Archives of Hydro-Engineering and Environmental Mechanics
Vol. 48 (2001), No. 4, pp. 23–37

Boundary Problem for Equations of Steady Gradually Varied Flow in Open Channels

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(Received December 3, 1999; revised October 23, 2000)

Abstract

In this paper the problem of solution of ordinary differential equations describing a steady, gradually varied flow is discussed. It is shown that, apart from the initial problem usually solved for open channels, the formulation of the boundary problem is necessary when water levels are imposed at ends of channel. This approach is the mathematically correct formulation of the solution problem for steady, gradually varied flow equations. It enables us to determine directly the water profile and discharge for a single channel, as well as for channel network, instead of the trial and error method usually used. Moreover the formulation of boundary problem with respect to the Manning coefficient and lateral inflow is presented. To solve the listed problems the finite differences method is used.

1. Introduction

The different flow problems in open channels are usually analysed using the Saint-Venant equations, the most general mathematical model widely applied in hydraulic engineering. In spite of this a solution of steady gradually varied flow (SGVF) seems necessary in many practical situations. A governing equation describing this kind of flow can be obtained in a different way (French 1985). SGVF as a particular case of unsteady flow is described by the equation which can be derived from the system of Saint-Venant equations. For steady flow this system can be simplified as derivatives over time do not exist. Consequently we have a continuity equation in the form of:

$$\frac{dQ}{dx} = q \tag{1}$$

and a momentum equation as follows:

$$\frac{d}{dx}\left(\frac{\alpha Q^2}{A}\right) + g A \frac{dH}{dx} = -g A S \tag{2}$$

where:

x	–	longitudinal distance,
$Q(x)$	–	discharge,
$H(x)$	–	water surface elevation,
q	–	lateral inflow,

$$S = \frac{n^2\,|Q|\,Q}{R^{4/3}A^2} \quad - \text{ slope friction,} \tag{3}$$

$A(x)$	–	wetted cross-sectional area,
$R(x)$	–	hydraulic radius,
n	–	Manning coefficient,
g	–	gravitational acceleration,
α	–	Coriolis coefficient.

In Eq. (2) the term representing the influence of lateral inflow is omitted. After differentiating its first term one obtains the expression:

$$\alpha\frac{Q}{A}\left(\frac{1}{A}\frac{d\,Q}{d\,x} - \frac{Q}{A^2}\frac{d\,A}{d\,x}\right) + g\frac{d\,H}{d\,x} = -g\,S \tag{4}$$

which with Eq. (1) can be rearranged to its typical form:

$$\frac{dH}{dx} = \frac{S + \frac{\alpha\,Qq}{gA^2}}{\frac{\alpha Q^2 B}{gA^3} - 1}. \tag{5}$$

A discharge $Q(x)$ can be obtained by integration of Eq. (1) over distance:

$$Q\,(x) = Q\,(x = 0) + \int_0^x q dr \tag{6}$$

where r is a dummy parameter.

For SGVF another form of governing equation can be derived. To this order Eq. (2) is rearranged as follows:

$$\frac{d}{d\,x}\left(H + \frac{\alpha\,Q^2}{2\,g\,A^2}\right) = -\,S. \tag{7}$$

Since, the expression in brackets represents the total flow energy E above an accepted datum, one finally obtains:

$$\frac{d\,E}{d\,x} = -\,S \tag{8}$$

with

$$E = H + \frac{\alpha Q^2}{2gA^2}. \tag{9}$$

This equation, like Eq. (5), describes a flow profile along the channel axis. A discharge variation is defined as preceding, by Eq. (6).

SGVF in an open channel can be considered as an initial value problem or a boundary value problem for ordinary differential equations. When discharge Q is imposed at one end of channel and lateral inflow $q$is given, an initial condition $H(x = x_0) = H_0$ should be known to compute $H(x)$ for $x > x_0$ by integration of Eq. (5) or Eq. (8). The problem formulated in this manner is a so-called initial problem for an ordinary differential equation. A typical example of its application in an open channel hydraulics is determination of the backwater effect due to a dam. The computation of SGVF is usually carried out by standard-step method (Chow 1959) or by numerical integration of governing ordinary differential equation (Chaudhry 1993). The results obtained in each way are almost identical (Schulte and Chaudhry 1987). It is obvious as it can be shown that both approaches are equivalent.

While the solution of an initial problem for SGVF in a single channel is trivial, it becomes more complicated in a channel network. The approach using a discrete energy equation was presented by Schulte and Chaudhry (1987) for a looped network. This method corresponds to the standard step method (Chow 1959) for a single channel completed by continuity and energy equations at the junctions. Recently, Naindu, Murty Bhallamudi and Narasimhan (1997) have proposed a procedure basing on the governing differential equation in its standard form i.e. without lateral inflow and with depth as a dependent variable. An analysis presented by these Authors shows that the presented method is computationally more efficient, but it can be used to compute the water surface profile in tree-type channel networks only.

Apart from an initial problem for SGVF equations, it seems useful to formulate a boundary problem. This approach enables to solve some flow cases directly instead of the usually applied trial and error method, though in a general case it is possible that the boundary problem has no solution. Let us consider SGVF in an open channel of length L connecting two reservoirs with different but time constant water levels (Fig. 1). In this case the function $H(x)$ should satisfy the governing equation and imposed conditions at upstream and downstream ends of the channel defined by water levels at the reservoirs. Moreover, the discharge Q is unknown. The problem formulated in this manner is a two points boundary problem for a system of ordinary differential equations (1) and (5) or (8) describing SGVF. The same problem can be formulated for a channel reach bounded by two control stations. If during SGVF the water levels are recorded in these stations and the Manning coefficient is known, the determination of the flow profile $H(x)$ and discharge Q for the considered channel reach is also a two points boundary problem. A similar approach can be formulated to determine the flow profile and Manning coefficient n or lateral inflow q.

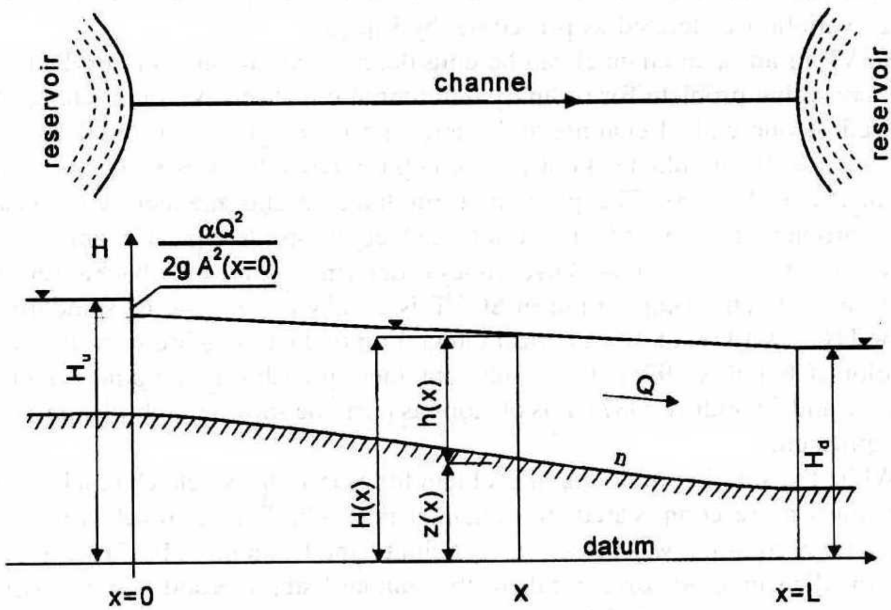

Fig. 1. Channel connecting two reservoirs

Very often the problems presented above are solved by integrating the equations of unsteady flow with the time constant boundary conditions (the constant water levels imposed at the upstream and downstream ends and time tending to infinity). This approach can be applied for a single channel as well as for channel network. Cunge et al. (1980) showed that the discretized system of Saint Venant equations for a steady flow does not correspond to the discrete form of the SVGF equation. Consequently the difference between the results calculated by the methods mentioned occurs.

To solve the boundary value problem for SGVF a shooting method can be used (Stoer and Burlisch 1980). It consists of successive solving of the initial problem for an SGVF equation with the initial condition $H(x = 0) = H_0$ and determining the values of Q for which the condition at boundary $x = L$ is satisfied. Unfortunately it can be used for a single channel only.

In this paper the solution of the boundary problem by the finite differences method is presented. Consequently one obtains an alternative approach to determine a flow profile $H(x)$ and discharge Q (or Manning coefficient or lateral inflow) directly for a single channel, as well as for tree type and looped network when at the channel's ends the water levels are imposed.

2. Determination of Water Profile and Flow Discharge for a Channel Connecting Two Reservoirs

As an example of a solution of the two points boundary problem, SGVF in a hypothetical channel connecting two reservoirs is considered (Fig. 1). This kind of flow has been investigated by Chow (1959), who solved it using a delivery curve.

There are some methods of solving the boundary problem for the ordinary differential equations (Bjorck and Dahlquist 1974). In this paper the difference method is applied. The channel of length $< 0, L >$ is divided by N nodes into $N-1$ intervals Δx_i. To avoid the difficulties occurring when the flow becomes critical it seems better to use Eq. (8) instead of Eq. (5). Neglecting lateral inflow Eq. (1) and Eq. (8) are approximated at the middle of each interval $x_i + \Delta x_i/2$ by a centered difference, coinciding with the implicit trapezoidal rule:

$$\frac{Q_{i+1} - Q_i}{\Delta x_i} = 0, \quad \frac{E_{i+1} - E_i}{\Delta x_i} + \frac{1}{2}(S_i + S_{i+1}) = 0 \qquad (10, 11)$$

where:

i – index of cross section,

Δx_i – length of interval number i.

It results from Eq. (10) that only one unknown value of Q exists as $Q_i = Q_{i+1} = Q = \text{const}$. Introducing the energy E and the friction slope S defined by expressions (9) and (3) into Eq. (11) yields:

$$\left(H_{i+1} + \frac{\alpha Q^2}{2g A_{i+1}^2}\right) - \left(H_i + \frac{\alpha Q^2}{2g A_i^2}\right) + \\ + \frac{\Delta x_i}{2}\left(\frac{n^2 Q^2}{R_i^{4/3} A_i^2} + \frac{n^2 Q^2}{R_{i+1}^{4/3} A_{i+1}^2}\right) = 0. \qquad (12)$$

Eq. (12) is well known the discrete energy equation usually applied to calculate water profile in natural channels (Chow 1959). Note, that it is simply the SGVF equation (7) numerically integrated by the implicit trapezoidal rule. Similar equations can be presented for each interval Δx_i ($i = 1, 2, \ldots, N-1$). In this manner one obtains a system of $N-1$ algebraic equations, in which $N+1$ unknowns occur. There are N water levels H_i at nodes and flow discharge Q. This system has to be completed by two additional equations resulting from imposed boundary conditions. Assuming a subcritical flow in the channel, the following conditions should be imposed at the ends of the channel:

$$E\ (x = 0) = H_1 + \frac{\alpha Q^2}{2g A_1^2} = H_u, \quad E\ (x = L) = H_N = H_d \qquad (13a, b)$$

where:

H_u, H_d – the imposed water levels at upstream and downstream reservoirs respectively,

H_1, H_N – the water levels at upstream and downstream ends of channel respectively.

As discharge Q is unknown the established boundary problem has a nonlinear boundary condition.

The final system of equations can be presented in the matrix form:

$$\mathbf{A}\,\mathbf{X} = \mathbf{b} \tag{14}$$

where:

A – matrix of coefficient,

$\mathbf{b} = (H_u, 0, \ldots, H_d, 0)^T$ – vector of the right hand side,

$\mathbf{X} = (H_1, H_2, \ldots, H_{N-1}, H_N, Q)^T$ – vector of unknowns,

T – transposition symbol.

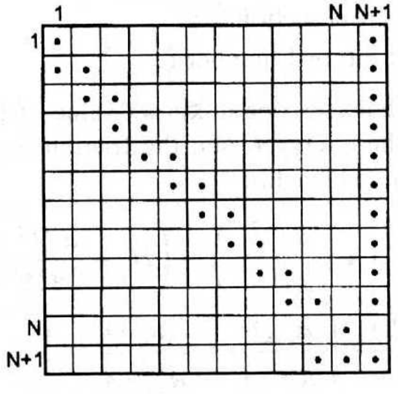

Fig. 2. Structure of **A** matrix (• – non-zero entry)

The matrix **A** of dimensions $(N+1) \times (N+1)$ is very sparse. Its structure is presented in Fig. 2 and its non-zero elements are defined as follows:

$$a_{1,1} = a_{N,N} = 1, \quad a_{1,N+1} = \frac{\alpha Q}{2g A_1^2}, \tag{15a, b}$$

$$a_{i,i} = 1, \qquad a_{i,i-1} = -1 \text{ for } i = 2, 3, \ldots, N-1, \tag{15c, d}$$

$$a_{i,N+1} = -\frac{\alpha Q}{2g A_{i-1}^2} + \frac{\alpha Q}{2g A_i^2} + \frac{\Delta x_{i-1}}{2}\left(\frac{n^2 |Q|}{R_{i-1}^{4/3} A_{i-1}^2} + \frac{n^2 |Q|}{R_i^{4/3} A_i^2}\right) \quad \text{for } i = 2, 3, \ldots, N-1, \tag{15e}$$

$$a_{NN} = 1, \tag{15f}$$

$$a_{N+1,N+1} = -\frac{\alpha Q}{2g A^2_{N-1}} + \frac{\alpha Q}{2g A^2_N} + \frac{\Delta x_{N-1}}{2}\left(\frac{n^2|Q|}{R^{4/3}_{N-1} A^2_{N-1}} + \frac{n^2|Q|}{R^{4/3}_N A^2_N}\right). \quad (15g)$$

This system of nonlinear algebraic equations should be solved by iterative method. The Newton method needs the determining of the Jacobian matrix in which some elements have to be calculated numerically when a natural channel is considered. In addition very often the Newton method does not ensure convergence. To avoid this, the following method is proposed:

$$\mathbf{A}^*\mathbf{X}^{(k+1)} = \mathbf{b} \quad (16)$$

where:

k	– is the iteration index.
$\Delta\mathbf{X}^{k+1} = \mathbf{X}^{k+1} - \mathbf{X}^k$	– is correction vector
$\mathbf{f}^k = \mathbf{A}^k\mathbf{X}^k - \mathbf{b}$	– is vector of residuals in Eq. (14)
$\mathbf{A}^* = \mathbf{A}\left(\frac{\mathbf{X}^{(k)} + \mathbf{X}^{(k-1)}}{2}\right)$	– is modified matrix in Eq. (14)

For $k = 0$ $\mathbf{A}* = \mathbf{A}(\mathbf{X}(0))$ is recommended. This, similar to the Newton method, has been derived by modification of the Picard method. After accepting the first estimation of the unknown vector $\mathbf{X}^{(0)}$, the iterative process is continued until two succeeding solutions satisfy the following criteria for convergence:

$$\left|X_i^{(k+1)} - X_i^{(k)}\right| \le \varepsilon_H \text{ for } i = 1, N \text{ and } \left|X_{N+1}^{(k+1)} - X_{N+1}^{(k)}\right| \le \varepsilon_Q \quad (17, 18)$$

where: ε_H and ε_Q represent the specified tolerances for water level H_i and discharge Q respectively.

In each iteration the system of linear algebraic equations is solved by the **LU** decomposition method using non-zero elements of matrix $\mathbf{A}^*$ only.

To demonstrate the method presented the solution of SGVF in idealised channel is analysed. The length of the channel is $L = 5000$ m, the bed slope $s = 0.0005$, the cross-section is trapezoidal with 10 m of bed width and side slopes 1:1. The Manning roughness coefficient is $n = 0.030$. The channel is divided by $N = 51$ nodes into 50 intervals of the constant length $\Delta x = 100$ m. The bed elevation above a datum changes linearly from $z(x = 0) = 5.0$ m at the upstream end to $z(x = L) = 2.5$ m at the downstream end.

As shown by Chow (1959), the form of flow profile $H(x)$ depends on the relation between the water levels in the reservoirs and the normal depth h_n. At first, similar to Chow, a head velocity at the upstream boundary is omitted. Therefore at the upstream reservoir, as well as upstream end of channel the constant stage

$H_1 = H_u = 10.0$ m, is accepted. It corresponds to the depth $h_u = 5.0$ m. Calculations have been performed for the following stages at the downstream end: $H_N = H_d = 8.75$ m and 6.25 m. They correspond to the depths $h_d = 6.25$ m and $h_d = 3.75$ m. The results of computation are presented in Fig. 3. The shape of flow profiles agree with the results obtained by Chow (1959). The calculated flow discharge for $H_d = 8.75$ m is 102.24 m^3/s whereas for $H_d = 6.25$ m it is $Q = 123.07$ m^3/s.

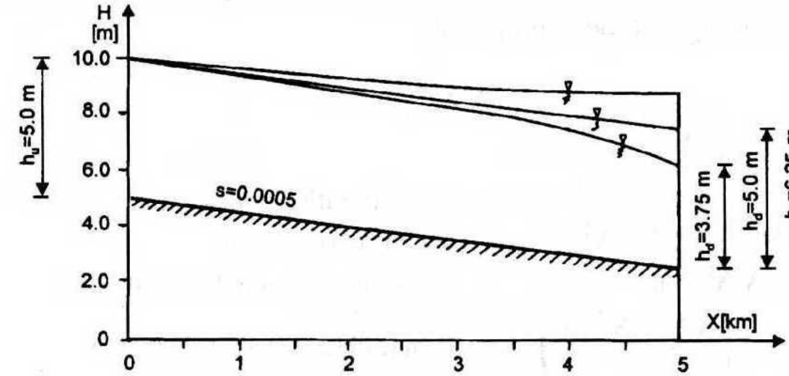

Fig. 3. Flow profile in a channel with subcritical flow and constant H_u without head velocity

Next calculations have been performed for the same data, but including a head velocity. Therefore, the boundary condition at the upstream end was in the form of Eq. (13a). The obtained head velocity at the upstream end of the channel (node 1) is 0.092 m for $H_d = 8.75$ m and 0.134 m for $H_d = 6.25$ m whereas the calculated flow discharges are $Q = 98.022$ m^3/s and 117.129 m^3/s respectively.

The computations performed showed great efficiency of the proposed method. Irrespective of the first estimation of the flow profile and starting value of discharge Q a solution with a tolerance of $\varepsilon_H = 0.0001$ m and $\varepsilon_Q = 0.001$ m^3/s was obtained after several iterations for both types of boundary condition at the upstream end.

3. Boundary Problem for an Open Channel Network

Consider the channel network as in Fig. 4. Subcritical flow in all arms is assumed. All trapezoidal channels are divided into 10 reaches of constant length. The characteristics of accepted network are presented in Table 1.

The bed elevation at the upstream end (node 1) is 10.000 m, whereas at the downstream end (node 77) it is 9.000 m. The total number of nodes is $N = 77$. As boundary conditions the water levels at upstream (u) and downstream end (d) of the network are specified: $H_1 = H_u$, $H_N = H_d$ (Fig. 4). The water levels at all internal nodes, as well as the discharges in all channels are unknowns.

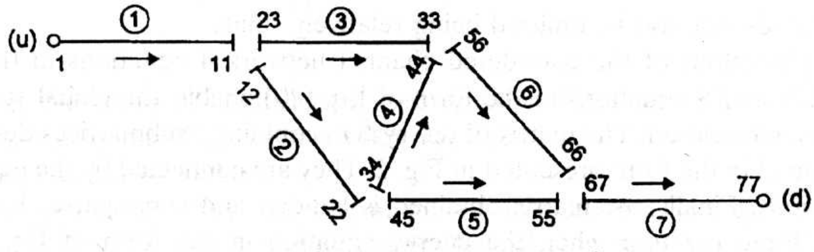

Fig. 4. Considered channel network

Table 1. Channel characteristics for network

Channel	Length [m]	Bed width [m]	Side slope	Bed slope	n	Reach [m]
1	400	5.0	1.5	0.001	0.030	40
2	300	3.0	1.5	0.001	0.035	30
3	400	3.0	1.5	0.001	0.035	40
4	300	2.0	1.5	0.00033	0.025	30
5	400	3.0	1.5	0.0005	0.035	40
6	300	3.0	1.5	0.00033	0.035	30
7	400	5.0	1.5	0.0005	0.030	40

For each channel, a set of 10 equations in the form of Eq. (16) can be established.

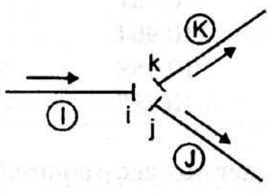

Fig. 5. Junction of three channels: I, J, K

The global system for a channel network consists of subsystems describing each channel individually. We therefore have 70 equations whereas the number of unknowns is equal to 82 (77 nodal values of water levels and 7 values of channel discharges completed by 2 boundary conditions). To close the global system of equations the additional relations should be introduced. Namely, at the junction of three channels I, J, K formed by nodes i, j, k (Fig. 5) the continuity equation:

$$Q_I = Q_J + Q_K \tag{19}$$

as well as the energy equation:

$$H_i + \frac{\alpha\, Q_I^2}{2g\, A_i^2} = H_j + \frac{\alpha\, Q_J^2}{2g\, A_j^2} = H_k + \frac{\alpha\, Q_K^2}{2g\, A_k^2} \tag{20}$$

can be written. In the above equation the losses are neglected. Sometimes the velocity heads can also be omitted being relatively small.

For 4 junctions of the considered channel network 4 equations in the form of Eq. (19) and 8 equations in the form of Eq. (20) enable the global system of equations to be closed. The matrix of this system contains 7 submatrices describing each channel in the form presented in Fig. 2. They are connected by the equations for junctions. Finally the matrix obtained is banded and very sparse. Each row contains three (or four when the energy equation in the form of Eq. (20) is used) non-zero elements only. The accepted boundary conditions were as follows: $H_1 = 11.500$ m, $H_{77} = 10.500$ m. The results of calculations are presented in Table 2. The numerical experiments show that irrespective of the first estimation of flow profiles and discharges in channels, the solution with tolerances $\varepsilon_H = 0.0005$ m and $\varepsilon_Q = 0.0005$ m^3/s was obtained after less than 20 iterations. Because the applied subroutine solving a linear system of equations uses non-zero elements only the final algorithm is computationally very efficient.

Table 2. Results of solution for considered network

Channel	Discharge [m^3/s]	Upstream water level [m]	Downstream water level [m]
1	11.709	11.500	11.113
2	6.093	11.142	10.958
3	5.616	11.147	10.963
4	0.795	10.981	10.979
5	5.298	10.964	10.813
6	6.412	10.958	10.805
7	11.709	10.785	10.500

The relatively small computer storage required by this algorithm, as well as small time consumed by iterative process, ensures fast computing of SGVF even for complex networks using PC.

4. Determination of Flow Profile and Manning Roughness Coefficient or Lateral Inflow

In the preceding case the flow profile and discharge were unknowns. It is possible to establish another boundary problem accepting Q as known. If the Manning coefficient n is known also the boundary problem is overdetermined and in general there is no solution for the arbitrary value of n. However, for certain special values of n Eq. (8) does have a solution. It is possible to reduce this problem to the standard case by introducing a new dependent variable n = const. Consequently, we can add another differential equation for a channel reach satisfied by n. Therefore, for the accepted discharge Q the following equations should be solved:

$$\frac{dn}{dx} = 0, \quad \frac{dE}{dx} = -S \tag{21a, b}$$

to obtain the value of coefficient n and function $H(x)$ satisfying the imposed boundary conditions. This trick is often used solving two points boundary value problem (Press et al. 1992).

To present this problem an experiment was carried-out (Geringer 1997). In a rectangular flume SGVF was reproduced (Fig. 6). The length of the flume was $L = 10$ m, width $B = 0.385$ m and bed slope $s = 0.0005$. The measurements and calculations were performed for the reach of length $L = 8$ m, which was divided into 16 intervals of length $\Delta x = 0.5$ m = const. Knowing recorded flow discharge Q and the depth at both ends, it is possible to determine a flow profile and Manning coefficient n by solving a boundary problem. To this order some modification of the described algorithms is needed because now Q is imposed and n is unknown.

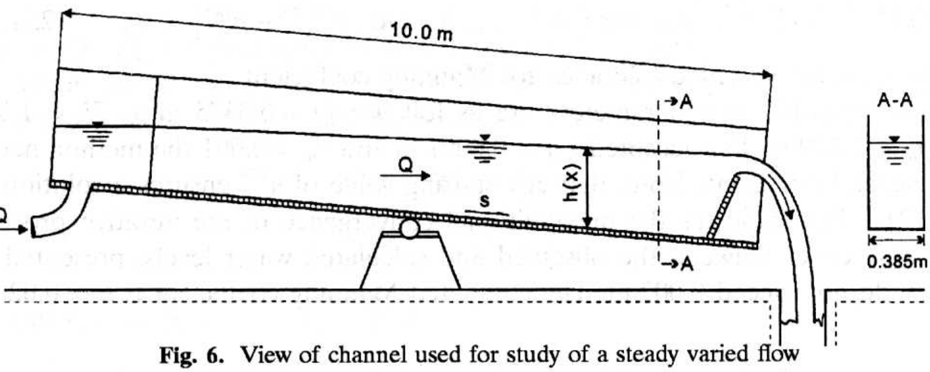

Fig. 6. View of channel used for study of a steady varied flow

In this case the elements of matrix **A**, and the vectors **X** and **b** occurring in Eq. (16) should be changed. The vector **X** has the following form:

$$\mathbf{X} = (H_1, H_2, \ldots, H_N, n)^T. \tag{22}$$

The elements of matrix **A** are defined as follows:

$$a_{1,1} = 1, \tag{23a}$$

$$a_{i,i} = 1, \quad a_{i,\, i-1} = -1 \quad \text{for } i = 2, 3, \ldots, N-1, \tag{23b, c}$$

$$a_{i,N+1} = \frac{\Delta x_{i-1}}{2}\left(\frac{n\,Q^2}{R_{i-1}^{4/3}\,A_{i-1}^2} + \frac{n\,Q^2}{R_i^{4/3}\,A_i^2}\right) \quad \text{for } i = 2, \ldots, N-1, \tag{23d}$$

$$a_{N,N} = 1, \tag{23e}$$

$$a_{N,N+1} = \frac{\Delta x_{N-1}}{2}\left(\frac{n\,Q^2}{R_{N-1}^{4/3}\,A_{N-1}^2} + \frac{n\,Q^2}{R_N^{4/3}\,A_N^2}\right). \tag{23f}$$

Whereas vector $\mathbf{b} = (b_1, b_2, \ldots, b_N, b_{N+1})^T$ has the elements:

$$b_1 = H_u, \tag{24a}$$

$$b_i = \frac{\alpha\,Q^2}{2g\,A_{i-1}^2} - \frac{\alpha\,Q^2}{2g\,A_i^2} \quad \text{for } i = 2, \ldots, N-1, \tag{24b}$$

$$b_N = H_d, \tag{24c}$$

$$b_{N+1} = \frac{\alpha\,Q^2}{2g\,A_{N-1}^2} - \frac{\alpha\,Q^2}{2g\,A_N^2}. \tag{24d}$$

The iterative process is stopped when

$$\left|H_i^{(k+1)} - H_i^{(k)}\right| \le \varepsilon_H \quad \text{for } i = 1, \ldots, N \quad \text{and} \quad \left|n^{(k+1)} - n^{(k)}\right| \le \varepsilon_n \tag{25a, b}$$

where ε_n is the specified tolerance for Manning coefficient.

The recorded flow parameters are as follows: $Q = 0.0375$ m^3/s, $H_u = 1.314$ m, $H_d = 1.309$ m. For accepted $\varepsilon_H = 0.0001$ m and $\varepsilon_n = 0.001$ the method needs only several iterations. Note, that any starting value of $n^{(0)}$ ensures a solution of Eq. (21). This indicates the unconditional convergence of the iterative process. The differences between the observed and calculated water levels, presented in Fig. 7, do not exceed 0.002 m. The computed Manning coefficient is $n = 0.0201$.

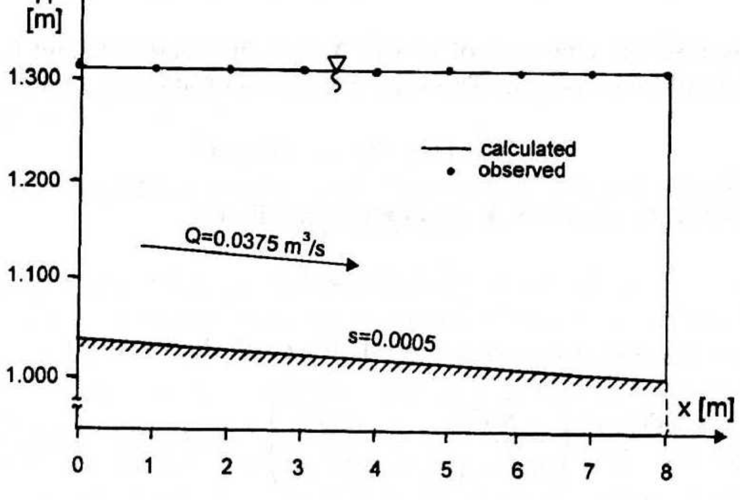

Fig. 7. Comparison of calculated and observed water levels in a channel

The same problem was solved for another set of data namely $Q = 0.0137$ m^3/s, $H_u = 1.248$ m and $H_d = 1.247$ m giving the value of $n = 0.019$. Also in this case the differences between the calculated and recorded water levels do not exceed 0.002 m.

The boundary value problem can be established as well as when a lateral inflow q, considered as constant, is unknown. In this case a discharge at one end of the channel and the water levels at two ends should be imposed. To calculate a lateral inflow q and flow profile the following system of equations should be solved:

$$\frac{dQ}{dx} = q, \quad \frac{dE}{dx} = -S. \tag{26a, b}$$

With variable Q Eq. (12) for any channel reach of length Δx_i takes the form:

$$H_{i+1} + \frac{\alpha Q_{i+1}^2}{2g A_{i+1}^2} = H_i + \frac{\alpha Q_i^2}{2g A_i^2} - \frac{\Delta x_i}{2}\left(\frac{n^2 Q_{i+1}^2}{R_{i+1}^{4/3} A_{i+1}^2} + \frac{n^2 Q_i^2}{R_i^{4/3} A_i^2}\right) \tag{27}$$

where: Q_i, Q_{i+1} – discharges in node i and $i+1$ respectively.

When the lateral inflow is assumed as constant over channel the discharge at the nodes can be expressed as follows:

$$Q_i = Q_1 + L_{i-1} q, \tag{28a}$$

$$Q_{i+1} = Q_1 + L_i q \tag{28b}$$

where: $L_{i-1} = \sum_{j=1}^{i-1} \Delta x_j, \quad L_i = \sum_{j=1}^{i} \Delta x_j,$

Q_1 – imposed discharge at the upstream end (node 1).

For the discharge imposed at the downstream end the expressions for Q_i and Q_{i+1} should be changed respectively. Introducing Eqs. (28a,b) into Eq. (27) gives the following expression:

$$-H_i + H_{i+1} - \frac{\alpha}{2g}\frac{Q_1^2 + 2Q_1 L_{i-1} q + L_{i-1}^2 q^2}{A_i^2} + \frac{\alpha}{2g}\frac{Q_1^2 + 2Q_1 L_i q + L_i^2 q^2}{A_{i+1}^2} + \\ + \frac{n^2 \Delta x_i}{2}\frac{Q_1^2 + 2Q_1 L_{i-1} q + L_{i-1}^2 q^2}{R_i^{4/3} A_i^2} + \frac{n^2 \Delta x_i}{2}\frac{Q_1^2 + 2Q_1 L_i q + L_i^2 q^2}{R_{i+1}^{4/3} A_{i+1}^2} = 0. \tag{29}$$

Presenting similar equations for all reaches $i = 1, 2, \ldots, N-1$ one obtains a system of nonlinear algebraic equations. Its solution with imposed water levels at upstream and downstream ends gives the water profile $H(x)$ and lateral inflow q.

As an example of the application of the presented algorithm, SVGF in a single channel is considered. Its cross section has trapezoidal shape with bed with $b = 20$ m and side slope 1:1. The length of channel is $L = 9300$ m, bed slope is $s = 0.0005$ and Manning coefficient is $n = 0.035$. At the upstream end depth $h_u = 3.0$ m and discharge $Q_u = 15$ m^3/s were imposed whereas at the downstream end depth $h_d = 4$ m was accepted. The tolerance for water levels was $\varepsilon_H = 0.0005$ m and for lateral inflow $\varepsilon_q = 0.000001$ m^3/s/m. After 14 iterations the calculated lateral inflow was $q = 0.000245$ m^3/s/m giving a discharge at the downstream end $Q_d = 17.28$ m^3/s.

The boundary problem with respect to the Manning coefficient or lateral inflow, can be formulated for channel network also. However, it is possible to solve this only when a constant value of n or q is assumed for all branches and additionally a discharge at one end is imposed. If each channel has different roughness or lateral inflow, it is impossible to solve this problem directly for to many unknowns comparing with the number of equations which can be established.

5. Conclusions

To determine a flow profile for a steady gradually varied flow when water levels are imposed at the ends of a channel, a boundary problem should be formulated for the governing equations. The presented approach, enables direct calculation of the unknowns flow parameters instead of the trial and error method usually applied. Application of the finite difference method to solve a boundary problem leads to a non-linear system of algebraic equations with sparse matrix. The proposed method of its solution ensures unconditional and relatively rapid convergence of the iterative process. Its coupling with solver of linear system using non-zero elements only gave a very effective algorithm. The proposed approach can be applied for a single channel, as well as for a tree-type or looped network.

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