ANNUAL OF NAVIGATION 3/2001

Kajetan Jackowski
Gdynia Maritime University

A FORMAL DESCRIPTION OF NAVIGATIONAL PROCESS

ABSTRACT In this paper there is undertaken an attempt to present the formal description of the ship's navigation process. This formal description takes into account not only all the most important ship's characteristics that influence the ship's motion but also all the most important environmental characteristics influencing the ship's behaviour at sea. The formal description comprises the following issues: ship's navigation process, navigational information, dynamic of the ship's motion, equations describing the ship's navigation process, and mathematical models of ship's navigation process.

GENERAL DESCRIPTION OF NAVIGATIONAL PROCESS

In this paper the term NAVIGATIONAL PROCESS [10, 12, 14,15] will comprise all the functions and procedures constituting **NAVIGATION** [7, 16, 17, 28, 36] including **STEERING** [7, 21, 26, 27] a ship and considered together with **SHIP'S MOVEMENT** [4, 7, 12, 15, 18, 21, 26, 27, 28, 36] along the adopted route from the point of departure to the point of destination.

Of all those navigational functions and procedures the following can be considered the principal ones:

- planning ship's route for the voyage;
- steering (and maneuvering) to keep the ship on the required route and avoid any encountered hazard or danger;
- monitoring ship's track (positioning), her course, speed and distance covered;
- incessant verification and (if necessary) modification of planned route according to current requirements and changing circumstances.

The basis for performing the procedures is process of gaining, processing and utilizing navigational information (measurements and received data). The information enables navigator to make adequate decisions in navigating and controlling the ship.

Navigational task

The term SHIP will denote every sea-going vessel or watercraft (while on a long-range sea trip) such as cargo vessels, fishing vessels, passenger and training ships, specialized ships, salvage craft, hydrographic and research vessels, ocean-going tugs as well as warships and coastguard craft (if on a voyage longer than several hours).

It is obvious that designation of such comprehensively meant SHIP as well as the ways of her employment is inevitably very different. One thing, however, is always in common: every ship is to fulfil her assigned task while observing some additional requirements, which make the performance of the task most effective. Examples of the assigned ship's task may be:

- carrying cargo from a loading port to discharging ports, delivery of cargo and getting alongside in another port to take next cargo on board before the laydays expire;
- getting as soon as possible from berth to a fishing area and then fishing (for a specified period of time) and delivering the fish to a factory ship in this area;
- towing a drilling platform from shipyard onto a precisely determined position;
- taking over hydrographic or military or scientific research duties on a given moment and at an appointed position or in a nominated area on a given date.

The fundamental and inherent component of every ship's task is NAVIGATIONAL TASK that is **safe and expedient** passage of ship along the entire route, from the place of departure to intended destination. Safety of sea passage means avoiding actual dangers (mostly permanent obstructions of any kind) and minimization of navigational hazard (i.e. potential perils resulting chiefly from adverse weather [11] conditions and vessel traffic, which are changeable factors). Expedience of passage means that the basic condition of safety is fulfilled with sufficient but not excessive margin of certainty, so that accomplishment of some additional requirements connected with the major task of voyage can be made optimal (i.e. carried out in a most effective way).

Some examples of above-mentioned additional requirements are:

- shortening the time of sea passage;
- minimizing fuel consumption;
- exact fulfillment of voyage schedule (i.e. reaching all call-at-points sharply on time) and many others, often contradictory to one another.

The realization of so defined navigational task is executed in navigational process. Two principal stages can be distinguished in the said process: planning of a voyage and its performance. Both stages are determined by ship's properties (characteristics) and by prevailing external conditions and circumstances.

Factors affecting navigation

There are three groups of factors affecting navigation of a ship:

I – **permanent** ("statical") conditions such as

I-1 – shape of coastline and depth of water (i.e. topography of sea bed or bathymetry)

I-2 – existing systems of positioning and systems of supporting navigation – GMDSS, Navtex, VTS, ship's routing systems/offices, landmarks and buoyage systems, traffic separation schemes, fairways, restricted areas and areas designated for special purposes; (although above listed items are liable to some changes and fluctuations they can be deemed constant factors for the considered time of a voyage – providing that all necessary allowances have been taken, that is the set of relevant charts and nautical publications has been updated in accordance with latest Notices to Mariners and navigation warnings in force)

II – **dynamic** (external) factors such as

II-1 – predominating weather conditions – force and direction of wind, state of sea, phase of tide, tidal streams, set and drift of current, visibility, temperature of air and sea water, humidity etc.)

II-2 – random (and sudden) changes of permanent conditions – especially if they pose and danger to navigation and immediately has to be taken into account

(II-3) – vessels traffic – in confined waters in particular

(II-4) – imperative orders and requirements (owner's or charterer's instructions, distress calls, commands of SAR coordinating center, VTS instructions etc.)

III – ship's **characteristics** with regard to her **maneuverability** and **stability**.

III-1 – chief characteristics describing ship's maneuverability are:

V(N) – speed as a function of propeller revolutions N that is

$$V = N\,\eta \qquad (1)$$

Where: η is actual propeller pitch

V_{max}, V_E – maximum and economical speed

NN, Nkr, Ndop – nominal, critical and allowable revolutions, respectively

η - actual pitch of propeller

$$\eta = \kappa\,\eta_N \qquad (2)$$

where: η_N is nominal (theoretical) propeller pitch

κ- propeller slip ratio, which depends on weather conditions – force of wind B^o, state of sea S^o, heading angle of wind kK_W and seas kK_f and present ship's displacement or her draught T as well as ship's speed V, that is altogether

$$\kappa=\kappa(B^o, kK_W, S^o, kK_f, T, V)$$

τ.(T) – stopping time as a function of draught T, that is for various loading conditions (displacements) and for various stopping commands (for instance from “FULL AHEAD” or to “FULL ASTERN”).

τ_V(T) – time of attaining required speed for various displacements and at given telegraph commands (SLOW AHEAD, HALF AHEAD, FULL AHEAD etc.)

$C_{p/s}$, C_{sb} – turning circle (tactical diameter) with the rudder put over hard-a-port ($C_{p/s}$) or hard-a- starboard (C_{sb}), generally Cr, where

$$Cr = Cr (\alpha, V, T)$$

which means that Cr depends on rudder angle α, speed V and draught T (i.e. vessel’s actual displacement)

$\omega(t,\alpha, V)$ – angular rate of turning as a function of time t, rudder angle (and vessel’s speed V (it may also occur $\omega_{p/s}$ (ω_{sb})

ρ, $\Delta\rho$ - aquat,

that is

$\rho(T,\Delta T, h, V)$ – change of mean draught with regard to depth of water h, initial mean draught T, trim ΔT and speed V,

as well as

$\Delta\rho(T, \Delta T, h, V)$ – analogical change of trim – and for the same variables (in very narrow passages an important parameter for squat determination is the width of channel of fairway)

$\Delta\Omega$ - angle of yawing at various weather conditions (dependant on similar variables as the coefficient κ)

III-2 – among most important stability characteristic [4, 18] there are

GM – metacentric height

GZ(Φ) – curve depicting righting lever (righting arm) as a function of angle of heal Φ

$\Phi_W{}^o$ – angle of heal due to steady wind pressure

$\Delta\Phi_A{}^o$ – angular amplitude of roll (due to action of sea waves) and also hull stresses (strength characteristics), namely

BM – bending moments

and

F_{SH} – shearing forces determined for the whole longitudinal section of hull or on chosen frames and bulkheads.

It should be noticed that not all above listed characteristics of a particular ship are known precisely and updated exactly. This pertains mostly to maneuvering characteristics, which ought to be determined experimentally, by sea trials. Practically they are often known in a very simple form, for a few chosen variables and for some of their values only. It leaves a margin of uncertainty, that should be taken into account while navigating and steering (maneuvering) the ship.

Permanent conditions are considered in planning a voyage and ship's route. All the other factors, either external ones as well as some variations of ship's characteristics (due to regular or random causes) influence navigation and my call for correction of planned route or exigent maneuvers.

Planning and performing sea voyage

The incipient stage of navigational process is voyage planning that is considering all expected conditions and factors affecting vessel to determine safe route and voyage speed. This means taking into account restrictions and limits resulting from permanent conditions (for instance, minimal depth of water, required safe distance to existing dangers and obstructions, assigned routes or traffic lanes) and variable factors (mostly predicted weather and currents). Also ship's characteristics shall affect voyage planning. Basic information for this stage of navigational process is contained in sea charts and nautical publications, in ship's data (her particulars and characteristics) and in any necessary correction of nautical or ship's data or allowance for forecast conditions or its changes. The initial information for ship's route planning shall also include all the requirements implied by the main task of intended voyage.

The planned route (trip TRAJECTORY), is a set of points, $\underline{w}_0$, $\underline{w}_1$,...., $\underline{w}_n$, where course is to be altered or speed to be changed (reduced/increased). The points are called WAY-POINTS and are connected by rhumb-line legs, that is courses, K_{01}, K_{12},..., $K_{n-1,n}$. If two waypoints are to be connected by a segment of great circle, this actually means necessity of an approximation of this segment by additional set of rhumb-line legs between intermediate waypoints [7, 28, 33, 37].

There are two ways of great circle approximation:

- on the "pole side", when with a chosen distance expressed in nautical miles or longitude difference, for instance for every 100 nM or $\Delta\lambda=5^o$ or 3°, the great circle course to the final point is calculated and this course is to be steered up to the next chosen way-point;
- on the "equator side", when with the assumed step, for instance $\Delta\lambda=5^o$ or for every consecutive 2^o alteration of great circle direction (azimuth), the coordinates (latitude) of the great circle points are calculated; these points, linked by rhumb-line courses, constitute intermediate way-points for given great circle segment.)

The set **speed** is usually constant for the entire voyage (V = const = V_0) and most often it is maximum or economical speed. Any alteration of true speed is mainly caused by weather changes, sea currents, anti-collision maneuvers, landfalling or approaching pilot station and sometimes due to owner's or charter's new instructions, distress calls, engine (or other) technical problems etc.

Restrictions and **limitations** are reflected in the choice of waypoints and assumed courses and speeds between them. They are implied by assumed safe distance to existing dangers and obstructions, by accuracy of positioning systems aboard and – during the voyage – by prevailing weather (and actual stability characteristics) as well as vessel traffic, if it involves a risk of collision or too close approach.

Mathematically all those constraints can be formulated as a range of allowable courses, all {K}, and speeds, all {V}, at a given instant and position.
The result of voyage planning can be written in the form of **initial route matrix**:

$$\mathbf{W}_0 = \begin{bmatrix} \underline{W}_0^T \\ \underline{W}_{01}^T \\ \bullet \\ \bullet \\ \bullet \\ \underline{W}_{0,n-1}^T \\ \underline{W}_{0,n}^T \end{bmatrix} = \begin{bmatrix} \underline{w}_0^T & V_{01} & K_{01} & all\{V_{01}\} & all\{K_{01}\} \\ \underline{w}_1^T & V_{12} & K_{12} & all\{V_{12}\} & all\{K_{12}\} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \underline{w}_{n-1}^T & V_{n-1,n} & K_{n-1,n} & all\{V_{n-1,n}\} & all\{K_{n-1,n}\} \\ \underline{w}_n^T & V_n & K_n & all\{V_n\} & all\{K_n\} \end{bmatrix} \tag{3}$$

In practice allowable courses and speeds, all $\{K_{i-1,i}\}$, all $\{V_{i-1,i}\}$, for any given i[th] leg are not anticipated. They are determined only in case of need – and even than it is usually the choice of one safe course and speed from the whole set of possibilities. With the beginning of voyage the initial route matrix becomes **current route matrix**:

$$\mathbf{W} = \begin{bmatrix} \underline{W}^T \\ \underline{W}_1^T \\ \bullet \\ \bullet \\ \bullet \\ \underline{W}_{,n-1}^T \\ \underline{W}_{,n} \end{bmatrix} = \begin{bmatrix} \underline{w}^T & V & K & all\{V\} & all\{K\} \\ \underline{w}_1^T & V_{12} & K_{12} & all\{V_{12}\} & all\{K_{12}\} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \underline{w}_{n-1}^T & V_{n-1,n} & K_{n-1,n} & all\{V_{n-1,n}\} & all\{K_{n-1,n}\} \\ \underline{w}_n^T & V_n & K_n & all\{V_n\} & all\{K_n\} \end{bmatrix} \tag{4}$$

where row $\underline{W}^T$ pertains to the present moment and the last row of matrix has usually the following form:

$$\underline{W}_n^T = [\underline{w}_n^T, 0, -, -, -].$$

On reaching each subsequent waypoint the route matrix is reduced (by one row) and on k[th] leg of planned trajectory it will be:

$$\mathbf{W}(k) = \begin{bmatrix} \underline{W}^T \\ \underline{W}_k^T \\ \underline{W}_{k+1}^T \\ \bullet \\ \bullet \\ \underline{W}_{n-1}^T \\ \underline{W}_n \end{bmatrix} \tag{5}$$

Executing sea voyage, that is covering the entire route $\underline{\mathbf{W}}_0 \rightarrow \underline{\mathbf{W}}$ from starting point $\underline{w}_0$ to final point $\underline{w}_n$ means navigating and steering the ship, that is carrying out the ship, that is carrying out the following procedures:

(a) - setting course K and speed V, including anti-collision and emergency maneuvers, adapting to weather conditions, correcting the course setting in order to keep to the assigned route or to deviate from it, if necessary;

(b) - recording ship's movement by dead reckoning (based on log and gyro indications) that is calculating DR positions [7, 28, 37] at any moment, if required;

(c) - identification of actual ship's track in the way of **positioning**, that is making (sufficiently often) position fixes PF [5, 7, 10, 16, 20, 28, 32, 37, 40];

(d) - incessant verification of ship's route and its modification, in case of need.

The above listed procedures are described formally by:

(e.c.) – equations of control – for (a,d), [15];
(e.s.) – equations of state – for (a,b), [12, 19, 23, 29, 41];
(e.o.) – equations of observation – for (c,d), [12, 19, 23, 29, 36, 41].

NAVIGATIONAL INFORMATION

Obtaining and processing navigational information [25, 30, 35] provides all decisions [39], procedures and operations in navigating [34, 35], controlling and maneuvering [21] a ship.

Sources of information

The information utilized in navigational process can be

- LOCAL, ("on the spot data"), if it is obtained directly from the sources placed aboard ship

or

- EXTERNAL, (i.e. global, regional, or "directional" – in case of being addressed to a specified ship), if it is received from the sources of information situated outside the ship (radio messages, satellite transmition, mail, fax, telegrams)

The LOCAL INFORMATION sources are:

1. Maritime POSITIONING SYSTMS and APPLIANCES;
2. LOGS (and devices for inertial navigation) to measure vessel's speed and distance covered;
3. GYRO- and MAGNETIC COMPASSES indicating ship's course (heading);
4. RADARS (and ARPA systems) enabling navigator to get position data from terrestrial [7] measurements as well as data concerning vessels traffic in the vicinity (already processed by ARPA with regard to collision avoidance aspect);
5. ECHOSOUNDERS gauging depth of water;
6. DIRECT (audiovisual) OBSERVATION of closest surroundings, that is observing ships and other objects within their optical range, watching state of sea, wind force, frequency of slamming, kind and intensity of hull vibrations etc.;

7. INSTRUMENTS and APPLIANCES measuring various physical and safety parameters (such as wind speed and direction, rudder angle, propeller revolutions, period of roll, angle;
8. RECORDERS of various type and design;
9. SET of SHIP's DATA, first of all her stability and maneuverability characteristics and other important current information (for example: loading condition, tanks condition, present ship's drafts and trim, fuel distribution and consumption, fresh water consumption, actual condition of steering gear and propelling machinery);
10. SET of BASIC DATA, that is set of charts, navigational aids and publications [7, 37], chosen records of previous observations and measurements as well as all information concerning present ship'' task (to be carried out) and some additional requirements or recommendations (if any).

The EXTERNAL INFORMATION is received from the following sources:

I. NAVIGATIONAL SAFETY SYSTEMS (and systems supporting maritime navigation) such as Navtex, GMDSS (already introduced and still being developed), numerous VTSes, maritime weather services and offices supervising landmark and buoyage systems;
II. DISPOSERs of the ship (her owner, or charterer, or – in some specific cases – center of SAR, coastguard or authorized representatives of the administration in the area);
III. other VESSELs.

The means of conveyance of external information are: radiophony, radiotelphony, radiotelegraphy, (including satellite commutations), data transmition, recently e-mail and mail (still a substantial part of shore-ship communication).
With respect kind of local information and type of its source it can be classified as:

- measuring data – which come from systems and appliances of positioning, from ship's movement indicators (logs and compasses) and other gauges providing required measurements of physical and safety parameters;
- observation data – from audiovisual watch, from observation or radar display as well as current output or various recorders;
- basic data (generally descriptive) – actual ship's characteristics, data describing her present condition and any records of previous observations or measuring data.

The original form and source of external information is usually unknown to navigator. For a recipient aboard ship it can be either a message or an order. Received messages (weather news and forecasts, navigation warnings, vessels traffic data) contain information about present or anticipated outer conditions.
Incoming orders (i.e. imperative commands or instructions) determine, directly or indirectly, a change of assigned task or modify the manner of carrying it into effect.

Processing and utilizing navigational data

The preliminary procedure, prior to processing the information, is making measurement or observation, or reception of a message, or choosing data of concern from a large set of data, which (especially in case of basic data) means also preselection [30]. The obtained data are then selected by its priorities [25, 30, 39]. The procedure of selection, however, is entirely at navigator discretion and cannot be described in a strict and formal way. It can only be asserted that decisions {D*} to be made depend on initial data $\underline{d}_0$, data from observations $\underline{\theta}$ and measurements $\underline{Y}$, imperative orders and instructions $\underline{D}$. (i.e. important decisions worked out beyond the ship) and that the assumed set of priorities {P*} is to be taken into account. Generally

$$\{D^*\} = D^* [\underline{d}_0, \underline{d}, \underline{Y}, \underline{\theta}, \{P^*\}, \underline{D}.] \qquad (6)$$

where operator D^* is a statistical decision function [39].

The key rule in granting priority to an information is that the more for ship's safety it is the higher its priority. Next come priorities securing optimal performance of navigational task (its effectiveness and expedience). If the order of sequence in which the incoming data are to be utilized, has no apparent significance for ship's safety and efficiency of task performance then data concerning ship's course (heading) and speed (distance covered) are usually used first and position data are considered as the second ones.

With respect to order and way of utilizing the navigational information in can be classified as preliminary or verifying or updating.

- PRELIMINARY (mostly basic initial data) is availed of at planning ship's route (trajectory) for the voyage;
- VERIRYING data (current observation and measurements) enable navigator to identify actual ship's track and compare it with the planned route, verify received forecasts and his own anticipations, correct course (and planned route itself) and – first and foremost – detect imminent threats and dangers and avoid them (by correcting the trajectory to be covered or by immediate maneuver); some of previously recorded data (results of observation or measured values) as well as results of their processing could prove useful at verifying current measuring data – for instance, a significant discrepancy between current position fix and corresponding dead reckoning position may be a signal of considerable error of log or gyro indications, or gross error of position fix (making it useless), or may mean that hydrometeorogical conditions have changed markedly (but the change can not be detected by log or compass or other gauging instruments aboard ship);
- UPDATING information (received warnings and weather forecasts as well as navigator's own conclusions and anticipations based on results of processing the verifying data) – it is directly applied to modify the set of navigational data in use.

A considerable part of incoming measurement results has to be processed before final usage; this processing includes application of eligible corrections (which are often liable to substantial changes), calculation (with analytical or graphical methods) and sometimes statistical averaging of gauged values.

Structural pattern of data processing

The flow chart or processing navigational data is shown on Fig. 1. Symbols used for marking sources of information:

- **(y)** → measurements
- **(θ)** → observations
- **(d)** → other navigational data (received messages or results of processing the measurements and observations)
- **(D)** → imperative orders and instructions

Other symbols:

D_T* - decision block of route planning and executing
D_S* - decision block of steering and maneuvering (setting bridge and engine controls)
$\underline{d}_0$ - vector of initial data
$\underline{d}$ - vector or current navigation data
$\underline{d}.$ - vector of processed data for collision avoidance
$\underline{D}.$ - vector of imperative orders and instructions
$\Delta\underline{d}$ - corrections of basic data (outcome of processing the verifying information)
$\mathbf{W}_0$ - initial matrix of assigned ship's route
$\mathbf{W}$ - actual matrix of assigned ship's route
$\underline{Y}$ - vector of measurements (gauged values)
$\underline{\theta}$ - vector of observations (values obtained from observation and evaluation)
$\Delta\underline{Y}$ - signal of measurement vector change (vector of measurements differences)
$\Delta\underline{\theta}$ - signal of a change of observations (observation results differences)
$\underline{u}_y$ - position fix, PF
$\underline{u}_z$ - dead reconed position, DR
$\underline{w}$ - waypoint
$\underline{s}_w$ - required vector of steering (vector of required settings)
$\underline{s}$ - actual vector of steering (current settings)
$\Delta\underline{u}_w$ - vector of deviation from planned route

$$\text{where } \Delta\underline{u}_w = w - \underline{u}_y \text{ or } \Delta\underline{u}_w = w - \underline{u}_z$$

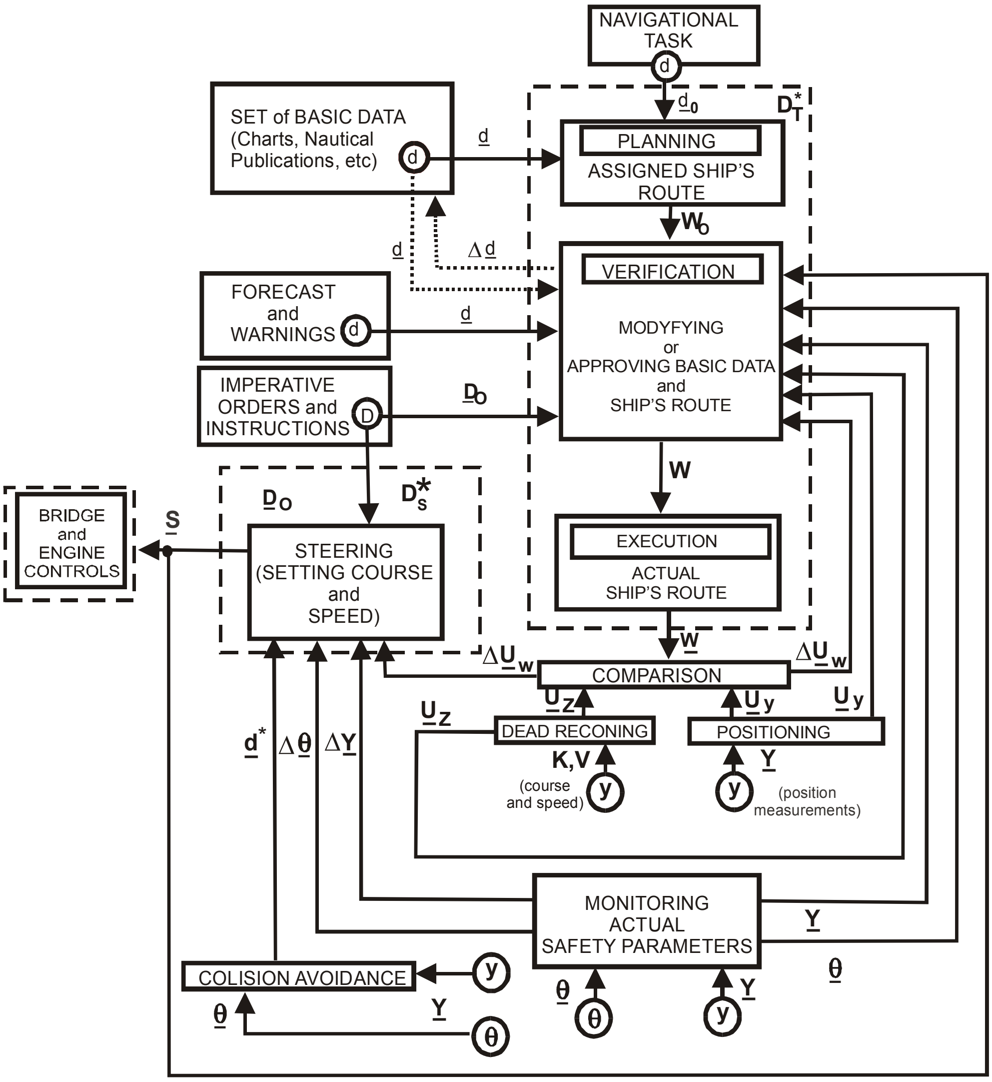

Fig. 1. Functional Diagram of navigational process

THE OBJECT OF THE PROCESS AND ITS DYNAMICS

A ship as an OBJECT [19, 23, 29, 41] of navigational process can be analyzed and described in different space-time scales. From practical point of view the extremes of these scales can be called GLOBAL SCALE and LOCAL SCALE. The term **global** means that the scale covers time intervals of many days and the whole problem can be reduced to considering the movement of a point on curvilinear surface of the Earth (the geoid) along a trajectory consisting of finite number of rhumbline legs.

The opposite extreme, the **local** scale applies to time intervals of minutes or even seconds. In such scale the shape of the Earth, its global size and surface curvature are of minor importance – but ship dimensions itself as well as acting forces (caused by operating propeller or rudder, or being exerted by wind and seas) and resultant accelerations affecting her hull become eligible for consideration. Ship's own and very particle of ship's body has six degrees of freedom: three for linear and three for angular motions (i.e. along or around each axis of the local coordinate system).

The linear ship's motions are:

L_1 – progression or headway (which is the effect of propeller driving force) and lengthwise shifts (i.e. fore and aft hull slips caused by sea waves action);

L_2 – transverse shifts (i.e. athworthship hull slips due to wind and sea action);

L_3 – vertical shifts or heaving and dipping (a result of upset equilibrium between ship's weight force and buoyancy force due to hull labor against sea waves).

The angular motions of ship's hull are:

Ω_1 – rolling (turning around longitudinal axis);

Ω_2 – pitching (turning around transverse axis);

Ω_3 – yawing (turning around vertical axis).

Global scale of vessel's motion

Global coordinates [2] relevant to a chosen mathematic representation of Earth's surface [2, 13, 33, 36] are adopted in description of ship's movement. Their general notation: (b, a). Most often it is geographical latitude and longitude, φ, λ, although other coordinates can be applied too (for example, departure instead of longitude, geocentric latitude or meridional part instead of geographical latitude, etc.) [2, 7, 13, 33, 37]. In general, ship's position ($\underline{u}$) and vector of speed ($\underline{v}$) upon the Earth surface can be presented as

$$\underline{u} = [u_b, u_a]^T \text{ and } \underline{v} = [v_b, v_a]^T \; ([u_b^I, u_a^I]^T = \underline{u}^I = d/dt\,(\underline{u}) \quad (7)$$

where components u_b, u_a are expressed in given global coordinates system and their time derivatives u_b^I, u_a^I are the components v_b, v_a, of speed vector.

Conversion of given global coordinates into another coordinate system is usually a non-linear operation [2, 13, 32, 33, 36]. For any chosen global coordinate system the simple equation:

$$\underline{u} = \underline{u}_0 + \Delta\underline{u} \quad (8)$$

is most general description of vessel movement on the Earth surface.

A slightly more evolved form of this equation is

$$\underline{u}(t) \equiv \underline{u}(t_0 + \Delta t) = \underline{u}(t_0) + \Delta\underline{u}(\Delta t) = \underline{u}_0 + \int_{t_0}^{t_0+\Delta t} \underline{v}(t)dt \quad (9)$$

Local scale of vessel's motion

For analysis of complex vessel motion (within very short time spells) it is convenient to choose a local coordinate system [4, 18, 21]. An additional problem is then conversion of coordinates [2, 33] from local to global system and vice versa. Most often the axes of local coordinate system are chosen as follows:

O_1 – longitudinal axis, parallel to ship's centerline
O_2 – transverse axis, perpendicular to ship's centerline
O_3 – vertical axis, perpendicular to horizontal plane, O_1, O_2 (or so called basic plane [4, 18] adopted in naval architecture).

For right-handed local coordinate system $O_0O_1O_2O_3$ with its central point 0_0 placed in ship's center of gravity general differential equations of vessel motion [4] have the forms:

$$\underline{h}^{I} + \underline{\omega} \times \underline{h} = \underline{F} \tag{10}$$

and

$$\underline{J}^{I} + \underline{\omega} \times \underline{J} + \underline{V} \times \underline{h} = \underline{M} \tag{11}$$

Where: $\underline{h}_{[3x1]}$ – vector of momentum (and $\underline{h}^{I}$ – its derivative)
$\underline{J}_{[3x1]}$ – moment of momentum relating to 0_o
(and vector of time derivative $\underline{J}^{I}$)
$\underline{V}_{[3x1]}$ – absolute velocity of system center 0_o
$\underline{\omega}_{[3x1]}$ – angular velocity of ship's hull (3-dimensional rate of turning)
$\underline{F}_{[3x1]}$ – resultant vector of external force
$M_{[3x1]}$ – resultant vector of external force moments (relating to 0_0)

(Mathematical symbols +, × in equations (11) denote vector sum and vector product.)

General equation of vessel motion [4] for given weight (mass) distribution **M** is:

$$\underline{F}_{M} = \mathbf{M}\,\underline{\gamma} \tag{12}$$

where

$$\mathbf{M} = \begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \tag{13}$$

is general mass matrix
m – is total ship's weight (scalar value),
matrices **I**, **0** have form:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

whereas elements of matrix **J** are moments of inertia I_{ii}
and moments of deviation I_{ik} (relating to axes i and k)
while the whole matrix of inertia

$$\mathbf{J} = \begin{bmatrix} I_{11} & 0 & -I_{13} \\ 0 & I_{22} & 0 \\ -I_{31} & 0 & I_{33} \end{bmatrix} \tag{14}$$

and finally vectors:

$$\underline{F}_M = \begin{bmatrix} \underline{F} \\ \underline{M} \end{bmatrix} \quad \underline{\gamma} = \begin{bmatrix} \underline{V}' \\ \underline{\omega}' \end{bmatrix} \tag{15}$$

where

$\underline{V}^I$, $\underline{\omega}^I$ are derivatives of (3-dimensional) linear velocity $\underline{V}$ and angular velocity $\underline{\omega}$, as in (10,11) – and make together vector $\underline{\gamma}$; resultant vectors of external forces $\underline{F}$ and moments $\underline{M}$, also given in (10,11), constitute vector $\underline{F}_M$ of external hull stimulation's (or "input function" in the equation of vessel motion).

EQUATIONS CIRCUMSCRIBING THE PROCESS

In a mathematic model of navigational process the state [19, 23, 24, 41] of considered object (that is a ship underway) can be deemed entirely determined by its actual position $\underline{u}(t)$ and vector of speed $\underline{v}(t)$. Thus the vector of state can be defined as

$$\underline{x} = \begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} \tag{16}$$

Alterations of state should be described adequately by equation of state [19, 23, 24, 41], abbreviation (e.s.), which is analyzed in subparagraph 4.2. In navigation practice any alteration of state $\underline{x}$, that is direct alteration of speed vector $\underline{v}(t)$ and subsequently vector of position $\underline{u}(t)$, is carried into effect through setting course K and speed of advance V, where $V = |\underline{v}|$ and course K is the angle between vector $\underline{v}$ and local meridian.

Thus vector of setting (or steering):

$$\underline{s} = \begin{bmatrix} K \\ V \end{bmatrix} \tag{17}$$

and equation of control – abbreviation (e.c.) – should comprise all major procedures of steering and manoeuvring [15, 35], a ship. It is discussed in subparagraph Equation of control.

Identification of state [23, 24, 29] is carried out through measuring process [12, 23, 24], where the measured values [31] are known functions of state vector

$$\underline{Y} = \underline{Y}(\underline{x}) \tag{18}$$

but most often separately for position

$$\underline{Y}_u = \underline{Y}(\underline{u}) \tag{19}$$

and for speed

$$\underline{Y}_v = \underline{Y}(\underline{v}) \tag{20}$$

An evolved description of (18) is equation of observation [1, 12, 14, 19, 23, 24, 29, 41], briefly marked as (e.o.), which is considered in subparagraph Equation of observation.

Equation of control

For the vector of setting

$$\underline{s}(t) = \begin{bmatrix} K(t) \\ V(t) \end{bmatrix} \tag{21}$$

the vector of set speed, that is undisturbed vector of ship's movement $s\underline{v}$, is a direct transformation of vector $\underline{s}$ (vector of current settings):

$$s\underline{v} = \underline{v}[\underline{s}(t)] = \begin{bmatrix} B \ \ V(t) \ \ \cos K(t) \\ A \ \ V(t) \ \ \sin K(t) \end{bmatrix} \tag{22}$$

where B, a are factors converting given units of speed V into corresponding with chosen coordinates, b, a [33]; the factors are variables dependant on geographical latitude φ, or its equivalent, b, in adopted global coordinate system:

B = B(φ) or B = B(b) and a = A(φ) or a = A(b).

Excluding temporarily from considerations the short spells of maneuvers, it can be said that along each rhumbline leg of route (between succeeding waypoints $\underline{w}_k$, $\underline{w}_{k+1}$) the vector of steering (i.e. set course K and speed V) is steady

$$\underline{s}(t) = \text{const} = \underline{s}_{k+1,k} = \begin{bmatrix} K_{k+1,k} \\ V_{k+1,k} \end{bmatrix} = \begin{bmatrix} K_k \\ V_k \end{bmatrix} = \underline{s}_k \tag{23}$$

and, as a rule, the set speed during the voyage (from start to end of sea passage) is unaltered and equals its initial value V_0 (i.e. sea speed at beginning of sea passage)

$$V_{k+1,k} \equiv V_k = \text{const} = V_0 \tag{24}$$

This means that

$$s\underline{v}_{k+1,k} = \begin{bmatrix} B\,V_{k+1,k} \cos K_{k+1,k} \\ A\,V_{k+1,k} \sin K_{k+1,k} \end{bmatrix} = \begin{bmatrix} B\,V_0 \cos K_k \\ A\,V_0 \sin K_k \end{bmatrix} = s\underline{v}_k \tag{25}$$

Any change $\Delta\underline{v}$ of set speed $s\underline{v}$ caused by alteration of steering $\Delta\underline{s}$ (that is course or speed alteration, or both, ΔK, ΔV), generally

$$s\underline{v} + \Delta\underline{v} = \underline{v}\,[\underline{s} + \Delta\underline{s}] \tag{26}$$

is a <u>maneuver</u> in formal description of navigational process.

According to (17) there can be three kinds of maneuver:
(I) – altering course only, (ΔK), which is of most frequent occurrence;
(II) – altering solely speed, (ΔV);
(III) – simultaneous alteration of course and speed, (ΔK ΔV), which is a general case.

Any considerable course alteration (couple tens of degree or more) virtually always means a general case of maneuver (ΔK, ΔV) – for with the rudder put to port or starboard rather heavily (from several degrees up to hard-a-port or hard-a-starboard) vessel's speed drops (due to an increase in water medium resistance experienced by hull and rudder) and returns to its initial value (answering propeller revolutions) some time after completion of turn. Hence such maneuver is in fact a combination of (ΔK, ΔV) with (ΔV).

For the general case of maneuver beginning at the instant t_k and lasting for time $\Delta t_{k+1,k}$ until the moment $t_{k+1} = t_k + \Delta t_{k+1,k}$, the speed and course become variable

$V(t) = var$ and $K(t) = var$ for $t_k \in \langle t_k, t_{k+1} \rangle$,

or $V(\Delta t) = var$ and $K(\Delta t) = var$ for $0 \le \Delta t \le \Delta t_{k+1,k}$,

and this implies that $V^I(t) \equiv V^I(\Delta t) \neq 0$ and $K^I(t) \equiv K^I(\Delta t) \neq 0$ for $0 \le \Delta t \le \Delta t_{k+1,k}$.
The course derivative

$$K^I = \omega(t) = \omega(\Delta t) \tag{27}$$

Is the rate of turn, while V^I is acceleration or deceleration of vessel's movement.
Directly form (1) and (2) if follows that

$$V = \kappa\, N\, \eta \tag{28}$$

Where: η- pitch of propeller,
N – revolutions (per minute or hour),
κ - propeller slip ratio.

As vessel's speed is variable during the time of maneuver

$$V = V(t) = \kappa(t) \quad N(t) \;\; \eta(t) \equiv \kappa(\Delta t) \quad N(\Delta t) \quad \eta(\Delta t) \tag{29}$$

so

$$V^{I} = d/dt\,(\kappa\ N\ \eta) \tag{30}$$

that is

$$V^{I} = \kappa^{I}\ N\ \eta + \kappa\ N^{I}\ \eta + \kappa\ N\ \eta^{I} \tag{31}$$

Right on completion of maneuver the effect of course alternation is

$$\Delta K_{k+1,k} = \int_{0}^{\Delta t_{k+1,k}} \omega(\Delta t)\,d\Delta t = \int_{t_k}^{t_{k+1}} \omega(t)\,dt \tag{32}$$

the total alteration of speed

$$\Delta V_{k+1,k} = \int_{0}^{\Delta t_{k+1,k}} V^{I}(\Delta t)\,d\Delta t = \int_{t_k}^{t_{k+1}} V^{I}(t)\,dt \tag{33}$$

and change of steering altogether

$$\Delta \underline{s}_{k+1,k} = \int_{t_k}^{t_{k+1}} \underline{s}(t)\,dt = \int_{t_k}^{t_{k+1}} \begin{bmatrix} K^{I}(t) \\ V^{I}(t) \end{bmatrix} dt = \begin{bmatrix} \Delta K_{k+1,k} \\ \Delta V_{k+1,k} \end{bmatrix} \tag{34}$$

which means that eventually

$$\underline{s}_{k+1} = \underline{s}(t_{k+1}) = \underline{s}(t_k) + \Delta\underline{s}(\Delta t_{k+1,k}) = \underline{s}_k + \Delta\underline{s}_{k+1,k} \tag{35}$$

Thus in general case of maneuver, ($\Delta K \Delta V$), its final outcome is a new value of set speed

$$s\underline{V}_{k+1} = \begin{bmatrix} B \bullet (V_k + \Delta V_{k+1,k}) \bullet \cos(K_k + \Delta K_{k+1,k}) \\ A \bullet (V_k + \Delta V_{k+1,k}) \bullet \sin(K_k + \Delta K_{k+1,k}) \end{bmatrix} \tag{36}$$

To get right formulae for other cases it suffices to put $\Delta V = 0$ in (36) for the case (I) of maneuver (ΔK), or to assume that $\Delta K = 0$ in case (II) of speed alteration (ΔV). Application of (32, 33), however, would be impracticable without some additional simplifying assumptions. If angular velocity during the turn is almost constant, $\omega \cong$ const, or its average value can be correctly estimated (from a relevant maneuvering characteristic), then naturally

$$\underset{0 \le \Delta t \le \Delta t_{k+1,k}}{\forall} \qquad \Delta K(\Delta t) \cong \Delta t\ \omega \tag{37}$$

Similarly, for speed alteration ΔV (within the same time interval Δt of a maneuver), if it is acceptable to assume that in (28) one factor only has changed essentially, then

- for altered revolutions $(N + \Delta N)$ of propeller (of uniform pitch)

$$\Delta V \cong \kappa\, \Delta N\, \eta \tag{38}$$

- for altered propeller pitch $(\eta + \Delta\eta)$ of adjustable pitch propeller

$$\Delta V \cong \kappa\, N\, \Delta\eta \tag{39}$$

and in the case of essential reduction of speed during the turn, in spite of almost constant revolutions and unchanged propeller pitch, that is

- for variable slip ratio $(\kappa+\Delta\kappa)$ of propeller

$$\Delta V \cong \Delta\kappa\, N\, \eta \tag{40}$$

Simplified relationships (37) and (38) can be used (together with given maneuverability characteristics) for evaluating a particular run of various maneuvers and its final outcome.

Changes of set speed $s\underline{v}$, as defined by (36), subsequent to alterations of steering $\Delta\underline{s}$ given in (34), are carried out to secure

- proceeding along the planned route, which means a course alteration (ΔK) at each waypoint or altering course and speed $(\Delta K \Delta V)$ when land falling or approaching pilot station, or passing through restricted waters;
- keeping to plotted track (within associated safety margin), which usually requires small course corrections (ΔK) only;
- collision avoidance, which may involve any kind of maneuver, (ΔK) or (ΔV) or $(\Delta K \Delta V)$;
- making allowances for and adjustments to prevailing weather conditions – from small course corrections (ΔK) for wind and drift till large course alternations and considerable reduction of speed, $(\Delta K \Delta V)$, while heaving to on a stormy sea;
- executing imperative instruction or answering distress calls.

Aforementioned set speed $s\underline{v}$, defined by (22), (25) and (36), differs from the actual vector of speed $\underline{v}$, which is influenced by external disturbances:

$$\underline{v}(t) = s\underline{v}(t) + v\underline{q}(t) \tag{41}$$

where $v\underline{q}(t)$ is vector of additive interference caused by random factors (such as inaccuracy of steering, fluctuation of wind and current, impact of rolling and yawing on average value of heading, etc. as well as systematic errors (such as undetected permanent drift, set of unknown current, errors of setting course or speed, etc).

For the whole time interval $<t_k, t_{k-1}>$ the actual vector of speed can be given as an average

$$\underline{v}_{k,k-1} = s\underline{v}_{k,k-1} + v\underline{q}_{k,k-1} \tag{42}$$

where

$$\underline{vq}_{k,k-1} = \left(\Delta t_{k,k-1}\right)^{-1} \int_{t_{k-1}}^{t_k} \underline{vq}\left(\Delta t\right) d\Delta t \tag{43}$$

and

$$s\underline{v}_{k,k\text{-}1} = \left(\Delta t_{k,k-1}\right)^{-1} \int_{t_{k-1}}^{t_k} s\underline{v}\left(t\right) dt \tag{44}$$

while at the instant t_k

$$v\underline{q}_k = \int_{t_{k-1}}^{t_k} v\underline{q}'\left(t\right) dt + v\underline{q}_{k-1} \tag{45}$$

$$s\underline{v}_k = \int_{t_{k-1}}^{t_k} s\underline{v}'\left(t\right) dt + s\underline{v}_{k-1} \tag{46}$$

In equation (42) the vectors $\underline{v}$, $s\underline{v}$, $v\underline{g}$ cease to be instantaneous values, as in (41), and become mean values for the time interval $\Delta t_{k,k\text{-}1} = t_k - t_{k\text{-}1}$. If for the said time $\Delta t_{k,k\text{-}1}$ the vector of steering does not change, $s\underline{v}(t)$ = const = $s\underline{v}_{k\text{-}1}$, then

$$\underline{v}_{k,k\text{-}1} = s\underline{v}_{k\text{-}1} + v\underline{q}_{k,k\text{-}1} = s\underline{v}_k + s\underline{q}_{k,k\text{-}1} \tag{47}$$

and the effect on position is:

$$\Delta\underline{u}_{k,k\text{-}1} = \Delta t_{k,k\text{-}1}\underline{v}_{k,k\text{-}1} = \Delta t_{k,k\text{-}1}\left(s\underline{v}_{k\text{-}1} + v\underline{q}_{k,k\text{-}1}\right) \tag{48}$$

If a maneuver begins at the moment t_k and lasts till t_{k+1}, that is if the vector of steering

$$s\underline{v}(t) = \text{var} = s\underline{v}_k + \delta\underline{v}(t) \quad \text{for} \quad t_k \le t \le t_{k+1} \tag{49}$$

where

$$\delta\underline{v}(t) = \underline{v}\,[\underline{s}(t_k) + \Delta\underline{s}(t)] - s\underline{v}(t_k) = s\underline{v}(t) - s\underline{v}_k \tag{50}$$

then the resultant change of position

$$\Delta\underline{u}_{k+1,k} = \Delta t_{k+1,k}\left(s\underline{v}_k + v\underline{q}_{k+1,k}\right) + \delta\underline{u}_{k+1,k} \tag{51}$$

where

$$\delta\underline{u}_{k+1,k} = \int_{t_k}^{t_{k+1}} \delta\underline{v}\left(t\right) dt \tag{52}$$

is an additional by-effect of this maneuver.

Equation of state

In order to differentiate the spells of maneuver from time intervals of unchanged settings of speed and course the differential equation of state [19, 23, 24, 29, 41] can be written in the following form

$$\underline{x}^{I} = \mathbf{F}^{(I)}\, \underline{x} + \mathbf{F}_{(I)}\, (\delta\underline{x} + \delta\underline{q}) \tag{53}$$

where

- actual vector of state

$$\underline{x} = [\underline{u}^{T}, \underline{v}^{T}]^{T} \tag{54}$$

- time derivative of state vector

$$\underline{x}^{I} = [(\underline{u}^{I})^{T}, (\underline{v}^{I})^{T}]^{T} \tag{55}$$

- controlled alteration of speed

$$\delta\underline{x} = \begin{bmatrix} s\underline{v}(t) - s\underline{v}(t_0) \\ s\underline{v}^{I}(t) \end{bmatrix} \tag{56}$$

- vector of instantaneous interference (additive noise vector)

$$\delta\underline{g} = \begin{bmatrix} v\underline{q}(t) - v\underline{q}(t_0) \\ v\underline{q}^{I}(t) \end{bmatrix} \tag{57}$$

and

- matrices of system dynamics

$$\mathbf{F}^{(I)} = \begin{bmatrix} \mathbf{0}_{[2x2]} & \mathbf{I}_{[2x2]} \\ \mathbf{0}_{[2x2]} & \mathbf{0}_{[2x2]} \end{bmatrix}, \qquad \mathbf{F}_{(I)} = \begin{bmatrix} \mathbf{0}_{[2x2]} & \mathbf{0}_{[2x2]} \\ \mathbf{0}_{[2x2]} & \mathbf{I}_{[2x2]} \end{bmatrix} \tag{58}$$

in which block elements, zero matrix $\mathbf{0}_{[2x2]}$ and unit matrix $\mathbf{I}_{[2x2]}$, as in (13)
The solution of (53) is

$$\underline{x}(t) = \mathbf{\Phi}(t,t_0)\, \underline{x}(t_0) + s\underline{x}(t,t_0) + \underline{q}(t,t_0) \tag{59}$$

and in this equation

$$\Phi(t, t_0) = \Phi(t - t_0) = \begin{bmatrix} \mathbf{I} & (t - t_0)\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{60}$$

is fundamental transition matrix [12, 24, 24, 29, 41], where **I**, **0** are same as in (58)

$$s\underline{x}(t, t_0) = \begin{bmatrix} s\underline{u}(t) - s\underline{u}(t_0) - (t - t_0)s\underline{v}(t_0) \\ s\underline{v}(t) - s\underline{v}(t_0) \end{bmatrix} \tag{61}$$

is controlled alteration of state (due to maneuvering) and

$$\underline{q}(t, t_0) = \begin{bmatrix} u\underline{q}(t) - u\underline{q}(t_0) - (t - t_0)\, v\underline{q}(t_0) \\ \\ v\underline{q}(t) - v\underline{q}(t_0) \end{bmatrix} \tag{62}$$

is total random disturbance of state vector, where vectors $s\underline{u}$, $u\underline{g}$ in (61,62) are defined as follows

$$s\underline{u}(t) = \int s\underline{v}(t)dt \qquad \text{and} \qquad u\underline{q}(t) = \int v\underline{q}(t)dt \tag{63}$$

Solution (59) of (53) also implies [22] the discrete equation of state

$$\underline{x}(t) = \Phi_{k+1,k}\, \underline{x}_k + s\underline{x}_{k+1,k} + \underline{q}_{k+1,k} \tag{64}$$

(for any chosen period $\Delta t_{k+1,k} = t_{k+1} - t_k$)
with transition matrix

$$\Phi_{k+1,k} = \begin{bmatrix} \mathbf{I} & \Delta t_{k+1,k}\,\mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{65}$$

and remaining components

$$s\underline{x}_{k+1,k} = \begin{bmatrix} s\underline{u}_{k+1} - s\underline{u}_k - \Delta t_{k+1,k}\, s\underline{v}_k \\ s\underline{v}_{k+1} - s\underline{v}_k \end{bmatrix} = \begin{bmatrix} \Delta t_{k+1,k}\left(s\underline{v}_{k+1,k} - s\underline{v}_k\right) \\ s\underline{v}_{k+1} - s\underline{v}_k \end{bmatrix} \tag{66}$$

$$\underline{q}_{k+1,k} = \begin{bmatrix} u\underline{q}_{k+1} - u\underline{q}_k - \Delta t_{k+1,k} v\underline{q}_k \\ \\ v\underline{q}_{k+1} - v\underline{q}_k \end{bmatrix} = \begin{bmatrix} \Delta t_{k+1,k}\left(v\underline{q}_{k+1,k} - v\underline{q}_k\right) \\ \\ v\underline{q}_{k+1} - v\underline{q}_k \end{bmatrix} \tag{67}$$

where
svk+1, k, v$\underline{q}$k+1, k are mean values (for time interval Δtk+1, k) given by (43, 44).
Allowing for relations (66, 67, 68) the discrete state vector can be rewritten as

$$\underline{x}_{k+1} = \begin{bmatrix} \underline{u}_{k+1} \\ \\ \underline{v}_{k+1} \end{bmatrix} = \begin{bmatrix} \underline{u}_k + \Delta t_{k+1,k}\left(s\underline{v}_{k+1,k} + u\underline{q}_{k+1,k}\right) \\ \\ s\underline{v}_{k+1} + v\underline{q}_{k+1} \end{bmatrix} \tag{68}$$

If in the iteration k → (k+1) no maneuver is carried out (i.e. $s\underline{v}_{k+1,k} \equiv s\underline{v}_{k+1} \equiv s\underline{v}_k$) then directly from (64) and (66) ensues the formula

$$\underline{x}_{k+1} = \mathbf{\Phi}_{k+1,k}\, \underline{x}_k + \underline{q}_{k+1,k} \tag{69}$$

which can be deemed describing ship's movement along entire trajectory, on condition however, that every transition of state in waypoints and any other speed or course alteration is instantaneous, and that inaccuracies resulting from this assumption are negligible, $\delta\underline{u} \approx \underline{0}$, or included into vector of disturbances $\underline{q}$.

Equation of observation

In general (e.o.) is non-linear [10, 12, 14, 15, 23, 24, 29, 41]

$$\underline{Y}(t) = \underline{G}(\underline{x}(t),t) + \underline{\varepsilon}(t) \tag{70}$$

and in some particular cases – linear [10, 12, 19, 29]

$$\underline{Y}(t) = \boldsymbol{L}(t)\underline{x}(t) + \underline{\varepsilon}(t) \tag{71}$$

Components of vector $\underline{Y}$ in (70) and (71) are measured quantities $\{y_i\}$ and $\underline{($is vector of measuring errors $\{e_i\}$. For a sequence of chosen moments, $t = t_k$, $k = 1, 2,...,$ and subsequently to linearization [10, 19, 29] of function $\underline{G}$ in (70) both cases of (e.o.) can be written in the same form:

$$\underline{Y}_k = \mathbf{G}_k\, \underline{x}_k + \underline{\varepsilon}_k \tag{72}$$

Where **G** is gradient matrix [6] of measured quantities

$$G = [G_U G_V] = \begin{bmatrix} u\underline{G}_1^T & v\underline{G}_1^T \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ u\underline{G}_m^T & u\underline{G}_m^T \end{bmatrix} \tag{73}$$

and elements of each row $\underline{G}^T$ are components of the following vectors:

$$u\underline{G}_i^T = [\partial y_i / \partial \underline{u}]^T \text{ and } v\underline{G}_j^T = [\partial y_j / \partial \underline{v}]^T \tag{74}$$

The formula (72) is fully adequate for linear equation (71), because Gk $\equiv L(t_k)$, whereas in most typical case of (70) it makes only a linear approximation, which is, however, sufficiently accurate for all practical applications (i.e. for every reasonably requested neighborhoods of an appropriately chosen point of linearization).

Positioning measurements and measurements of speed V or course K in marine navigation are mutually independent and applied separately. For this reason the equation (72) can be replaced by two independent (e.o.), for $\underline{u}$ and $\underline{v}$ separately

$$\underline{Y}_U = \mathbf{G}_U\underline{u} + \underline{\varepsilon}_U \tag{75}$$

or

$$\underline{Y}_V = \mathbf{G}_V \underline{v} + \underline{\varepsilon}_V \tag{76}$$

and what is more, speed V, course K or actual vector of ship movement $\underline{v}$ can be gauged directly:

$$\underline{Y}_V = \underline{v} + \underline{\varepsilon}_V \text{ or } \underline{Y}_V = \begin{bmatrix} K + \varepsilon K \\ V + \varepsilon V \end{bmatrix} \tag{77}$$

(Symbols $\underline{\varepsilon}_u$, $\underline{\varepsilon}_v$, εK, εV in (75-77) denote relevant errors of measurements).

Thus the problem of solving (e.o.) reduces itself to finding solution of (75) or possibly, a solution of general equation (72). For a vector of simultaneous measurements or measurements brought to a common instant

$$\underline{Y} = \mathbf{G}\, \underline{x} + \underline{\varepsilon} \tag{78}$$

or

$$\underline{Y}_U = \mathbf{G}\, \underline{u} + \underline{\varepsilon}_U \tag{79}$$

the respective solutions are:

- observed vector of state

$$\underline{x}^0 = \mathbf{H}\, \underline{Y} \tag{80}$$

or

- position fix

$$\underline{u}^0 = \mathbf{H}\, \underline{Y}_U \tag{81}$$

The matrix of solution, **H**, exists if dimension of vector $\underline{Y}$ is equal or greater than dimension of $\underline{x}$ or $\underline{u}$, respectively. The case of equal dimensions, $\underline{Y}_{[4x1]}$ for $\underline{x}$ or $\underline{Y}_{[2x1]}$ for $\underline{u}$, is the simplest and unambiguous one. The solution matrix is then

$$\mathbf{H} = \mathbf{G}^{-1} \tag{82}$$

If taken measurements outnumber searched coordinates (components of state vector x or coordinates of position u), which means redundant measuring information, then – in order to utilize all the data – a statistical method [3, 5, 8, 9, 10, 13, 16, 20, 29, 31, 36] shall be applied.

The principal one is the least square method, which in its simplest formula yields the following solution

$$\mathbf{H} = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T \tag{83}$$

If more sophisticated statistical methods [3, 10, 12, 23, 29] are to be availed of, then generally

$$\mathbf{H} = \mathbf{H}(\mathbf{G}, \mathbf{Y})_{(M)} \tag{84}$$

where subscript (M) denotes chosen method – and

$$\mathbf{Y} = \mathrm{E}\{\underline{\varepsilon}\ \underline{\varepsilon}^T\} \tag{85}$$

is covariance matrix of measured vector.
In particular, for the least square method with weights [3, 10, 16, 20, 29, 36] the solution is

$$\mathbf{H} = (\mathbf{G}^T\, \Gamma\, \mathbf{G})^{-1} \mathbf{G}^T\, \Gamma \tag{86}$$

where Γ is weight matrix (a specific from of this matrix depends on recognition of statistical parameters of measuring process [3, 8, 24].

Statistically, the best result can be obtained by putting

$$\Gamma = \mathbf{Y}^{-1} \tag{87}$$

and then

$$\mathbf{H} = (\mathbf{G}^T\, \mathbf{Y}^{-1}\, \mathbf{G})^{-1} \mathbf{G}^T \mathbf{Y}^{-1} \tag{88}$$

If matrix Y is diagonal, $\mathbf{Y} \equiv \mathbf{D} \equiv \mathbf{D}_0\mathbf{D}_0$, then method of solving (e.o.) can be simplified [10] by replacing $\mathbf{G}$ with $\mathbf{G}_D = \mathbf{D}_0^{-1}\mathbf{G}$ and $\underline{Y}$ with $\underline{Y}_D = \mathbf{D}_0^{-1}\underline{Y}.$ In consequence

$$\mathbf{H} = (\mathbf{G}_D^T\, \mathbf{G}_D)^{-1} \mathbf{G}_D^T\, \mathbf{D}_0^{-1} = (\mathbf{G}^T\, \mathbf{D}^{-1}\, \mathbf{G})^{-1}\, \mathbf{G}^T\, \mathbf{D}^{-1} \tag{89}$$

Which is a solution identical to (88).

Measurements, which are not simultaneous, have to be brought to a common time instant. Usually, it is the moment of latest measurement or current instant – but any other convenient moment can be chosen as well. Virtually, the updating of measurements concerns solely the (e.o.) for position, (79). Speed and course are being measured (and indicated) incessantly and can be checked at any moment.

Let vector $\underline{Y}^y = [y_{01}, y_{02},..., y_{0,m-1}, y_{0,m}]^T$ denote original series of measurements taken at chosen moments $t_1, t_2,..., t_m$ in position $\underline{u}_1, \underline{u}_2,..., \underline{u}_m$, respectively, where $\underline{u}_i = \underline{u}(t_i)$, and let $\underline{Y} = [y_1, y_2,...y_{m-1}, y_m]^T$ be the vector of simultaneous measurements at given moment t in position $\underline{u} = \underline{u}(t)$, or a vector of updated measurements (that is brought to **this** moment and position). Then

$$\underline{Y} = \underline{Y}^y + \Delta\underline{Y} \cong \mathbf{G}_U\, \underline{u} \tag{90}$$

(with accuracy up to the vector of errors $\underline{\varepsilon}$) which means that for each measurement

$$y_i = y_{0i} + \Delta y_i \cong \underline{G}_i^T\, \underline{u} \quad \text{where} \quad i = 1, 2,..., m \tag{91}$$

(with accuracy up to the error ε_i)

Symbols Δy_i, $\Delta\underline{Y}$ in (90) and (91) stand for true differences of measured (between their original values and the value in reference position $\underline{u}$). Similarly, all the positions of original measurements can be compared with the reference position

$$\underline{u} = \underline{u}_i + \Delta\underline{u}_i, \quad \text{for each} \quad i = 1, 2,..., m \tag{92}$$

If during the time $\Delta t = t\text{-}t_1$ of taking all the measurements $y_{01}, y_{02}, \ldots, y_{0,m}$ ship's true speed does not vary, $\underline{v}$ = const, then

$$\underline{u} = \underline{u}_i + (t - t_i)\,\underline{v} = \underline{u}_i + \Delta t_i\,\underline{v} \tag{93}$$

and subsequently

$$y_i = \underline{G}_i^T\,(\underline{u}_i + \Delta t_i \underline{v}), \quad i = 1, 2, \ldots, m \tag{94}$$

If also gradient of measurand in reference position $\underline{u}$ does not differ from gradients in positions $u_1, u_2, \ldots, u_m$ (or differences are negligible),
that is $\forall \underline{G}_i \cong \underline{G}_{0i}$ and $\underline{G}_i^T \underline{u}_i \cong (\underline{G}_{0i})^T \underline{u}_i \equiv y_{0i}$, then

$$y_i\ (y_{0i} + \Delta t_i \underline{G}_i^T\,\underline{v} \tag{95}$$

for each i = 1, 2,..., m,
and finally, vector of updated measurements

$$\underline{Y} = \underline{Y}^y + \Delta\mathbf{T}\,\mathbf{G}_U\,\underline{v} = \underline{Y}^y + \Delta\underline{Y}_0 \tag{96}$$

where matrix of time differences:

$$\Delta\mathbf{T} = \begin{bmatrix} \Delta t_1 & 0 & ----- & 0 \\ 0 & \Delta t_2 & 0--- & 0 \\ 0 & -------- & & 0 \\ 0----- & & \Delta t_{m-1} & 0 \\ 0------ & & 0 & \Delta t_m \end{bmatrix} \tag{97}$$

For the updated measurements the vector of errors is

$$\underline{\varepsilon} = \underline{\varepsilon}^y + \Delta\mathbf{T}\,\mathbf{G}_U\,v\underline{q} \tag{98}$$

and its covariance matrix is

$$\mathbf{Y} = \mathbf{Y}^y + \Delta\mathbf{T}\,\mathbf{G}_U\,\mathbf{Q}_V\,\mathbf{G}_U^T\,\Delta\mathbf{T} \tag{99}$$

where $\mathbf{Y}^y = E\,\{\underline{\varepsilon}^y\,(\underline{\varepsilon}^y)^T\}$, $\mathbf{Q}_V = E\,\{v\underline{g}\,(v\underline{g})^T\}$ and $\mathbf{Y}$ as in (85). (Equality (98) requires the assumption that $E\,\{v\underline{g}\,(\underline{\varepsilon}^y)^T\}$ ($E\,\{\underline{\varepsilon}^y\,(v\underline{g})^T\} = 0$.)

If however, differences of measured gradients in given positions $\underline{u}, \underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{m-1}, \underline{u}_m$ can not be neglected, $\underline{G}_i = \underline{G}_{0i} + \Delta\underline{G}_i$ and $\Delta\underline{G}_i \neq 0$, (which means that apart from bringing measurements to common time instant they also have to be brought to reference position $\underline{u}$) and if average speed $\underline{v}_i$ within time intervals $\Delta t_{\ i+1,i} = t_{\ i+1} - t_i$ can not be deemed equal to mean speed vector $\underline{v}$ for entire time Δt of taking all measurements, i.e. $\underline{v}_i = \underline{v} + \Delta\underline{v}_i$ ($\Delta\underline{v}_i \neq 0$), then additional corrections must be applied to get the right vector of updated measurements:

$$\underline{Y} = \underline{Y}^y + \Delta\underline{Y}_0 + \Delta\underline{Y}_V + \Delta\underline{Y}_G \tag{100}$$

where (as before)

$\Delta \underline{Y}0 = \Delta \mathbf{T}\, \mathbf{G}_U\, \underline{v}$ (101)

whereas components of vector $\Delta \underline{Y}_V$ are corrections

$\Delta t_i\, \underline{G}_i^T\, \Delta \underline{v}_i$ (102)

and components of $\Delta \underline{Y}_G$

corrections $\Delta \underline{G}_i\, \underline{u}_i$ (103)

Eventually

$y_i = y_{0i} + \Delta t_i\, \underline{G}_i^T\, \underline{v} + \Delta t_i\, \underline{G}_i^T\, \Delta \underline{v}_i + \Delta \underline{G}_i^T\, \underline{u}_i$ (104)

(for every i = 1, 2,... m-1, m)
In case (104) the elements of covariance matrix **Y** will increase due to errors of finding the differences $\Delta \underline{G}_i$ and $\Delta \underline{v}_i$ or – most often in practice – because of omitting them.

If estimation methods [1, 3, 10, 12, 14, 24, 29, 36] are applied in navigation, then:
(I) for vector of state $\underline{x}$ its estimated value $\underline{x}^x$ is generally given by equation

$\underline{x}^x = \underline{x}^z + \mathbf{K}\, \underline{\Delta}$ (105)

where prediction $\underline{x}^z = \underline{x}^x$ (106)

signal of deviation $\underline{\Delta} = \underline{x}^0 - \underline{x}^z$ (107)

and where **K** is matrix of filter amplification [1, 3, 10, 12, 29].
(II) analogically for position $\underline{u}$ its estimate

$\underline{u}^x = \underline{u}^z + \mathbf{K}\, \underline{\Delta}$ (108)

where prediction $\underline{u}^z = \underline{u}^x + \Delta t\, \underline{v}$ (109)

whereas signal of deviation can be

$\underline{\Delta} = \underline{\Delta}_U = \underline{u}^0 - \underline{u}^z$ (110)

or

$\underline{\Delta} = \underline{\Delta}_Y = \underline{Y} - \underline{Y}^z$ for $\underline{Y}^z = \mathbf{G}\, \underline{u}^z$ (111)

(This, of course, means different matrices, $\mathbf{K}_U$, $\mathbf{K}_Y$, for $\underline{\Delta}_U$, $\underline{\Delta}_Y$, respectively.)

In somewhat more detail:
ad (I) – if filtering consecutive vectors of state, then

$\underline{x}^x_k = \underline{x}^z_k + \mathbf{K}_k\, (\underline{x}^0_k - \underline{x}^z_k)$, for $\underline{x}^z_k = \mathbf{\Phi}_{k,k-1}\, \underline{x}^x_k$ (112)

ad (II) – in case of filtering positions

a) if each position fix $\underline{u}^0_k$ is determined from only two measurements, then

$\underline{u}^x_k = \underline{u}^z_k + \mathbf{K}_k\, (\underline{u}^0_k - \underline{u}^z_k)$ and $\underline{u}^z_k = \underline{u}^x_{k-1} + \Delta t_{k,k-1}\, \underline{v}_{k,k-1}$ (113)

b) if measurements (simultaneous or updated) given by $\underline{Y}_k$ are applied altogether, then

$$\underline{u}^x{}_k = \underline{u}^z{}_k + \mathbf{K}_k (\underline{Y}_k - \mathbf{G}_k \underline{u}^z{}_k) \tag{114}$$

c) if single measurements y_k are used successively, then

$$\underline{u}^x{}_k = \underline{u}^z{}_k + \underline{K}_k (y_k - \underline{G}_k{}^T \underline{u}^z{}_k) \tag{115}$$

where $\underline{u}^z{}_k$ as above, or – in case of simultaneous measurements – just $\underline{u}^z{}_k = \underline{u}^x{}_{k-1}$ (and where $\underline{K}_k$, $\underline{G}_k$ are column vectors).
Amplification matrices **K** in (I), (II a, b), and vector $\underline{K}$ for scalar y in (IIc), are subsequent to chosen method of estimation or filtering [1, 3, 5, 10, 12, 23, 24, 29, 36]. Their detailed form depends on statistical parameters of considered process.

Statistical parameters

The navigational process in consideration is a **stochastic** process [8, 12, 24, 29], and its outcome is a **realization** of stochastic process. This also refers to all subprocesses of navigational process, which are formally described by aforementioned (e.s.), (e.c.) and (e.o.).

A random constituent of $\underline{x}$ is vector $\underline{q}$, so

$$\mathbf{Q} = E\ \{\underline{q}\underline{q}\} \tag{116}$$

is covariance matrix of vector x (covariance of state)

If vector of disturbances $\underline{g}$ can be considered the sum

$$\underline{q} = \Delta\underline{q} + c\underline{q} \tag{117}$$

of purely random constituent $\Delta\underline{g}$ and systematic constituent $c\underline{g}$ (unknown permanent drift)
then

$$\mathbf{Q} = \Delta\mathbf{Q} + \mathbf{C} \tag{118}$$

where

$$\Delta\mathbf{Q} = E\ \{\Delta\underline{q}\ \Delta\underline{q}^T\} \tag{119}$$

and

$$\mathbf{C} = E\ \{c\underline{q}\ c\underline{q}^T\} = c\underline{q}\ c\underline{q}^T \tag{120}$$

For the observed vector of state x0 and its error $\underline{r} = \underline{x}^0 - \underline{x}$,
for the prediction of state $\underline{x}^z$ and its error $\underline{z} = \underline{x}^z - \underline{x}$,
and for the state estimate $\underline{x}^x$, and its error $\underline{e} = \underline{x}^x - \underline{x}$
the relevant covariance matrices are

$$\mathbf{R} = E\ \{\underline{r}\ \underline{r}^T\} \tag{121}$$

$$\mathbf{Z} = E\ \{\underline{z}\ \underline{z}^T\} \tag{122}$$

$$\mathbf{P} = E\ \{\underline{e}\ \underline{e}^T\} \tag{123}$$

Covariance matrix **Y** of measurements $\underline{Y}$ (with errors $\underline{\varepsilon}$) has already been defined by (85). Because

$$\underline{r} = \mathbf{H}\,\underline{\varepsilon} \qquad (124)$$

where **H** according to (82), (83), (84) and (86), then it follows that

$$\mathbf{R} = \mathbf{H}\,\mathbf{Y}\,\mathbf{H}^{T} \qquad (125)$$

(For instance, in the simplest case (82) formula (125) yields

$$\mathbf{R} = \mathbf{G}^{-1}\,\mathbf{Y}\,(\mathbf{G}^{-1})^{T} \qquad (126)$$

(and similarly for other cases).

If observed vector of state $\underline{x}^0$ is plain combination of position $\underline{u}^0$ and observed speed $\underline{v}^0$ obtained from independent and direct speed measurements (which, however, are not possible yet for most ship's course and speed gauges in use and of very seldom occurrence even in case of Doppler logs), then covariance matrix of observation:

$$\mathbf{R} = \begin{bmatrix} \mathbf{U} & \mathbf{R}_C \\ \mathbf{R}_C & \mathbf{V} \end{bmatrix} \qquad (127)$$

where

$$\mathbf{U} = E\,\{r\underline{u}\; r\underline{u}^{T}\} \text{ for } r\underline{u} = \underline{u}^0 - \underline{u} \qquad (128)$$

$$\mathbf{V} = E\,\{r\underline{v}\; r\underline{v}^{T}\} \text{ for } r\underline{v} = \underline{v}^0 - \underline{v} \qquad (129)$$

and

$$\mathbf{R}_C = E\,\{r\underline{u}\; r\underline{v}^{T}\}\ (E\,\{r\underline{v}\; r\underline{u}^{T}\} \qquad (130)$$

Usually, position and speed errors, $r\underline{u}$ and $r\underline{v}$, are not correlated, so $\mathbf{R}_C = \mathbf{0}$, and then

$$\mathbf{R} = \begin{bmatrix} \mathbf{U} & \mathbf{0} \\ \mathbf{0} & \mathbf{V} \end{bmatrix} \qquad (131)$$

In case (still prevailing in navigation) of indirect finding $\underline{v}^0$ on the grounds of consecutive position fixes

$$\underline{v}^0_{k,k-1} = (t_k - t_{k-1})^{-1}\,(\underline{u}^0_k - \underline{u}^0_{k-1}) = (\Delta t_{k,k-1})^{-1}\,\Delta\underline{u}^0_{k,k-1} \qquad (132)$$

the observed speed $\underline{v}^0$ (which is an average value for the time interval $\Delta t_{k,k-1}$, between fixes $\underline{u}^0_{k-1}$ and $\underline{u}^0_k$) has the error

$$r\underline{v}_{k,k-1} = (\Delta t_{k,k-1})^{-1}\,(r\underline{u}_k - r\underline{u}_{k-1}) \qquad (133)$$

and in effect

$$\mathbf{R}_k = \begin{bmatrix} \mathbf{U}_k & , & (\Delta t_{k,k-1})^{-1}(\mathbf{U}_k - c\mathbf{R}_{k,k-1}) \\ (\Delta t_{k,k-1})^{-1}(\mathbf{U}_k - c\mathbf{R}_{k,k-1}) & , & (\Delta t_{k,k-1})^{-2}(\mathbf{U}_k + \mathbf{U}_{k-1} - 2c\mathbf{R}_{k,k-1}) \end{bmatrix} \qquad (134)$$

Frequently in navigation practice it can also be assumed that there is no correlation between errors of successive fixes, ($c\mathbf{R}_{k,k-1} = \mathbf{0}$), especially if they are obtained from independent systems of positioning or (in case of the same system) if the lapse of time between them is long enough. This simplifies (135) transforming it into

$$R_k = \begin{bmatrix} \mathbf{U}_k & , & (\Delta t_{k,k-1})^{-1}\,\mathbf{U}_k \\ (\Delta t_{k,k-1})^{-1} & , & (\Delta t_{k,k-1})^{-2}\,(\mathbf{U}_k + \mathbf{U}_{k-1}) \end{bmatrix} \tag{135}$$

The basis for actual prediction $\underline{x}$ can be last observation of state

$$\underline{x}^z_k = \mathbf{\Phi}_{k,k-1}\,\underline{x}^0_{k-1} \tag{136}$$

or last estimation of state

$$\underline{x}^z_k = \mathbf{\Phi}_{k,k-1}\,\underline{x}^x_{k-1} \tag{137}$$

and thus covariance matrix of prediction is either

$$\mathbf{Z}_k = \mathbf{\Phi}_{k,k-1}\,\mathbf{R}_{k-1}\,\mathbf{\Phi}_{k,k-1}^T + \mathbf{Q}_{k,k-1} \tag{138}$$

or

$$\mathbf{Z}_k = \mathbf{\Phi}_{k,k-1}\,\mathbf{P}_{k-1}\,\mathbf{\Phi}_{k,k-1}^T + \mathbf{Q}_{k,k-1} \tag{139}$$

The elements of matrices **R** and **P** are variables dependent on measurement accuracy and a number of redundant measurements as well as on the way of their averaging, that is on the applied method of estimation or data filtering.

MATHEMATICAL MODELS OF THE PROCESS

In navigation, a global description of considered process is the principal one, as it comprises all the sea voyage – not only a short segment of it. A description on local scale level can solely be a supplement. From practical point of view it is pure theoretical description having no reference to navigation practice – where, at present, there do not exist possibilities of determining 3-dimensional vector of ship movement and gauging (separately and precisely) linear or angular accelerations affecting vessel's hull.

A description based on equations (10,11) and (12) could prove useful at simulating the effects of wind and sea action and by-effects of operating propeller and rudder (in order to find out their real impact on ship's movement) as well as at simulation of various maneuvers in different conditions (to assess more accurately their final outcome). It might broaden the possibilities of ascertaining and updating ship's characteristics of maneuverability.

Discrete models

A basis for a formal description of sea voyage is the set of equations: (7), (8) and 9). The most general and at the same time the simplest version of such description is

FUNDAMENTAL DISCRETE LINEAR MODEL

resolving itself to movement of point along trajectory consisting of final number of rhumbline legs, where alterations of speed vector (at waypoints) are instantaneous.

The equations constituting this model are: (7), (8), (16), (17), (25), (72), (75, 76, 77) and (80).

The inclusion of maneuver yields

DISCRETE LINEAR MODEL with MANOEUVRE

described by the same equations as the fundamental model adding equations (36) and (32, 33) or (36) together with (37) and (38, 39, 40) (which is a simpler variant – but maybe more practical).

Models with continuous time

Next step towards more detailed models (i.e. further shift from global to local scale of processes description) means transition to non-linear models with continuous time. The starting point for their construction is relation between speed and position $\underline{u}^{I} = \underline{v}$, that is (9), which immediately implies a differential (e.s).

MODEL with DIFFERENTIAL EQUATION of STATE

has similar set of descriptive equations as the model with maneuver but with principal substitution of (59) and (22) instead of (36). Still, however, the (e.o.) with discrete time, (72), is applicable – as it reflects the specific of maritime navigation, where a system of continuous positioning does not exist yet.

The existing satellite and radionavigation systems provide position fixes frequently but not continuously (although it would be feasible for them). They can be considered the systems of continuous positioning in the global scale of vessel's movement (i.e. within intervals of many hours or days).

Supplementing the differential model with equations

(72) and (71) makes COMPLETE CONTINUOUS MODEL

where in equation (72) the discrete time variable is to be replaced by continuous one.

General solution (80) remains valid for (72) with continuous time, providing that all measurements constituting vector $\underline{Y}$ (its components) are taken simultaneously.

It must be noted, however, that for navigation systems and instruments being in use nowadays a model with continuous (e.o.) is not adequate. Such equation describes rather variations of measured value than a real measuring process.

Possible extensions of a model

A mathematical model of NAVIGATIONAL PROCESS could also include equations (9) and (12) – though in a simplified form, for some chosen cases of external stimulating forces and with solutions reduced to 2-dimensions.
For the time being a full and complex application of these equations is restricted (in maritime navigation) by the scope and practical possibilities of taking measurements. Therefore there is no necessity to extend presented models of navigational process by introducing 3-dimensional vectors of speed and position or 3-dimensional global coordinates.

LIST of SYMBOLS and ABBREVIATIONS

In general, underlined letters denote vectors, bold capital letters – matrices, other letter symbols stand for scalar values or are some abbreviations.
(Superscript ^{T} marks transposition of a vector or matrix.)

Letter symbols:

α - rudder angle
b, a - global coordinates
B, a - coefficients of coordinate conversion
C - matrix of permanent drift
Cr - turning radius
$\underline{d}$, $\underline{d}_0$, d. - navigation data
d, ∂ - differential, partial differential
Δ, δ, - difference, increment, error
D, D₀ - diagonal covariance matrices
D*{}, $\underline{D}$*, $\underline{D}$. – decision function/operator, decision, imperative orders
$\underline{E}$ - estimate error
E{} - expected value operator (mathematical expectation function)
ε, $\underline{\varepsilon}$ - measuring error (scalar), vector of measuring errors
$\underline{F}_M$ - vector of external moments and forces (external stimulation vector)
$\underline{F}$ - resultant force vector
φ,λ - geographic coordinates
F - matrix of system dynamics
Φ - transition matrix
Φ, $\Delta\Phi$ - angle of heal, angular amplitude of roll
$\underline{G}$ () - non-linear vector function
$\underline{G}$, **G** - gradient of measured quantity matrix of gradients
$\underline{\gamma}$ - vector of (linear or angular) acceleration
Γ - matrix of statistic weights
$\underline{h}$ - momentum vector
H - matrix of statistic weights
I - moment of inertia, moment of deviation
I - unit matrix
$\underline{J}$ - moment of momentum

$\underline{J}$	- matrix of moments of inertia
K, kK	- course, heading angle
$\underline{K}$, **K**	- filter amplification: vector, matrix
κ	- propeller slip ratio
L	**-** linear matrix in equation of observation
m, **M**	- ship's weight (mass), general mass matrix
$\underline{M}$	- vector of moments
N	- propeller revolutions
η	- propeller pitch
ΔN, $\Delta\eta$	- increase or reduction of revolutions, alteration of propeller pitch
$\underline{0}$, **0**	- null vector, zero matrix (with all elements equal zero)
0_0	- centre of local coordinate system
O_1, O_2, O_3 – axes of local coordinate system	
ω, $\underline{\omega}$	- angular speed (rate of turning)
Ω, $\Delta\Omega$	- angular displacement (angle of turn), angle of yawing
P	- estimate covariance matrix
P*	- priority
$\underline{g}$, **Q**	- vector of disturbances and its covariance matrix
$\underline{r}$, **R** -	- error of observed (gauged) state coordinates and its covariance matrix
$\underline{s}$, $\Delta\underline{s}$	- steering (i.e. setting course and speed), alteration of steering
t, Δt	- time, time interval (time difference)
τ, $\Delta\tau$	- time constants
Δ**T**	**-** matrix of time differences
T, ΔT	- ship's draught, trim
$\underline{\theta}$	- vector of observations (monitored but not gauged quantities, approximate values or qualitative data)
ρ, $\Delta\rho$	- ship's squat, change of trim (due to squat)
$\underline{u}$, $\underline{u}^0$, $\underline{u}^z$, $\underline{u}^x$ – true position, position fix, dead reconed position, estimated position	
$\underline{U}$	- position covariance matrix
$\underline{v}$, $\underline{v}^0$	- two-dimensional speed vector: true, observed (gauged)
V, $\underline{V}$	- scalar speed, three-dimensional vector of speed
V	- covariance matrix of speed vector
$\underline{w}$	- way point
$\mathbf{W}_0$	- initial route (trajectory) matrix
W	- current route (trajectory) matrix
$\underline{W}^T$, $\underline{W}$	- row, column of route matrix
$\underline{x}$, $\underline{x}^0$, $\underline{x}^z$, $\underline{x}^x$ – vector of state: true, observed (in a way of measuring), predicted, estimated	
y, $\underline{Y}$, $\underline{Y}^y$, $\underline{Y}^z$ – measurement (scalar), vector of measurements: simultaneous, non-simultaneous, predicted	
Y, $\mathbf{Y}^y$	- covariance matrices of simultaneous and non-simultaneous measurements
$\underline{z}$, **Z**	**-** error of dead reconing or prediction and its covariance matrix

Abbreviations:

BM – bending meoments
F_{SH} – shearing forces
GM – metacentric height
DR – dead reconed position
PF – position fix
(e.c.) – equation of control
(e.s.) – equation of state
(e.o.) – equation of observation
SAR – search and rescue
GMDSS – global maritime distress and safety system
GPS – global positioning system

(Symbols B^0 and S^0 stand for wind force scale and sea state scale – respectively.)

REFERENCES

1. Anderson B.D.O., Moor J.B, Optimal filtering (in Polish). WNT, Warszawa 1984.
2. Biernacki F., Introduction to the cartographic projections (in Polish). PWN, Warszawa 1973.
3. Deutsch R., Theory of estimation (in Polish). PWN, Warszawa 1969.
4. Dudziak J., Theory of the ship (in Polish). WM, Gdańsk 1988.
5. Filipowicz W., The using of last square method for integrated navigation purposes (in Polish). Zeszyty Naukowe WSM, Gdynia 1980.
6. Gantmacher F.R., The theory of matrices, v.1 & 2, Chelsea Publishing Company, New York 1959.
7. Giertowski J., Meissner T., Terrestrical navigation (in Polish). WM, Gdynia 1963.
8. Goldberger A., Theory of the econometrics (in Polish). PWE, Warszawa 1972.
9. Goodwin E.M., Kemp J.F., Marine statistics theory & practice. Stanford Maritime, London 1979.
10. Górski S., Jackowski K., Urbański J., Estimation of the accuracy of the navigation (in Polish). WSM, Gdynia 1990.
11. Holec M., Tymański K., Introduction to the meteorology and meteorological navigation (in Polish). WM, Gdańsk 1973.
12. Jackowski K., The analyse and evaluation of possibilities of applying navigational algorithms of filtration and prediction (in Polish). (doctoral thesis), PG, Gdańsk 1986.
13. Jackowski K., The method's errors in marine navigation (in Polish). WSM, Gdynia 1983.
14. Jackowski K., Algorithms of filtering in the process of navigation (in Polish). Zeszyty Naukowe PG, Gdańsk 1982.
15. Jackowski K., Urbański J., About the formal description of the process of navigation (in Polish). IX Konferencja N-T Sekcji Komitetu Geodezji PAN, AMW, Gdynia 1994.
16. Jurdziński M., Szczepanek Z., Astronavigation. WM, Gdańsk 1978.
17. Jurdziński M., Szymoński M., Satellite radionavigation. WM, Gdańsk 1978.
18. Kabaciński J., Stability and sunk-resistibility of the vessel (in Polish). WSM, Szczecin 1992.

19. Kaczorek T., Theory of the multidimensional linear, dynamic systems (in Polish). WNT, Warszawa 1983.
20. Krasavtsev B., Khlustin B., Nautical astronomy (in Russian). Mir Publishers, Moskwa 1970.
21. Lekki W., Guide for manoeuvring of the ship (in Polish). WM, Gdańsk 1977.
22. Leavy H., Lessman F., Differential equation (in Polish). PWN, Warszawa 1966.
23. Mańczak K., Methods of identification of multidimensional objects of steering (in Polish). WNT, Warszawa 1977.
24. Nowakowski J., Porębski W., Stochastic theory of steering (in Polish). Wydawnictwo PG, Gdańsk 1981.
25. Nowakowski J., Sobczak W., Theory of information. WNT, Warszawa 1971.
26. Nowicki A., Knowledge about the manoeuvring of the sea vessels (in Polish). WM, Gdańsk 1978.
27. Perepeczko A., Ship steering gears (in Polish). WM, Gdańsk 1983.
28. Posiła J., Urbański J., Navigation in difficult conditions (in Polish). WSMW, Gdynia 1977.
29. Riwkin S.S., Iwanowskij R.I., Kostrov A.W., Statistical optimisation of the navigation systems (in Russian). Sudostrojenije, Leningrad 1976.
30. Sobczak W., Malina W., Methods of selection of the information (in Polish). WNT, Warszawa 1978.
31. Szydłowski H., Theory of the measurement (in Polish). PWN, Warszawa 1981.
32. Urbański J., Banachowicz A., Methods of computation of the coordinates of the position in marine navigation (in Polish). Geodezja i Kartografia, t. XXXVIII, no. 155, 1989.
33. Urbański J., Czapczyk M., Foundation of navigational cartography and geodesy (in Polish). WSM, Gdynia 1988.
34. Urbański J., Jackowski K., About the flow and navigation data processing (in Polish). IX Konferencja N-T Sekcji Komitetu Geodezji PAN, AMW, Gdynia 1994.
35. Urbański J., Kopacz Z., Posiła J., Assessment of the accuracy of the navigation (in Polish). WSMW, Gdynia 1977.
36. Urbański J., Kopacz Z., Posiła J., Marine navigation (in Polish). WM, Gdańsk 1979.
37. Wald A., Statistical decision functions. J. Wiley & Sons, New York 1958.
38. Wereszczyński J., Mathematical foundation of the radiodetermination of the position (in Polish). PWN, Warszawa 1974.
39. Wierzbicki A., Models and sensibility of the steering systems (in Polish). WNT, Warszawa 1977.

Received March 2001
Reviewed August 2001