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MANOEUVRING TO REQUIRED APPROACH PARAMETERS - DISTANCE, TIME AND BEARINGS

ABSTRACT Formulae for calculation of own speed and course to achieve a required distance, time and bearings in respect to a selected object are derived and their graphical interpretation is provided.

INTRODUCTION

The predicted object distance of approach D and the time interval to its occurrence T_D has been first defined in [Lenart, 1990] as new collision avoidance parameters "safe distance" and "time to safe distance". The scope of this paper is aimed at the problem which although it can be and it is connected with collision avoidance manoeuvres, but it is rather reversed and can be applied for intentional approaches or in naval tactical manoeuvres - what own speed and/or course manoeuvre should be undertaken to achieve the required distance and/or time and the true or relative bearing to the object?

ASSUMPTIONS AND INPUT PARAMETERS

For the purposes of this analysis, own vessel and extraneous objects of interest are regarded as if the mass of each object was concentrated at a point. It will be assumed that all moving external objects are travelling at constant speed and course. In the movable plane tangential to the Earth's surface Cartesian coordinates system Ox, Oy (Fig. 1) with Oy pointing North O is at the present position of own vessel. It will also be assumed that manual plots or the radar processing and tracking has yielded the present relative position of the extraneous object X, Y and components of its true V_{tx}, V_{ty} or relative V_{rx}, V_{ry} speed. The relationship of the own and the object speeds can be described by equations

$$V_{tx} = V_{rx} + V_x \qquad (1)$$

$$V_{ty} = V_{ry} + V_y \qquad (2)$$

where: V_x, V_y - own speed components,

$$V_x = V \sin \psi \qquad (3)$$

$$V_y = V \cos \psi \qquad (4)$$

$$V = \sqrt{V_x^2 + V_y^2} \qquad (5)$$

ψ- own course (the angle measured clockwise from Oy to V).

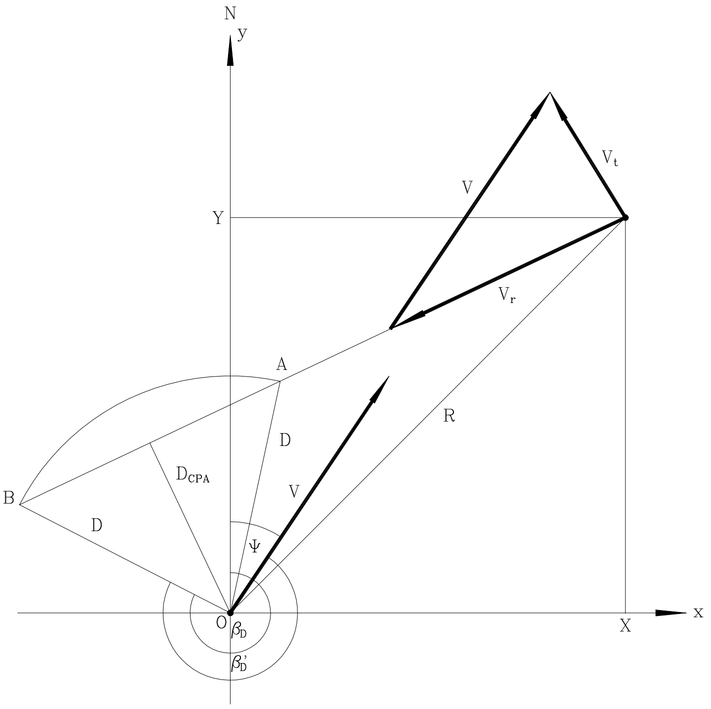

Fig. 1. Input parameters

The own and the object's motion parameters should be either ground or sea referenced and a drift angle is assumed to be zero.

EQUATION OF RELATIVE MOTION

The relative position of an extraneous object, at time t, is given by

$$X(t) = X + V_{rx}\,t \qquad (6)$$

$$Y(t) = Y + V_{ry}\,t \qquad (7)$$

and the relative distance D at time T_D

$$D = \sqrt{X^2(T_D) + Y^2(T_D)} \qquad (8)$$

therefore equation of relative motion has the form

$$V_r^2 T_D^2 + 2(XV_{rx} + YV_{ry})T_D + R^2 - D^2 = 0 \qquad (9)$$

where:

$$V_r = \sqrt{V_{rx}^2 + V_{ry}^2} \qquad (10)$$

$$R = \sqrt{X^2 + Y^2} \qquad (11)$$

DERIVATION OF EQUATION D = f(V_{RX}, V_{RY}, T_D)

From equation (9) we get

$$D = \sqrt{R^2 + (V_r T_D)^2 + 2(XV_{rx} + YV_{ry})T_D} \qquad (12)$$

This equation yields the distance D to the selected object after time T_D.

DERIVATION OF EQUATION T_D = f(V_{rx}, V_{ry}, D)

Solving a quadratic equation in T_D (9) we obtain

$$T_D = \frac{-(XV_{rx} + YV_{ry}) \pm \sqrt{(DV_r)^2 - (XV_{ry} - YV_{rx})^2}}{V_r^2} \qquad (13)$$

or [Lenart, 1990]

$$T_D = T_{CPA} \pm \frac{\sqrt{D^2 - D_{CPA}^2}}{V_r} \qquad (14)$$

where: D_{CPA}, T_{CPA} - distance and time to the closest point of approach,
and real solutions exist if

$$D \geq D_{CPA} \qquad (15)$$

Equation (13) gives time T_D to achieve the distance D to the selected object.

EQUATION OF TRUE MOTION

From equations (1) through (4)

$$V_{rx} = V_{tx} - V \sin\psi \qquad (16)$$

$$V_{ry} = V_{ty} - V \cos\psi \qquad (17)$$

Substitution in equation (9) and rearranging yields equation of true motion

$$(V^2+V_t^2)T_D^2-2[(X+V_{tx}T_D)\sin\psi+(Y+V_{ty}T_D)\cos\psi]VT_D+2(XV_{tx}+YV_{ty})T_D+R^2-D^2=0$$

(18)

where:

$$V_t = \sqrt{V_{tx}^2 + V_{ty}^2} \tag{19}$$

DERIVATION OF EQUATION D = f(V, ψ, T_D)

From equation (18) we get

$$D = \sqrt{(VT_D)^2 - 2(A_{Td}\sin\psi + B_{Td}\cos\psi)VT_D + C_{Td}} \tag{20}$$

where:

$$A_{Td} = X + V_{tx}\,T_D \tag{21}$$

$$B_{Td} = Y + V_{ty}\,T_D \tag{22}$$

$$C_{Td} = R^2 + (V_t T_D)^2 + 2(XV_{tx} + YV_{ty})T_D \tag{23}$$

This equation is an equivalent to equation (12) but for true motion.

DERIVATION OF EQUATION T_D = f(V, ψ, D)

Solving a quadratic equation in T_D (18) we obtain

$$T_D = \frac{-B_{V\psi} \pm \sqrt{B_{V\psi}^2 - A_{V\psi}(R^2 - D^2)}}{A_{V\psi}} \qquad 24)$$

where:

$$A_{V\psi} = V^2 + V_t^2 - 2V(V_{tx}\sin\psi + V_{ty}\cos\psi) = V_r^2 \tag{25}$$

$$B_{V\psi} = XV_{tx} + YV_{ty} - (X\sin\psi + Y\cos\psi)\,V \tag{26}$$

Equation (24) is an equivalent to equation (13) but for true motion.

DERIVATION OF EQUATION V = f(ψ, D, T_D)

Solving a quadratic equation in V (18) we obtain

$$V = A_{VTd}\sin\psi + B_{VTd}\cos\psi \pm \sqrt{(A_{VTd}\sin\psi + B_{VTd}\cos\psi)^2 - C_{VTd}} \tag{27}$$

where:

$$A_{VTd} = V_{tx} + \frac{X}{T_D} = \frac{A_{Td}}{T_D} \tag{28}$$

$$B_{VTd} = V_{ty} + \frac{Y}{T_D} = \frac{B_{Td}}{T_D} \tag{29}$$

$$C_{VTd} = V_t^2 + \frac{2(XV_{tx} + YV_{ty})}{T_D} + \frac{R^2 - D^2}{T_D^2} = \frac{C_{Td} - D^2}{T_D^2} \qquad (30)$$

Real solutions exist if

$$(A_{VTd} \sin\psi + B_{VTd} \cos\psi)^2 \geq C_{VTd} \qquad (31)$$

Equation (27) can yield up to two speeds $V \geq 0$ which own vessel must adopt to achieve the required distance D at the required time T_D (in respect to the selected object) for different assumed own courses ψ.

A graphical interpretation of solutions given by equation (27) can be obtained in Cartesian coordinates of own speed V_x, V_y substituting in equation (18) equations (3) and (4)

$$(V_x - A_{VTd})^2 + (V_y - B_{VTd})^2 = \left(\frac{D}{T_D}\right)^2 \qquad (32)$$

The above equation reveals that the locus of points for which D and T_D is a constant is a circle centred at (A_{VTd}, B_{VTd}) and having radius $|D/T_D|$.

A conventional PPI displays the position of each object by plotting them in polar (r, ψ) or Cartesian (x, y) coordinates. If we apply a scaling factor τ to the speed coordinates (V, ψ) or (V_x, V_y) such that

$$r = V\tau \qquad (33)$$

$$x = V_x \tau \qquad (34)$$

$$y = V_y \tau \qquad (35)$$

then the position and speed coordinates can be plotted on a common display.

In the combined coordinates frame for plotting position and speed can also be plotted positions and speed vectors of objects and the own speed vector (real or simulated). Figure 2 illustrates a family of circles (32) for various required D and T_D and an exemplary object as well as, for comparison, circles T_{CPA}=const. and line D_{CPA}=0 derived in [Lenart, 1983, 1986 or 1999].

It can be proved [Lenart, 1983, 1986], that if for an assumed own course ψ exists V=f(ψ, D_{CPA}=0) with T_{CPA}>0 (own speed which will lead to a collision) then for V<V=f(ψ, D_{CPA}=0) the object will pass ahead (D_c>0), and for V>V=f(ψ, D_{CPA}=0) the object will pass astern (D_c<0).

It should be noted from equations (13), (14) and (24) that there can exist two times of approach at distance D: shorter - approach at point A (Fig. 1) and longer - approach at point B. If the earlier (the first) approach is interesting for us only then, for this time the condition T_{CPA}>T_D for selected own motion parameters V, ψ should be fulfilled. This criterion fulfils points of circle D, T_D which lie inside the circle T_{CPA} for the same time (marked in Fig. 2 by the thicker line).

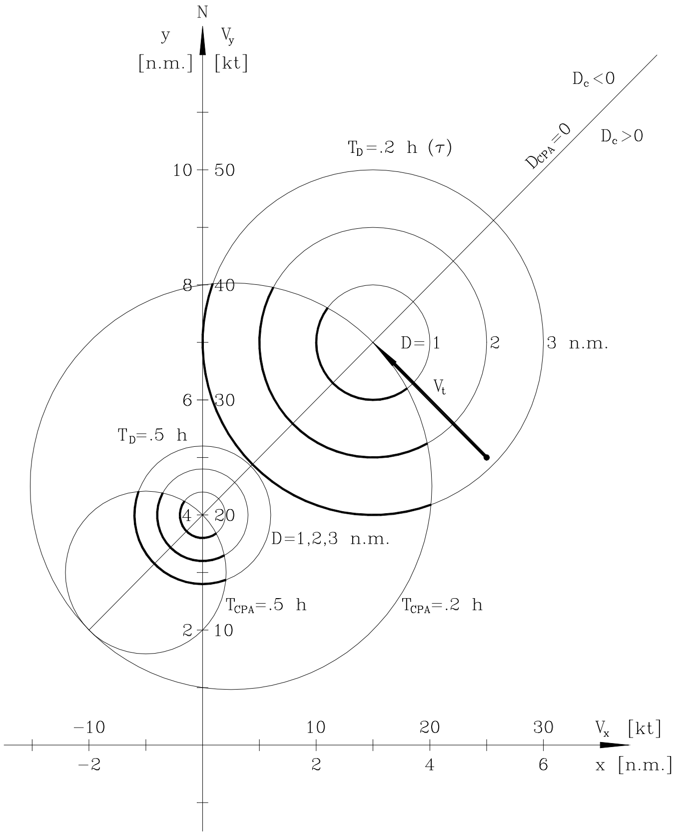

Fig. 2. Circles D, T_D=const.
τ=0.2 h, X=Y=5 n.m., V_{tx}= -10 kt, V_{ty}=10 kt

DERIVATION OF EQUATION $\psi = g(V, D, T_D)$

If we search for own course (which will lead to the required distance D at the required time T_D and at an assumed own speed V then we can get an inverse function ψ=g(V, D, T_D) to the function V=f(ψ, D, T_D) by substitution in equation (18) trigonometric identities

$$\sin\psi = \frac{2\tan\frac{\psi}{2}}{1+\tan^2\frac{\psi}{2}} \tag{36}$$

$$\cos\psi = \frac{1-\tan^2\frac{\psi}{2}}{1+\tan^2\frac{\psi}{2}} \tag{37}$$

which will result in equation

$$(V^2 + 2B_{VTd} + C_{VTd})\tan^2\frac{\psi}{2} - 4A_{VTd}V\tan\frac{\psi}{2} + (V^2 - 2B_{VTd} + C_{VTd}) = 0 \quad (38)$$

and its solution

$$\tan\frac{\psi}{2} = \frac{A_{VTd} \pm \sqrt{A_{VTd}^2 + B_{VTd}^2 - C_{\psi VTd}^2}}{B_{VTd} + C_{\psi VTd}} \quad (39)$$

where:

$$C_{\psi VTd} = \frac{1}{2V}\left(V^2 + V_t^2 + \frac{2(XV_{tx} + YV_{ty})}{T_D} + \frac{R^2 - D^2}{T_D^2}\right) = \frac{1}{2}\left(V + \frac{C_{VTd}}{V}\right) \quad (40)$$

Real solutions exist if

$$A_{VTd}^2 + B_{VTd}^2 \geq C_{\psi VTd}^2 \quad (41)$$

and equation (39) can give up to two own courses ψ which will lead to the required distance D at the required time T_D and at an assumed own speed V.

DERIVATION OF EQUATIONS β_D, $\beta'_D = g(V, \psi, D, T_D)$

It may be noted from Fig. 2 that there is unlimited number of sets of own speed and course V, ψ (or V_x, V_y) which can lead to the required distance D at the required time T_D, therefore we can consider the next approach parameters such as true and relative bearings to the object at the distance D.

At the distance D the relative position of the object (in respect to our vessel) is (X_D, Y_D) or in polar coordinates (D, β_D) or (D, β'_D) where β_D and β'_D are true and relative bearings to the object at this distance respectively. These parameters are given by equations

$$X_D = X + V_{rx} T_D = X + (V_{tx} - V \sin\psi) T_D \quad (42)$$

$$Y_D = Y + V_{ry} T_D = Y + (V_{ty} - V \cos\psi) T_D \quad (43)$$

$$D = \sqrt{X_D^2 + Y_D^2} \quad (44)$$

$$\tan\beta_D = \frac{X_D}{Y_D} \quad (45)$$

$$\beta'_D = \beta_D - \psi \quad (46)$$

therefore

$$\tan\beta_D = \frac{X + (V_{tx} - V\sin\psi)T_D}{Y + (V_{ty} - V\cos\psi)T_D} \quad (47)$$

and

$$\tan(\beta'_D + \psi) = \frac{X + (V_{tx} - V\sin\psi)T_D}{Y + (V_{ty} - V\cos\psi)T_D} \tag{48}$$

Equations (47) and (48) calculate true and relative bearings to the object at the distance D determinated by equation (40) or (20) after time T_D.

DERIVATION OF EQUATIONS V, ψ = f(D, T_D, β_D)

Equations (42) and (43) may be rewritten as

$$X + V_{rx} T_D = D \sin\beta_D \tag{49}$$

$$Y + V_{ry} T_D = D \cos\beta_D \tag{50}$$

and from the above system of equations

$$V_{rx} = \frac{D\sin\beta_D - X}{T_D} \tag{51}$$

$$V_{ry} = \frac{D\cos\beta_D - Y}{T_D} \tag{52}$$

and with regard to equations (1) through (5)

$$V_x = V_{tx} + \frac{X - D\sin\beta_D}{T_D} \tag{53}$$

$$V_y = V_{ty} + \frac{Y - D\cos\beta_D}{T_D} \tag{54}$$

$$V = \sqrt{V_x^2 + V_y^2} \tag{55}$$

$$\tan\psi = \frac{V_x}{V_y} \tag{56}$$

Equations (53) through (56) give own speed V and own course ψ which will lead to the required distance D at the required time T_D and the required true bearing β_D.

DERIVATION OF EQUATIONS V, ψ = f(D, T_D, β'_D)

Equations (42) and (43) may be rewritten as

$$X + (V_{tx} - V\sin\psi)\, T_D = D \sin(\beta'_D + \psi) \tag{57}$$

$$Y + (V_{ty} - V\cos\psi)\, T_D = D \cos(\beta'_D + \psi) \tag{58}$$

This system of equations has solutions

$$\tan\frac{\psi}{2} = \frac{-B_{Td} \pm \sqrt{A_{Td}^2 + B_{Td}^2 - (D\sin\beta'_D)^2}}{A_{Td} + D\sin\beta'_D} \tag{59}$$

$$V = \frac{B_{Td} - D\cos(\beta'_D + \psi)}{T_D \cos\psi} \tag{60}$$

or

$$V = \frac{(B_{Td} + D\cos\beta')\tan^2\frac{\psi}{2} + 2D\sin\beta'\tan\frac{\psi}{2} + B_{Td} - D\cos\beta'}{(1-\tan^2\frac{\psi}{2})T_D} \tag{61}$$

Real solutions exist if

$$A_{Td}^2 + B_{Td}^2 \geq (D\sin\beta'_D)^2 \tag{62}$$

Equations (59) through (61) can give own speed $V \geq 0$ and own course ψ which will lead to the required distance D at the required time T_D and the required relative bearing β'_D.

TIME TO MANOEUVRE

It has to be emphasised that the calculated above manoeuvres are kinematic and should be undertaken immediately. If we require to have the time lapse Δt for calculations, for the decision to initiate a manoeuvre and for the execution of the calculated manoeuvre then X, Y in the above equations should be replaced by $X_{\Delta t}$, $Y_{\Delta t}$ respectively, given by equations

$$X_{\Delta t} = X + V_{rx}\Delta t \tag{63}$$

$$Y_{\Delta t} = Y + V_{ry}\Delta t \tag{64}$$

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