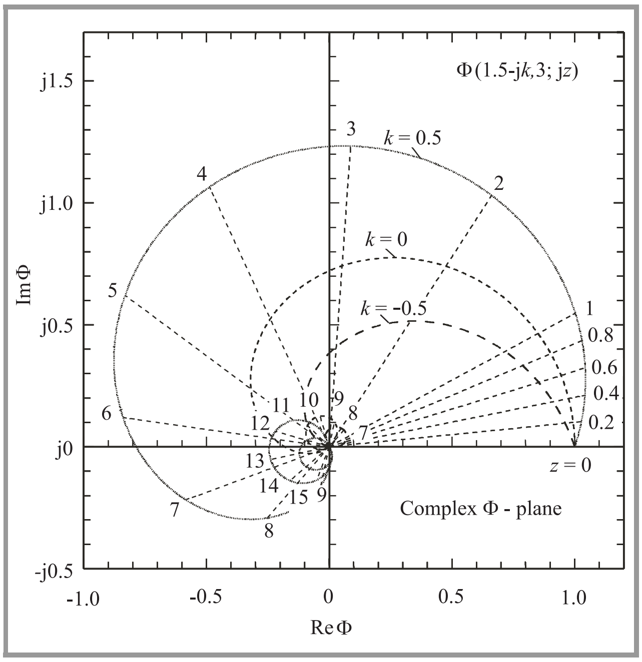

Fig. 1. Loci curves of $\Phi(1.5 - \mathrm{j}k, 3; \mathrm{j}z)$ in the complex plane for $k = +0.5$, 0 and -0.5.

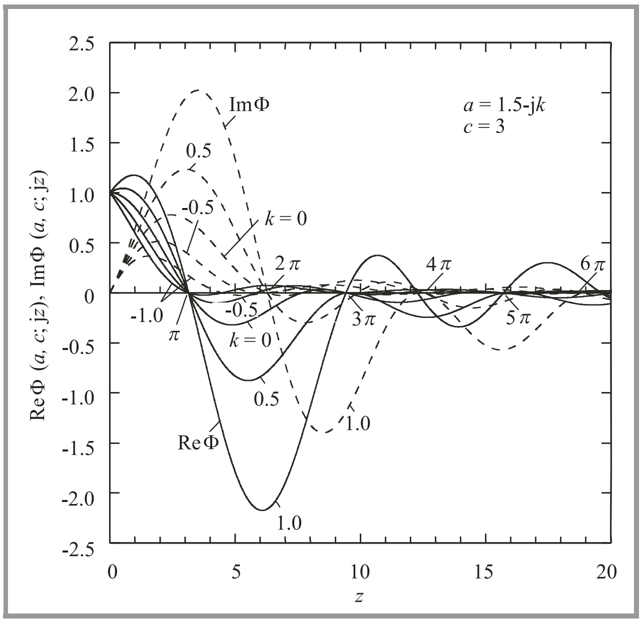

Fig. 2. Real and imaginary parts of Kummer function $\Phi(1.5 - \mathrm{j}k, 3; \mathrm{j}z)$ against z for $k = 0, \pm 0.5$ and ± 1.0.

the variation of $\mathrm{Re}\,\Phi$ (solid lines) and $\mathrm{Im}\,\Phi$ (dashed lines) versus z for $k = 0, \pm 0.5, \pm 1.0$ and Fig. 3 gives the dependence of modulus and argument of Φ on z for $k = +0.5$, 0 and -0.5 (solid, dotted and dashed lines, re-

Table 1
First six positive purely imaginary zeros $\zeta_{k,n}^{(3)}$ of $\Phi(1.5-\mathrm{j}k, 3; \mathrm{j}z)$ for $k = -1.0\,(0.2) + 1.0$

k	$\zeta_{k,1}^{(3)}$	$\zeta_{k,2}^{(3)}$	$\zeta_{k,3}^{(3)}$	$\zeta_{k,4}^{(3)}$	$\zeta_{k,5}^{(3)}$	$\zeta_{k,6}^{(3)}$
−1.0	4.4750 5671	9.5777 9569	15.0744 6601	20.7758 5770	26.6000 3381	32.5053 0790
−0.8	4.9618 8564	10.3259 3914	15.9980 9339	21.8286 7627	27.7540 5190	33.7420 5957
−0.6	5.5218 6556	11.1477 3249	16.9911 7329	22.9469 7930	28.9703 0361	35.0384 2135
−0.4	6.1595 3442	12.0428 8636	18.0516 0729	24.1278 4699	30.2454 3063	36.3907 7149
−0.2	6.8751 0735	13.0069 8966	19.1734 8573	25.3647 3201	31.5725 5798	37.7920 4131
0.0	7.6634 1194	14.0311 7334	20.3469 3627	26.6473 8388	32.9412 6801	39.2317 1702
0.2	8.5142 1018	15.1029 6417	21.5590 7859	27.9628 4223	34.3385 7601	40.6968 5232
0.4	9.4140 5779	16.2082 5362	22.7959 6241	29.2973 5379	35.7509 1422	42.1739 8392
0.6	10.3489 2135	17.3336 0506	24.0447 2652	30.6384 5569	37.1660 9203	43.6511 3385
0.8	11.3063 8822	18.4679 5058	25.2951 0103	31.9763 7998	38.5746 8212	45.1191 1960
1.0	12.2767 8251	19.6032 3531	26.5398 8420	33.3044 4623	39.9703 5445	46.5718 6228

Table 2
First positive purely imaginary zeros $\zeta_{k,1}^{(3)}$ of $\Phi(1.5-\mathrm{j}k, 3; \mathrm{j}z)$ and products $|k|\zeta_{k,1}^{(3)}$ and $|a|\zeta_{k,1}^{(3)}$ for large negative k

k	$\zeta_{k,1}^{(3)}$	$\lvert k\rvert\zeta_{k,1}^{(3)}$	$\lvert a\rvert$	$\lvert a\rvert\zeta_{k,1}^{(3)}$
−10000	0.00065 93654 06232	**6.59365 4**0623	10 000.00011 25000	**6.59365 41**365
−20000	0.00032 96827 04784	**6.59365 4**0956	20 000.00005 62500	**6.59365 41**142
−40000	0.00016 48413 52600	**6.59365 410**40	40 000.00002 81250	**6.59365 410**86
−60000	0.00010 98942 35093	**6.59365 410**55	60 000.00001 87500	**6.59365 410**76
−80000	0.00008 24206 76327	**6.59365 4106**1	80 000.00001 40625	**6.59365 410**72
−100000	0.00006 59365 41063	**6.59365 4106**2	100 000.00001 12500	**6.59365 410**70

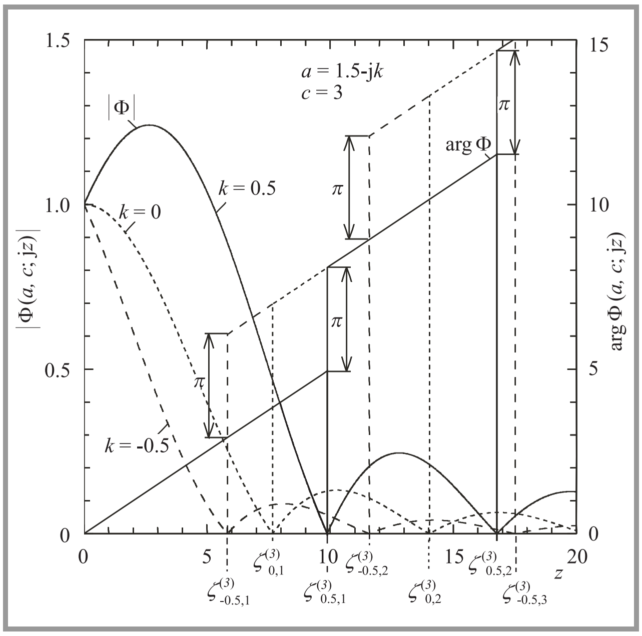

Fig. 3. Modulus and argument of Kummer function $\Phi(1.5-\mathrm{j}k, 3; \mathrm{j}z)$ versus z for $k = +0.5$, 0 and -0.5.

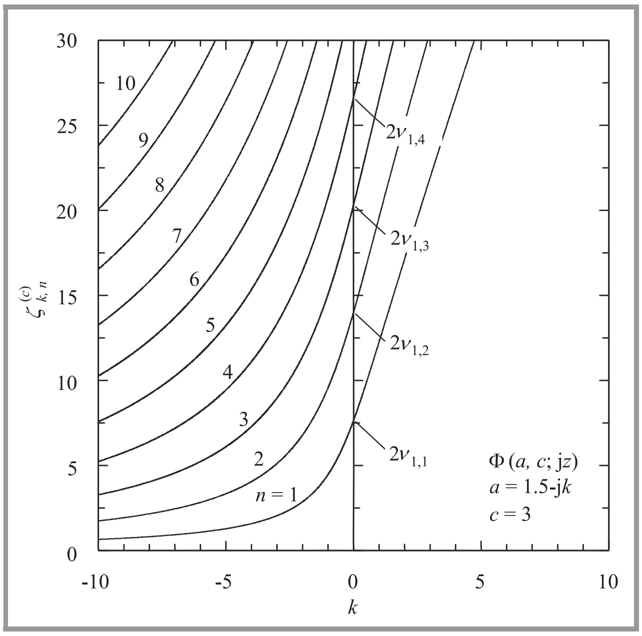

Fig. 4. Distribution of the first ten positive purely imaginary zeros of Kummer CHF $\Phi(1.5-\mathrm{j}k, 3; \mathrm{j}z)$ with k.

spectively). The distribution of the first ten zeros of Φ with k is plotted in Fig. 4. The curves intersect the ordinate axis ($k = 0$) at points $\zeta_{0,n}^{(3)} = 2\nu_{1,n}$ [$\nu_{1,n}$ is the nth zero of Bessel function $J_1(x)$] which could be proved, using the second Kummer theorem [1–4, 7, 9, 10, 13, 16]. Values of $\zeta_{k,n}^{(c)}$ for small and large $|k|$ are listed in Tables 1 and 2. The analysis shows that it is true: $\lim_{k\to-\infty} \zeta_{k,n}^{(c)} = 0$ and $\lim_{k\to+\infty} \zeta_{k,n}^{(c)} = +\infty$. The products $|k|\zeta_{k,n}^{(c)}$ and $|a|\zeta_{k,n}^{(c)}$ are of special interest, if k gets very large negative (see Table 2). It is valid [72]:

Lemma 5: If $\zeta_{k,n}^{(c)}$ is the nth positive purely imaginary zero of Kummer function $\Phi(a, c; x)$ in x ($n = 1, 2, 3\ldots$) provided $a = c/2 - \mathrm{j}k$ – complex, $c = 2\,\mathrm{Re}\,a$ – restricted positive integer, $x = \mathrm{j}z$ – positive purely imaginary, z – real, positive, $k = \mathrm{j}(a - c/2)$ – real, then the infinite sequences of positive real numbers $\{\zeta_{k,n}^{(c)}\}$, $\{|k|\zeta_{k,n}^{(c)}\}$ and $\{|a|\zeta_{k,n}^{(c)}\}$ are convergent for $k \to -\infty$ (c, n – fixed). The limit of the first sequence is zero and the limit of the second and third ones is the same. It equals the finite positive real number L, where $L = L(c, n)$. It holds:

$$\lim_{k\to-\infty} |k|\zeta_{k,n}^{(c)} = L(c, n)\,, \tag{7}$$

$$\lim_{k\to-\infty} |a|\zeta_{k,n}^{(c)} = L(c, n)\,. \tag{8}$$

For any $|k|$ and relevant $|a|$ it is true $|k|\zeta_{k,n}^{(c)} < L(c, n) < |a|\zeta_{k,n}^{(c)}$. In case $k \to +\infty$, $\{\zeta_{k,n}^{(c)}\}$, $\{|k|\zeta_{k,n}^{(c)}\}$, and $\{|a|\zeta_{k,n}^{(c)}\}$ also tend to $+\infty$. Results for complex Φ – function can be found in [55, 57–59, 72], too.

6. Some properties of the real Kummer function

6.1. Properties due to the analytical study by F. G. Tricomi

Tricomi has proved that if $\hat{a}, \hat{c}, \hat{x}$ are real, $\hat{x} > 0$ and $\hat{c} > 0$:

- the Kummer CHF $\Phi(\hat{a}, \hat{c}, \hat{x})$ has real positive zeros only if $\hat{a} < 0$;
- the number of zeros $\hat{l} = abs[\hat{a}]$ is finite, $[\hat{a}]$ is the largest integer less or equal to $\hat{a}$, i.e., $[\hat{a}] \le \hat{a}$;
- at the point $\hat{a} = [\hat{a}] = -\hat{n}$ ($\hat{n} \le \hat{l}$ – a positive integer, $\hat{n} = 1, 2, \ldots, \hat{l}$) a new zero appears [1–3, 7, 9, 10, 44].

6.2. Properties due to the numerical study

The case $\hat{a} = \hat{c}/2 + \hat{k}$ – real, $\hat{c}$ – positive integer, $\hat{k}$ – real ($\hat{k} = \hat{a} - \hat{c}/2$), $\hat{x}$ – real, positive, is treated. Computations of the function $\Phi(1.5 + \hat{k}, 3; \hat{x})$ have been performed, making use of series (1). Figures 5 and 6 represent Φ versus $\hat{x}$ for $\hat{k} > 0$ (solid lines), $\hat{k} = 0$ (dotted curve) and $\hat{k} < 0$ (dashed lines). The monotonous (oscillating) character of curves for $\hat{k} > -1.5$ ($\hat{k} < -1.5$) is in agreement with above analytical results. Values of the real zeros $\hat{\zeta}_{\hat{k},\hat{n}}^{(\hat{c})}$ of the same function are given in Tables 3–5 for different intervals of variation of $\hat{k}$. The distribution of the first eight zeros of Φ against $\hat{k}$ is drawn in Fig. 7. The numerical analysis indicates

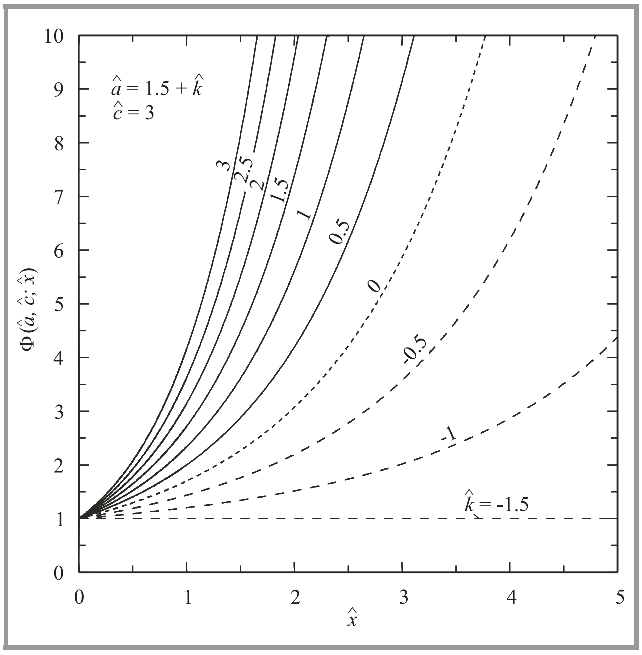

Fig. 5. Kummer function $\Phi(1.5 + \hat{k},\ 3;\ \hat{x})$ against $\hat{x}$ for $\hat{k} = -1.5(0.5)3$.

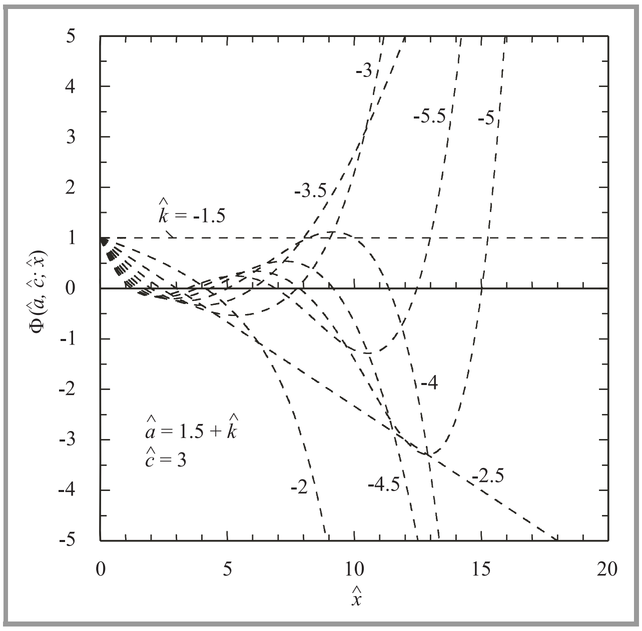

Fig. 6. Kummer function $\Phi(1.5 + \hat{k},\ 3;\ \hat{x})$ versus $\hat{x}$ for $\hat{k} = -5(0.5) - 1.5$.

that it holds: $\lim_{\hat{k}\to-\infty} \hat{\zeta}_{\hat{k},\hat{n}}^{(\hat{c})} = 0$ and $\lim_{\substack{\hat{k} \to -(\hat{n}-1)-\hat{c}/2 \\ \hat{k} < -(\hat{n}-1)-\hat{c}/2}} \hat{\zeta}_{\hat{k},\hat{n}}^{(\hat{c})} = +\infty$.

The products $|\hat{k}|\hat{\zeta}_{\hat{k},\hat{n}}^{(\hat{c})}$ and $|\hat{a}|\hat{\zeta}_{\hat{k},\hat{n}}^{(\hat{c})}$ are of interest, if $\hat{k}$ is very large negative (Table 5). It is true:

Lemma 6: If $\hat{\zeta}_{\hat{k},\hat{n}}^{(\hat{c})}$ is the $\hat{n}$th positive real zero of Kummer function $\Phi(\hat{a}, \hat{c}; \hat{x})$ in $\hat{x}$ ($\hat{n} = 1, 2, \ldots, \hat{l}$, $\hat{l} = abs[\hat{a}]$) provided $\hat{a}, \hat{c}, \hat{x}$ are real, $\hat{c}$ – restricted positive integer and $\hat{k} = \hat{a} - \hat{c}/2$ – real ($\hat{a} = \hat{c}/2 + \hat{k}$), then

Table 3

First six positive real zeros $\hat{\zeta}^{(3)}_{\hat{k},\hat{n}}$ of $\Phi(1.5+\hat{k}, 3; \hat{x})$ for $\hat{k} = -\left[(2\hat{n}+1)/2 + 1.10^{-\hat{s}}\right]$ and $\hat{s} = 2(1)10$

$\hat{s}$	$\hat{\zeta}^{(3)}_{\hat{k}(\hat{s}),1}$	$\hat{\zeta}^{(3)}_{\hat{k}(\hat{s}),2}$	$\hat{\zeta}^{(3)}_{\hat{k}(\hat{s}),3}$	$\hat{\zeta}^{(3)}_{\hat{k}(\hat{s}),4}$	$\hat{\zeta}^{(3)}_{\hat{k}(\hat{s}),5}$	$\hat{\zeta}^{(3)}_{\hat{k}(\hat{s}),6}$
10	32.6943 6952	39.2832 5273	45.2869 3680	50.9779 8646	56.4636 0593	61.8003 1150
9	30.1381 7435	36.6025 7840	42.4984 2791	48.0931 9869	53.4911 8502	58.7470 5165
8	27.5553 2227	33.8846 6661	39.6641 8688	45.1555 0085	50.4595 9370	55.6290 4210
7	24.9390 3482	31.1204 5777	36.7733 9084	42.1526 1734	47.3553 1077	52.4316 6489
6	22.2793 6643	28.2967 9449	33.8104 1524	39.0668 9583	44.1589 4739	49.1340 0339
5	19.5607 1308	25.3932 7731	30.7511 9691	35.8712 6945	40.8408 6789	45.7041 6336
4	16.7561 8418	22.3751 7209	27.5550 4993	32.5201 8558	37.3513 6214	42.0887 8082
3	13.8134 2126	19.1750 2405	24.1432 3161	28.9257 1322	33.5946 6329	38.1852 0648
2	10.6181 4852	15.6405 7545	20.3351 4451	24.8844 5526	29.3480 2383	33.7537 3550

Table 4

First six positive real zeros $\hat{\zeta}^{(3)}_{\hat{k},\hat{n}}$ of $\Phi(1.5+\hat{k}, 3; \hat{x})$ for $\hat{k} = -2(-1)-10$

$\hat{k}$	$\hat{\zeta}^{(3)}_{\hat{k},1}$	$\hat{\zeta}^{(3)}_{\hat{k},2}$	$\hat{\zeta}^{(3)}_{\hat{k},3}$	$\hat{\zeta}^{(3)}_{\hat{k},4}$	$\hat{\zeta}^{(3)}_{\hat{k},5}$	$\hat{\zeta}^{(3)}_{\hat{k},6}$
−2	4.1525 7778					
−3	2.3908 7384	7.7342 0261				
−4	1.7240 3430	4.9963 8913	11.3550 3906			
−5	1.3562 4234	3.8054 2722	7.8425 2881	15.0185 8200		
−6	1.1202 9295	3.0969 9425	6.1880 6299	10.8491 1987	18.7168 8187	
−7	0.9552 6444	2.6191 1978	5.1554 0981	8.7786 0273	13.9709 0761	22.4429 7395
−8	0.8330 6998	2.2725 0326	4.4346 8551	7.4417 8723	11.5221 5873	17.1799 4235
−9	0.7388 2652	2.0086 2555	3.8982 3676	6.4852 2265	9.9005 3270	14.3837 0603
−10	0.6638 7020	1.8005 8410	3.4814 0975	5.7592 2215	8.7176 8903	12.4948 1718

Table 5

First positive real zeros $\hat{\zeta}^{(3)}_{\hat{k},1}$ of $\Phi(1.5+\hat{k}, 3; \hat{x})$ and products $|\hat{k}|\hat{\zeta}^{(3)}_{\hat{k},1}$ and $|\hat{a}|\hat{\zeta}^{(3)}_{\hat{k},1}$ for large negative $\hat{k}$

$\hat{k}$	$\hat{\zeta}^{(3)}_{\hat{k},1}$	$\lvert\hat{k}\rvert\hat{\zeta}^{(3)}_{\hat{k},1}$	$\hat{a}$	$\lvert\hat{a}\rvert\hat{\zeta}^{(3)}_{\hat{k},1}$
−10000	0.00065 93654 15127	**6.59365 41**512	−9998.5	**6.59**266 51031
−20000	0.00032 96827 05895	**6.59365 41**179	−19998.5	**6.593**15 95938
−40000	0.00016 48413 52739	**6.59365 410**95	−39998.5	**6.593**40 68475
−60000	0.00010 98942 35134	**6.59365 410**80	−59998.5	**6.593**48 92666
−80000	0.00008 24206 76343	**6.59365 410**74	−79998.5	**6.593**53 04764
−100000	0.00006 59365 41072	**6.59365 410**72	−99998.5	**6.593**55 52023

the infinite sequences of positive real numbers $\left\{\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\right\}$, $\left\{|\hat{k}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\right\}$ and $\left\{|\hat{a}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\right\}$ are convergent for $\hat{k} \to -\infty$ ($\hat{c}, \hat{n}$ – fixed). The limit of the first sequence is zero and the limit of the second and third ones is the same. It equals the finite positive real number $\hat{L}$, where $\hat{L} = \hat{L}(\hat{c}, \hat{n})$. It is valid:

$$\lim_{\hat{k}\to-\infty} |\hat{k}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}} = \hat{L}(\hat{c}, \hat{n}), \tag{9}$$

$$\lim_{\hat{k}\to-\infty} |\hat{a}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}} = \hat{L}(\hat{c}, \hat{n}). \tag{10}$$

For any $|\hat{k}|$ and corresponding $|\hat{a}|$ it holds $|\hat{a}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}} < \hat{L}(\hat{c}, \hat{n}) < |\hat{k}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}$. If $\hat{k} \to +\infty$, $\left\{\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\right\}$, $\left\{|\hat{k}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\right\}$ and $\left\{|\hat{a}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\right\}$ go to $+\infty$, too.

Lemma 7: Let $\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}$ is the $\hat{n}$th positive real zero of Kummer function $\Phi(\hat{a}, \hat{c}; \hat{x})$ in $\hat{x}$ ($\hat{n} = 1, 2, \ldots, \hat{l}$, $\hat{l} = abs[\hat{a}]$) provided $\hat{a}, \hat{c}, \hat{x}$ are real, $\hat{c}$ – restricted positive integer and $\hat{k} = \hat{a} - \hat{c}/2$ – real ($\hat{a} = \hat{c}/2 + \hat{k}$). If $\hat{k} = -\left[(2\hat{n}+1)/2 + 1.10^{-\hat{s}}\right]$, and $\hat{s} = 1, 2, 3, \ldots$ is a positive integer, then the dif-

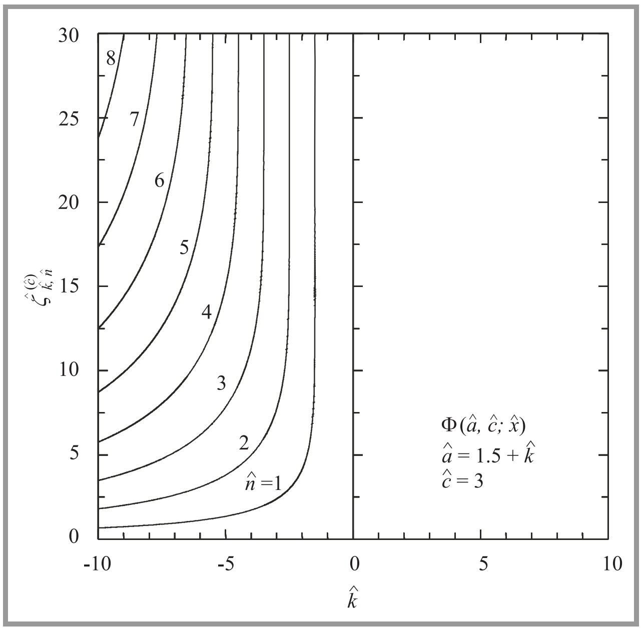

Fig. 7. Distribution of the first eight positive real zeros of Kummer function $\Phi(1.5+\hat{k}, 3; \hat{x})$ with $\hat{k}$.

ferences $\hat{\Delta}_{\hat{s}+1,\hat{s},\hat{n}} = \hat{\zeta}^{(\hat{c})}_{\hat{k}(\hat{s}+1),\hat{n}} - \hat{\zeta}^{(\hat{c})}_{\hat{k}(\hat{s}),\hat{n}}$ and $\hat{\Delta}^2_{\hat{s}+2,\hat{s}+1,\hat{s},\hat{n}} = \hat{\Delta}_{\hat{s}+1,\hat{s},\hat{n}} - \hat{\Delta}_{\hat{s}+2,\hat{s}+1,\hat{n}}$ where $\hat{k}(\hat{s}+1)$ and $\hat{k}(\hat{s})$ are any two neighbouring parameters for certain $\hat{n}$, tend to a finite real positive number and zero, respectively, especially if $\hat{s}$ gets large and $\hat{n}$ is small. Accordingly, the zeros $\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}$, situated close to the points $\hat{k} = -(2\hat{n}+1)/2$ can be computed from the approximate formula:

$$\hat{\zeta}^{(\hat{c})}_{\hat{k}(\hat{s}+2),\hat{n}} \approx \hat{\zeta}^{(\hat{c})}_{\hat{k}(\hat{s}+1),\hat{n}} + \hat{\Delta}_{\hat{s}+1,\hat{s},\hat{n}} = 2\hat{\zeta}^{(\hat{c})}_{\hat{k}(\hat{s}+1),\hat{n}} - \hat{\zeta}^{(\hat{c})}_{\hat{k}(\hat{s}),\hat{n}}. \quad (11)$$

This relation permits to obtain the subsequent zero, if the values of the preceding two are known (Table 4). Results for real Φ – function are available also in [10, 11, 13, 14].

7. A theorem for the identity of zeros of certain Kummer functions

Theorem 1: If $\zeta^{(c)}_{k,n}$ is the nth positive purely imaginary zero of complex Kummer function $\Phi(a,c;x)$ in x ($n = 1,2,3,\ldots$) provided $a = c/2 - \mathrm{j}k$ – complex, $c = 2\mathrm{Re}\,a$ – restricted positive integer, $x = \mathrm{j}z$ – positive purely imaginary, z – real, positive, $k = \mathrm{j}(a - c/2)$ – real, and if $\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}$ is the $\hat{n}$th positive real zero of real Kummer function $\Phi(\hat{a},\,\hat{c};\,\hat{x})$ in $\hat{x}$ ($\hat{n} = 1,2,\ldots,\hat{l}$, $\hat{l} = abs[\hat{a}]$) provided $\hat{a}, \hat{c}, \hat{x}$ are real, $\hat{c}$ – restricted positive integer and $\hat{k} = \hat{a} - \hat{c}/2$ – real ($\hat{a} = \hat{c}/2 + \hat{k}$), then the infinite sequences of positive real numbers $\{\zeta^{(c)}_{k,n}\}$, $\{|k|\zeta^{(c)}_{k,n}\}$ and $\{|a|\zeta^{(c)}_{k,n}\}$ are convergent for $k \to -\infty$ (c, n – fixed), and the infinite sequences of positive real numbers $\{\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$, $\{|\hat{k}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$ and $\{|\hat{a}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$ are convergent for $\hat{k} \to -\infty$ ($\hat{c}, \hat{n}$ – fixed). The limits of $\{\zeta^{(c)}_{k,n}\}$ and $\{\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$ equal zero. The limits of $\{|k|\zeta^{(c)}_{k,n}\}$ and $\{|a|\zeta^{(c)}_{k,n}\}$ coincide. They equal the positive real number L, where $L = L(c,n)$. The same is fulfilled for the limits of $\{|\hat{k}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$ and $\{|\hat{a}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$ which equal the positive real number $\hat{L}$, where $\hat{L} = \hat{L}(\hat{c},\hat{n})$. On condition that $c = \hat{c}$ and $n = \hat{n}$, it is correct:

$$L(c,n) = \hat{L}(\hat{c},\hat{n}). \quad (12)$$

In addition, in case $k = \hat{k}$ – large negative, it is true:

$$\zeta^{(c)}_{k,n} \approx \hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}. \quad (13)$$

It holds $\zeta^{(c)}_{k,n} < \hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}$ and $|\hat{a}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}} < |k|\zeta^{(c)}_{k,n} < L(c,n) < |a|\zeta^{(c)}_{k,n} < |\hat{k}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}$ for any $c = \hat{c}$, $n = \hat{n}$, $|k| = |\hat{k}|$ and $|a| \approx |\hat{a}|$, ($|\hat{a}| < |a|$). The rate of convergence decreases as follows $\{|k|\zeta^{(c)}_{k,n}\}$, $\{|a|\zeta^{(c)}_{k,n}\}$, $\{|\hat{k}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$ and $\{|\hat{a}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$. When $c << |k|$ ($\hat{c} << |\hat{k}|$), the sequences $\{|\hat{a}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$ and $\{|a|\zeta^{(c)}_{k,n}\}$ converge faster. For $c = 3$ and $n = 1,2,\ldots,10$, it is valid $L(c,n) = \hat{L}(\hat{c},\hat{n})$ =6.593654107, 17.71249973, 33.75517722, 54.73004731, 80.6387791, 111.48189218, 147.25958974, 187.9719664, 233.61907045, 284.20092871. Assuming that $k \to +\infty$ ($\hat{k} \to +\infty$), $\{\zeta^{(c)}_{k,n}\}$, $\{|k|\zeta^{(c)}_{k,n}\}$ and $\{|a|\zeta^{(c)}_{k,n}\}$, ($\{\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$, $\{|\hat{k}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$ and $\{|\hat{a}|\hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}\}$) tend to $+\infty$. The proof of Theorem 1 is based on the numerical study of the zeros of Kummer CHF, respectively on Lemmas 5 and 6 (Tables 2 and 5).

8. Azimuthally magnetized circular ferrite waveguide

An infinitely long, homogeneous, perfectly conducting circular waveguide of radius r_0, entirely filled with lossless ferrite, magnetized in azimuthal direction to remanence by an infinitely thin switching wire, is considered. The anisotropic load has a scalar permittivity $\varepsilon = \varepsilon_0\varepsilon_r$ and a Polder permeability tensor $\overleftrightarrow{\mu} = \mu_0[\mu_{ij}]$, $i,j = 1,2,3$ of nonzero components $\mu_{ii} = 1$ and $\mu_{13} = -\mu_{31} = -\mathrm{j}\alpha$, ($\alpha = \gamma M_r/\omega, \gamma$ – gyromagnetic ratio, M_r – remanent magnetization, ω – angular frequency of the wave). The propagation of normal and slow rotationally symmetric TE modes in the structure is examined. The following quantities are used in the study of the fields of first type: β – phase constant of the wave in the guide, $\beta_f = \beta_1\sqrt{\mu_{eff}}$, $\beta_1 = \beta_0\sqrt{\varepsilon_r}$, $\beta_0 = \omega\sqrt{\varepsilon_0\mu_0}$ – natural propagation constants of the unbounded azimuthally magnetized ferrite and dielectric media of relative permittivity ε_r and of free space, respectively, $\mu_{eff} = 1 - \alpha^2$ – effective relative permeability and $\beta_2 = (\beta_f^2 - \beta^2)^{1/2}$ – transverse distribution coefficient. The expressions: $\overline{\beta} = \beta/(\beta_0\sqrt{\varepsilon_r})$, $\overline{\beta}_f = \beta_f/(\beta_0\sqrt{\varepsilon_r})$, $\overline{\beta}_2 = \beta_2/(\beta_0\sqrt{\varepsilon_r})$ and $\overline{r}_0 = \beta_0 r_0\sqrt{\varepsilon_r}$ provide universality of the results.

9. A microwave application of the complex Kummer function

9.1. Propagation problem for normal TE_{0n} modes in an azimuthally magnetized circular ferrite waveguide

The guided TE_{0n} waves in configuration described are normal, if $\bar{\beta}_2 = \left(\bar{\beta}_f^2 - \bar{\beta}^2\right)^{1/2}$ is real $\left(\bar{\beta}_f = \sqrt{\mu_{eff}}\right)$, i.e., $\bar{\beta} < \bar{\beta}_f$, $\left(\bar{\beta} > 0,\ \bar{\beta}_f > 0\right)$. They are governed by the following characteristic equation [54, 55, 57, 59–61, 63, 66, 69, 70, 72–74]:

$$\Phi\left(a, c; x_0\right) = 0\,, \tag{14}$$

where $a = 1.5 - \mathrm{j}k$, $c = 3$, $x_0 = \mathrm{j}z_0$, $k = \alpha\bar{\beta}/\left(2\bar{\beta}_2\right)$, $z_0 = 2\bar{\beta}_2\bar{r}_0$. It holds, provided $\bar{\beta}_2 = \zeta_{k,n}^{(c)}/\left(2\bar{r}_0\right)$ which defines the eigenvalue spectrum of the fields examined.

9.2. Phase characteristics

Using the roots $\zeta_{k,1}^{(3)}$ of Eq. (14) and the relations between barred quantities, the dependence of $\bar{\beta}$ on $\bar{r}_0$ with α as parameter for the normal TE_{01} mode in the ferrite waveguide is computed and plotted in Fig. 8. The solid (dashed) lines, corresponding to positive (negative) magnetization are of infinite (finite) length. Hence, transmission is possible for $\alpha_+ > 0$ $\left(\alpha_- < 0\right)$ in an unlimited from above (restricted from both sides) frequency band. The common starting point of the curves for the same $|\alpha|$ at the horizontal axis depicts the pertinent cutoff frequency $\bar{r}_{0cr} = \left[\zeta_{0,1}^{(3)}/2\right]/\left(1-\alpha^2\right)^{1/2}$. The ends of characteristics for $M_r < 0$ of co-ordinates ($\bar{r}_{0en-}$, $\bar{\beta}_{en-}$) form an envelope (dotted line), labelled with En_{1-}, limiting from the right the area of propagation for negative magnetization. The curve, marked with $\alpha = 0$ (the ferrite degenerates into isotropic dielectric) is infinitely long (transmission takes place in an unlimited from above frequency range). The characteristics for $\alpha_+ > 0$ ($\alpha_- < 0$) are single-valued (double-valued below cut-off, with an inversion point of abscissa $\bar{r}_{0i-}$). The envelope En_{1-} possesses a minimum $\min \bar{r}_{0right-} = 2L(3,1)$ at $\alpha_{-,\min} = -1\sqrt{2}$ and $\bar{\beta}_{right-,\min} = 1/\sqrt{2}$.

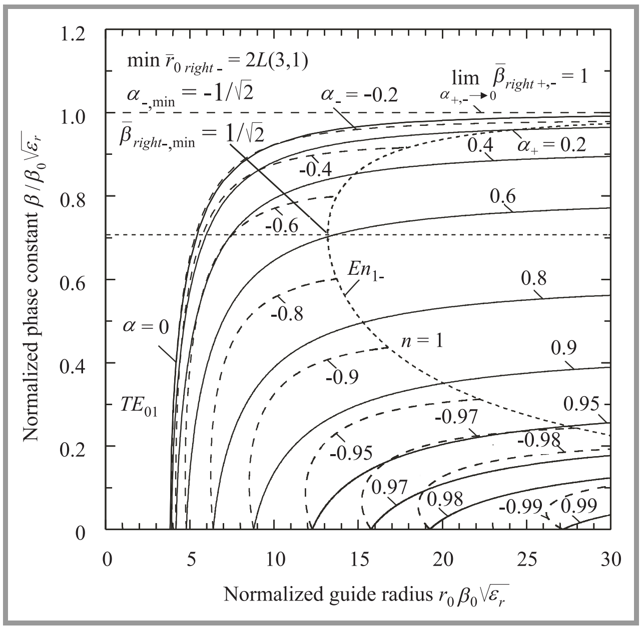

Fig. 8. Phase curves $\bar{\beta}(\bar{r}_0)$ of the normal TE_{01} mode in the circular ferrite waveguide.

9.3. Propagation conditions

Integrating the results of analysis of complex Kummer CHFs and of the problem studied, it turns out that the normal TE_{0n} waves propagate in one region whose boundaries for $M_r > 0$ and $M_r < 0$ are determined by the terms: $\alpha_{left+,-} < \alpha_{+,-} < \alpha_{right+,-}$, $k_{left+,-} < k_{+,-} < k_{right+,-}$, $\bar{\beta}_{left+,-} < \bar{\beta}_{+,-} < \bar{\beta}_{right+,-}$, $\bar{r}_{0left+,-} < \bar{r}_{0+,-} < \bar{r}_{0right+,-}$, where $\alpha_{left-} = -1$, $\alpha_{right-} = 0$, $\alpha_{left+} = 0$, $\alpha_{right+} = 1$, $k_{left-} = -\infty$, $k_{right-} = 0$, $k_{left+} = 0$, $k_{right+} = +\infty$, $\bar{\beta}_{left+,-} = 0$, $\bar{\beta}_{right+,-} = \left(1-\alpha_{+,-}^2\right)^{1/2}$, $\bar{r}_{0left+} = \bar{r}_{0cr}$, $\bar{r}_{0right+} = +\infty$, $\bar{r}_{0left-} = \bar{r}_{0i-}$, $\bar{r}_{0right-} = \bar{r}_{0en-} = L(c,n)/\left[|\alpha_-|\left(1-\alpha_-^2\right)^{1/2}\right]$. Moreover $\bar{\beta}_{right} = \bar{\beta}_{en-}$. The subscripts "*left*", "*right*" designate the limits of domain in which certain quantity varies and the ones "+", "–" show the sign of magnetization to which the latter is relevant.

9.4. Phaser operation

The waveguide may provide differential phase shift $\Delta\bar{\beta} = \bar{\beta}_- - \bar{\beta}_+$ for TE_{01} mode when latching M_r in the area of partial overlapping $\Delta = \bar{r}_{0right-} - \bar{r}_{0left+} = \bar{r}_{0en-} - \bar{r}_{0cr}$ of the intervals $\Delta_- = \left(\bar{r}_{0left-},\ \bar{r}_{0right-}\right)$, and $\Delta_+ = \left(\bar{r}_{0left+},\ \bar{r}_{0right+}\right)$, pertinent to $\bar{\beta}_-(\bar{r}_{0-})$ and $\bar{\beta}_+(\bar{r}_{0+})$ curves for the same $|\alpha|$ ($\Delta = \Delta_- \cap \Delta_+$, Fig. 8 and Fig. 1 [74]). Hence, the condition for the geometry to operate as phaser at fixed $|\alpha|$ (the working point $\bar{r}_0$ to be part of Δ), is $\bar{r}_{0cr} < \bar{r}_0 < \bar{r}_{0en-}$, or [69]:

$$\zeta_{0,1}^{(3)}|\alpha| < 2\bar{r}_0|\alpha|\sqrt{1-\alpha^2} < 2L(3,1)\,. \tag{15}$$

Save from the graphs, $\Delta\bar{\beta}$ could be computed also directly from structure parameters, using the formulae $\Delta\bar{\beta} = A|\alpha|$, $\Delta\bar{\beta} = B/\bar{r}_0$, $\Delta\bar{\beta} = (C/\bar{r}_0)|\alpha|$ [66, 74]. The values of factors A, B, C are tabulated in [66, 74]. If $\bar{r}_0 > \bar{r}_{0en-}$, the configuration has potentialities as current controlled switch or isolator.

10. A microwave application of the real Kummer function

10.1. Propagation problem for slow $\widehat{TE}_{0\hat{n}}$ modes in an azimuthally magnetized circular ferrite waveguide

The guided $\widehat{TE}_{0\bar{n}}$ waves examined are slow, if $\bar{\hat{\beta}}_2 = \left(\bar{\hat{\beta}}^2 - \bar{\hat{\beta}}_f^2\right)^{1/2}$ is real ($\bar{\hat{\beta}}_f^2 = \hat{\mu}_{eff}$, $\hat{\mu}_{eff} = 1 - \hat{\alpha}^2$), i.e., provided $\bar{\hat{\beta}}^2 > \bar{\hat{\beta}}_f^2$, ($\bar{\hat{\beta}} > 0$, $\bar{\hat{\beta}}_f^2 > 0$, $\bar{\hat{\beta}}_f^2 < 0$ or $\bar{\hat{\beta}}_f^2 = 0$). The solution of Maxwell equations subject to boundary condition at the wall $\bar{\hat{r}} = \bar{\hat{r}}_0$ yields the corresponding characteristic equation [69]:

$$\Phi\big(\hat{a}, \hat{c}; \hat{x}_0\big) = 0 \qquad (16)$$

with $\hat{a} = 1.5 + \hat{k}$, $\hat{c} = 3$, $\hat{x}_0 = 2\bar{\hat{\beta}}_2\bar{\hat{r}}_0$, $\hat{k} = \hat{\alpha}\bar{\hat{\beta}}/\big(2\bar{\hat{\beta}}_2\big)$. It is valid in case $\bar{\hat{\beta}}_2 = \hat{\zeta}^{(\hat{c})}_{\hat{k},\hat{n}}/\big(2\bar{\hat{r}}_0\big)$, giving the eigenvalue spectrum looked for. (Equation (16) could be obtained from (14) putting $k = \mathrm{j}\hat{k}$ and $\bar{\beta}_2 = -\mathrm{j}\bar{\hat{\beta}}_2$).

10.2. Propagation conditions

Combining the outcomes of the study of real Kummer CHFs and of the problem regarded, it is found that the slow $\widehat{TE}_{0\hat{n}}$ modes could be guided for $\hat{M}_r < 0$ solely in two areas, set by the criteria: $\hat{\alpha}^{(1),(2)}_{left-} < \hat{\alpha}^{(1),(2)}_{-} < \hat{\alpha}^{(1),(2)}_{right-}$, $\hat{k}^{(1),(2)}_{left-} < \hat{k}^{(1),(2)}_{-} < \hat{k}^{(1),(2)}_{right-}$, $\bar{\hat{\beta}}^{(1)}_{left-} < \bar{\hat{\beta}}^{(1)}_{-} < \bar{\hat{\beta}}^{(1)}_{right-}$, $\bar{\hat{\beta}}^{(2)}_{left-} > \bar{\hat{\beta}}^{(2)}_{-} > \bar{\hat{\beta}}^{(2)}_{right-}$, $\bar{\hat{r}}^{(1),(2)}_{left-} < \bar{\hat{r}}^{(1),(2)}_{0-} < \bar{\hat{r}}^{(1),(2)}_{0right-}$, with $\hat{\alpha}^{(1)}_{left-} = -1$, $\hat{\alpha}^{(1)}_{right-} = 0$, $\hat{\alpha}^{(2)}_{left-} = -\infty$, $\hat{\alpha}^{(2)}_{right-} = -(2\hat{n}+1)$, $\hat{k}^{(1)}_{left-} = -\infty$, $\hat{k}^{(1)}_{right-} = -(2\hat{n}+1)/2$, $\hat{k}^{(2)}_{left-} = \hat{\alpha}^{(2)}_{-}/2$, $\hat{k}^{(2)}_{right-} = -(2\hat{n}+1)/2$, $\bar{\hat{\beta}}^{(1)}_{left-} = \Big[1 - \big(\hat{\alpha}^{(1)}_{-}\big)^2\Big]^{1/2}$, $\bar{\hat{\beta}}^{(1)}_{right-} = \Big\{\Big[1 - \big(\hat{\alpha}^{(1)}_{-}\big)^2\Big]/\Big[1 - \Big(\hat{\alpha}^{(1)}_{-}/(2\hat{n}+1)\Big)^2\Big]\Big\}^{1/2}$, $\bar{\hat{\beta}}^{(2)}_{right-} = \Big\{\Big[\big(\hat{\alpha}^{(2)}_{-}\big)^2 - 1\Big]/\Big[\Big(\hat{\alpha}^{(2)}_{-}/(2\hat{n}+1)\Big)^2 - 1\Big]\Big\}^{1/2}$, $\bar{\hat{\beta}}^{(2)}_{left-} = +\infty$, $\bar{\hat{r}}^{(1)}_{0left-} = \hat{L}\big(\hat{c}, \hat{n}\big)/\Big\{\big|\hat{\alpha}^{(1)}_{-}\big|\Big[1 - \big(\hat{\alpha}^{(1)}_{-}\big)^2\Big]^{1/2}\Big\}$, $\bar{\hat{r}}^{(1)}_{0right-} = +\infty$, $\bar{\hat{r}}^{(2)}_{0left-} = 0$, $\bar{\hat{r}}^{(2)}_{0right-} = +\infty$. The superscripts (1), (2) designate the zone to which the corresponding quantity relates. Thus, the symbol $\widehat{TE}_{0\hat{n}}$ is a general notation for two waves $\widehat{TE}^{(1)}_{0\hat{n}}$ and $\widehat{TE}^{(2)}_{0\hat{n}}$, supported in the first and second regions, respectively.

10.3. Phase characteristics

Taking into account the propagation conditions and repeating the procedure, described in Subsection 9.2 with the roots $\hat{\zeta}^{(3)}_{\hat{k},1}$ of Eq. (16), the $\bar{\hat{\beta}}^{(1)}_{-}\big(\bar{\hat{r}}^{(1)}_{0-}\big)$ and $\bar{\hat{\beta}}^{(2)}_{-}\big(\bar{\hat{r}}^{(2)}_{0-}\big)$ – characteristics with $\hat{\alpha}^{(1)}_{-}$ and $\hat{\alpha}^{(2)}_{-}$ as parameters for the slow $\widehat{TE}^{(1)}_{01}$ and $\widehat{TE}^{(2)}_{01}$ modes, respectively in the structure are computed and presented with dashed curves of infinite length in Figs. 9 and 10, respectively. Thus, transmission takes place for $\hat{\alpha}_- < 0$ in two unlimited from above frequency bands. An envelope (dotted line), labelled with $\hat{E}n_{1-}$ (for the co-ordinates of the points of which $(\bar{\hat{r}}^{(1)}_{0en-}, \bar{\hat{\beta}}^{(1)}_{en-})$ it is valid $\bar{\hat{r}}^{(1)}_{0en-} = \bar{\hat{r}}^{(1)}_{0left-}$ and $\bar{\hat{\beta}}^{(1)}_{en-} = \bar{\hat{\beta}}^{(1)}_{left-}$), restricts from the left the area of propagation

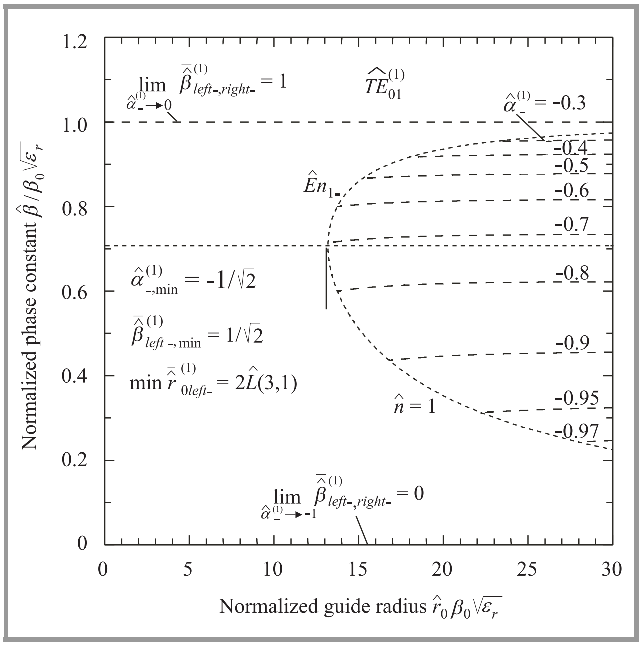

Fig. 9. Phase curves $\bar{\hat{\beta}}^{(1)}_{-}\big(\bar{\hat{r}}^{(1)}_{0-}\big)$ of the slow $\widehat{TE}^{(1)}_{01}$ mode in the circular ferrite waveguide for $-1 < \hat{\alpha}^{(1)}_{-} < 0$.

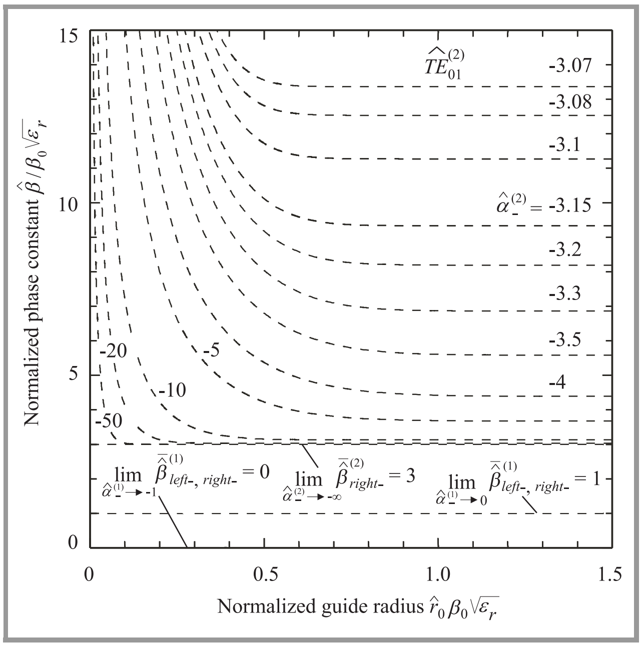

Fig. 10. Phase curves $\bar{\hat{\beta}}^{(2)}_{-}\big(\bar{\hat{r}}^{(2)}_{0-}\big)$ of the slow $\widehat{TE}^{(2)}_{01}$ mode in the circular ferrite waveguide for $\hat{\alpha}^{(2)}_{-} < -3$.

in case of weak anisotropy (Fig. 9). It has a minimum $\min \bar{\hat{r}}_{0left-} = 2\hat{L}\,(3,1)$ at $\hat{\alpha}^{(1)}_{-,\min} = -1\sqrt{2}$ and $\bar{\hat{\beta}}^{(1)}_{left-,\min} = 1\sqrt{2}$. A comparison of both sets of curves shows that a large slowing down is provided if the anisotropy is strong, especially in case $\bar{\hat{r}}^{(2)}_0$ is small (see Fig. 10). Ferrite switches and isolators are the possible applications of the structure.

11. Areas of TE_{01} mode propagation

The joint consideration of the results of analysis of the anisotropic waveguide shows that in case of positive (counterclockwise) magnetization there is one (densely hatched) area in which normal TE_{01} mode is supported (Fig. 11). If the magnetization is negative (clockwise), the areas are already three: one (densely hatched) of normal and two (sparsely hatched) of slow wave propagation (Fig. 12).

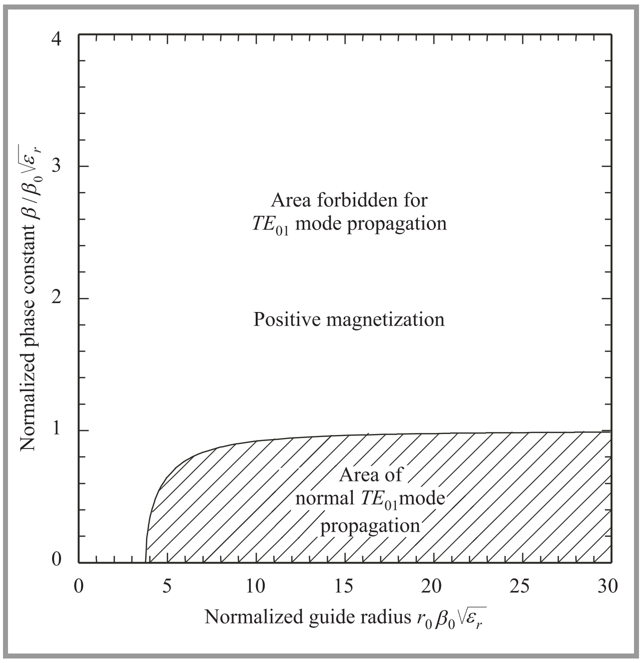

Fig. 11. Areas of TE_{01} mode propagation in case of positive magnetization.

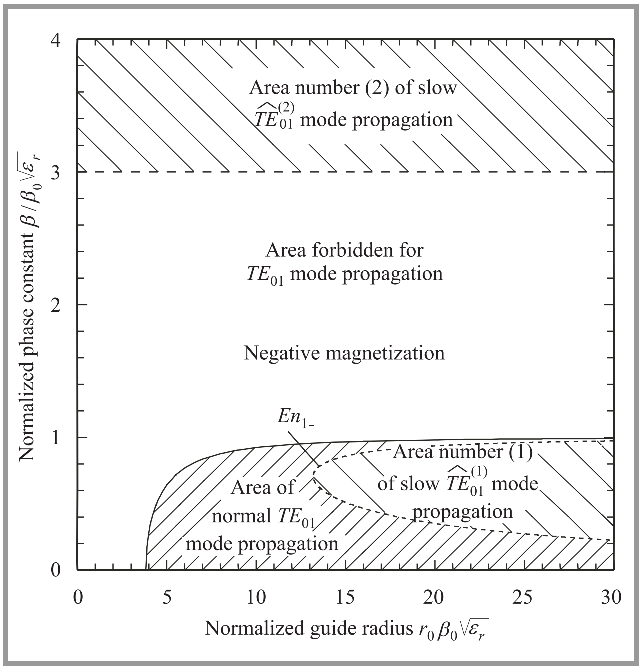

Fig. 12. Areas of TE_{01} mode propagation in case of negative magnetization.

An important corollary of Theorem 1 is the coincidence of the envelopes of characteristics for $M_r < 0$ of the normal (Fig. 8) and the slow mode (Fig. 9) in one curve (the dotted line in Fig. 12, labelled En_{1-}) which does not belong to any of the zones and serves as their common border, delimiting them. Indeed, since in view of Eq. (12) $L(3,1) = \hat{L}(3,1)$, the points $(\overline{r}_{0en-},\ \overline{\beta}_{en-})$ and $(\overline{\hat{r}}^{(1)}_{0en-},\ \hat{\beta}^{(1)}_{en-})$ in the $\overline{r}_0 - \overline{\beta}$ phase plane, forming the En_{1-} and $\hat{E}n_{1-}$ – characteristics for the TE_{01} and $\widehat{TE}^{(1)}_{01}$ modes, respectively, are identical for all values of parameters $\alpha_- = \hat{\alpha}^{(1)}_-$ whose intervals of variation, determined by the corresponding propagation conditions in Sections 9.3 and 10.2, are the same. Area number (2) for the slow wave is separated from aforesaid two ones by a region where no transmission is allowed.

12. Conclusions

Some basic concepts of the theory, the special cases and examples of the use of the CHFs in different fields of physics are considered. The opinion is declared that a universal mathematical procedure, based on them would successfully substitute the methods for analysis of a large number of tasks, utilizing the numerous functions which are their special cases. This approach would make possible to reveal the interior connections among plenty of phenomena and would facilitate the physical interpretation of the results from their description, as well as the process of their generalization.

The problems for normal and slow rotationally symmetric TE modes in the circular waveguide, uniformly filled with azimuthally magnetized ferrite are threshed out as a sphere of microwave application of the complex, and real Kummer CHFs rescpectively. The propagation conditions and phase characteristics of the structure are obtained, using various properties of the wave function, established analytically and/or numerically. The main result of the study is that for positive (negative) magnetization one area of normal (three areas – one of normal and two of slow) TE_{01} mode propagation exists (exist). The region of normal and the first one of slow waves transmission in case of negative magnetization are demarcated by an envelope curve which can be traced by means of a numerically proved theorem for identity of the zeros of certain Kummer functions. The areas mentioned are separated from the second one for slow wave propagation by a zone in which no fields can be sustained. The phase behaviour reveals the potentialities of the structure as a remanent phaser (for normal waves) or as a current controlled switch and isolator (for both kinds of waves). The criterion for phaser operation of waveguide is deduced as a direct corollary of the aforesaid theorem for the zeros. A large number of configurations, containing a central ferrite rod of azimuthally magnetized ferrite, coated by an arbitrary number of dielectric layers could be described, extending the analysis method based on the Kummer CHFs.

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