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Methods of Determination of Elastic Moduli of Particulate Materials – brief review

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Abstract

The methods of determination of elastic moduli for particulate materials with special attention to non-cohesive soils are discussed. Various exp erimental techniques to isolate elastic response of tested materials are described and analysed. Some short-comings are indicated and any inconsistencies with classical theory of elasticity are discussed. Attention is focused on the methods that can be applied using conventional laboratory apparatuses. In addition, some models that allow for calculation of elastic moduli are also presented.

1. Introduction

Much effort has been made in recent years to develop procedures for evaluating the response of particulate (granular, non-cohesive) materials, such as sand, gravel, steel balls, natural grains etc., under various loading conditions. There exists a large number of mechanical models which describe particular aspects of the behaviour of particulate materials however, their successful application is greatly influenced by the incorporation of representative material properties. A considerable part of the models involves the elastic response of the material as one of the possible behaviours under loading. The description of reversible part of the material deformation requires the determination of its elastic properties.

Experimental isolation of elastic response of particulate materials is, in general, a very difficult task since during loading both reversible and irreversible deformations occur and the stress-strain behaviour is rarely purely elastic. Even for a small closed stress path, irreversible strains are observed (Loret, 1985).

Elastic deformation is relatively small and often obscured by plastic deformation caused by slippage, rearrangement and crushing of particles subjected to higher stress states (Harding and Blandford, 1986). However, for many reasons despite its magnitude elastic deformations can not be neglected. First, the classic theory of elasticity is still very popular in geotechnical engineering. Secondly, accurate determination of elastic response is important for inverting the elasto-plastic strain-stress relations and for complete description of soil behaviour. Finally, experimental isolation of elastic deformation leads to the accurate determination of plastic response of the material investigated which constitutes the verification of any theoretical plasticity laws.

Elastic properties of a given material are described in terms of the elastic moduli which are assumed to be intrinsic properties of the material and therefore independent of the method of testing or particular configuration of the material (Selvadurai, 1979). These properties should be determined experimentally in apparatuses commonly used in geotechnical laboratories by measuring the elastic stiffness of the material along various effective stress paths. However, due to the nature of materials composed of discrete solid particles, despite various theoretical considerations and a large number of experimental studies resulting in a variety of methods proposed, the problem has not been satisfactorily solved. This may be the reason why the "substitutions" of the elastic moduli such as "tangent", "secant", "resilient" or "constrained" moduli gained high popularity in soil mechanics. For example, in geotechnical studies, secant moduli are very frequently used to calculate displacements of structures due to soil movements, which are assumed to represent the elastic behaviour of subsoil. However, such an interpretation of experimental results is inconsistent with classical elasticity theory.

Similarly, a typical violation of the theory fundamentals is the admission by some authors of values of Poisson's ratio greater than 0.5 (Duncan and Chang, 1970; Selvadurai, 1979; Lambe and Whitman, 1970).

Generally, in most of the methods it is assumed that with regard to an elastic behaviour, particulate materials exhibit isotropic response. However, some authors state that any irreversible strain will create an anisotropic structure which leads to an anisotropic elastic behaviour (Hicher, 1996) and suggest measuring the elastic response of the material in each of respective principal stress planes. The analysis of the experimental results on various types of sands (Rowe, 1971) shows that they exhibited isotropic behaviour when unloaded, even when the strains during loading indicate anisotropic behaviour. Lade and Nelson, 1987 state that sands with initial anisotropic fabric may exhibit anisotropic behaviour, but the elastic behaviour observed during unloading is isotropic for practical purposes.

For an isotropic material the elastic behaviour can be described by two independent material constants that may be Young's modulus E and Poisson's ratio ν or shear modulus G measured from simple shear or wave propagation tests. Thus determination of the elastic properties of a given non-cohesive soil can be reduced to the experimental determination of two independent elastic constants.

A study of past and recent developments in determining the elastic properties of particulate materials presented in the literature, leads to the conclusion that the methods proposed could be divided into three main groups:

- theoretical approach,
- methods based on a measurement of very small strains (standard geotechnical laboratory apparatuses adapted for measurement at very low strain levels, resonant columns, wave propagation methods carried out in both laboratory and field conditions),
- methods employing relatively larger strains of soil response (traditional laboratory devices such as triaxial apparatus, oedometer, hollow cylinder, cyclic simple shear boxes etc.).

The first group of methods concerns the calculations of elastic moduli from theoretical solutions for regular assemblages of elastic spheres. Such solutions have been developed in the late 40's and early 50's based on pioneering works on various configurations of sphere assemblages such as face-centred, (Duffy and Mindlin, 1957), simple cubic (Deresiewicz, 1958) and hexagonal, (Gassman, 1951; Duffy, 1959). Theoretical works were continued by Makhlouf and Stewart, 1967; Walton, 1987 and Chang, 1988. Recently, in times of almost unlimited access to the very fast and capable computers growing popularity was gained by computer simulation approach that solves for the assembly deformation based on governing equations for each particle interacting with its neighbours (e.g. see Serrano and Rodriguez-Ortiz, 1973; Kishino, 1988). However, due to the nature of non-cohesive soils as a special form of particulate material and very idealised specific configurations of assemblages of spheres, the theoretical solutions were obtained these values can be still treated as an approximation. In the paper attention will be basically focused on the experimental methods, thus the theoretical approach will not be discussed in details here.

The second group of methods is connected with the assumption that the elastic behaviour of particulate materials is restricted to the infinitesimal stress increments resulting in infinitesimal strains (Seed et al., 1985; Loret, 1985; Hardin and Blandford, 1989; Hicher, 1996). This approach will be discussed briefly in the next section.

The third group consists of most common methods where one assumes that despite the non-linear behaviour of stress-strain characteristics, part of it may be treated as the almost elastic or the elasticity of the material is described in the incremental form. Typical representations of this approach are various tangent, secant, unloading-reloading or resilient moduli. The modulus values are usually obtained on the basis of stress-strain diagrams from experiments carried out in conventional geotechnical devices such as triaxial apparatus, oedometer, hollow cylinder, three-dimensional cubical triaxial apparatus, simple shear box etc. Due to common access to the devices mentioned this group of methods will be most comprehensively discussed in the paper.

Another division of the methods for determining the elastic moduli is proposed by Loret, 1985, who distinguishes the following groups:

a) Measurement of the slope of the stress-strain curve just at load reversal for several stress states: if an elastic region exists, an infinitesimal unloading must be elastic. However, the elastic region is probably so small for soils that it makes this measurement very difficult for experiments on the triaxial apparatus,

b) On some paths, stress-controlled tests for soils with a large number of cycles lead the material to an adapted state: after a certain number of cycles the stress-strain relation does not depend on the direction of loading,

c) Unloading for the radial stress path, i.e. the stress path with constant principal stress ratios, is considered as elastic.

The purpose of this paper is a comprehensive review of the available information on the determination of elastic constants for particulate materials, particularly non-cohesive soils. Attention will mostly be focused on the experimental methods concerning traditional geotechnical devices such as triaxial apparatus and oedometer. The most common proposals are analysed and any inconsistencies with the classical theory of elasticity are discussed. The variety of different definitions of elastic moduli obtained by various methods is presented.

2. Experimental Procedures

2.1. Small Strains Tests

In order to satisfy the assumption of infinitesimal stress increments inducing infinitesimal strains, two different types of tests have been developed: static and dynamic. Static tests can be carried out in specially adapted laboratory apparatuses for the measurement of small deformations (Tatsuoka and Shibuya, 1992; Viggiani and Atkinson, 1995; Hicher, 1996). Apparatuses are usually equipped with very sensitive local gauges that enable very accurate measurement of strains. They are generally of the order of 10^{-6} to 10^{-5}, whereas the resolution of strain measurement in standard triaxial apparatuses is between 10^{-3} to 10^{-2}.

The dynamic tests are usually performed in resonant columns in which cylindrical samples suspended in a triaxial cell are subjected to forced harmonic torsional vibrations and values of Young's modulus are measured from resonant frequencies. Resonant column tests can be used to evaluate the stiffness of soils at shearing strains equal to 10^{-7}.

Another group of dynamic laboratory tests is related to the employment of bender elements that are installed in the bottom and top platens of the traxial cell so as to enter about 3 mm into the sample (Dyvik and Madshus, 1985; Viggiani and Atkinson, 1995). Bender elements are small electro-mechanical transducers that can transmit and receive shear waves propagating through the tested material. Bending of the element due to a change of voltage causes transmitting of shear waves and conversely mechanical bending of the elements can produce a change of voltage.

In field conditions the dynamic testing of the soils is made in terms of wave propagation methods (cross-hole or down-hole tests). In this case the propagation of different types of small amplitude waves through the porous medium serves to measure the different elastic moduli. The determination of the moduli is made indirectly using direct measurement of various wave velocities: shear wave velocity v_s to determine shear modulus $G = \rho v_s^2$, rod wave velocity v_r to determine Young's modulus $E = \rho v_r^2$ and pressure wave velocity v_p to determine constrained modulus $M = \rho v_p^2$ where ρ is the bulk density of the material (Hardin and Blandford, 1989). Potential elastic anisotropy can be measured by testing the sample in various directions.

Due to the introduction of new technologies in the measurement of strains the sample undergoes in laboratory conditions the accuracy of this measurement is continuously improving. However, it is still difficult to accept so high resolution of the measurement (the order of 10^{-6}) as a reliable value. It particularly concerns testing the samples of particulate materials with the gauges that are directly mounted on them. Particulate materials are very sensitive to any external impact. This can be observed during the process of sample formation, mounting the gauges, or adjustment of the load cell e.t.c. The strains which usually occur in the sample during this phase are several orders of magnitude higher than 10^{-6}. It regards both adapted apparatuses as well as dynamic tests where one deals with indirect measurement of strains. Additionally, such laboratory equipment is not widely used in standard laboratories due to the sophisticated way of testing and relatively high price.

2.2. Large Strains Tests

Typical results of conventional triaxial compression tests on dense Lubiatowo sand are shown in Fig. 1. The main physical characteristics of the sand are as follows: minimum void ratio $e_{min} = 0.49$, maximum void ratio $e_{max} = 0.74$, coefficient of uniformity $c_u = 1.5$. In the test the sample of dry fine sand of relative density $D_r = 76\%$ has been subjected to four successive cycles of loading and unloading. For the sake of convenience the following units have been incorporated: stress unit – 10^5 N/m^2, strain unit – 10^{-3}, Young's modulus unit – 10^8 N/m^2. The small deviator at zero strains was mainly caused by adjusting the load cell to the top of the sample at the very beginning of the test.

During the test that was carried out at constant confining pressure $p = 200$ kPa both axial and radial strains were measured. The measurement of radial strains was performed by a special gauge fixed to the sample half way up its height. This gauge makes use of the Hall effect phenomenon. The detailed description

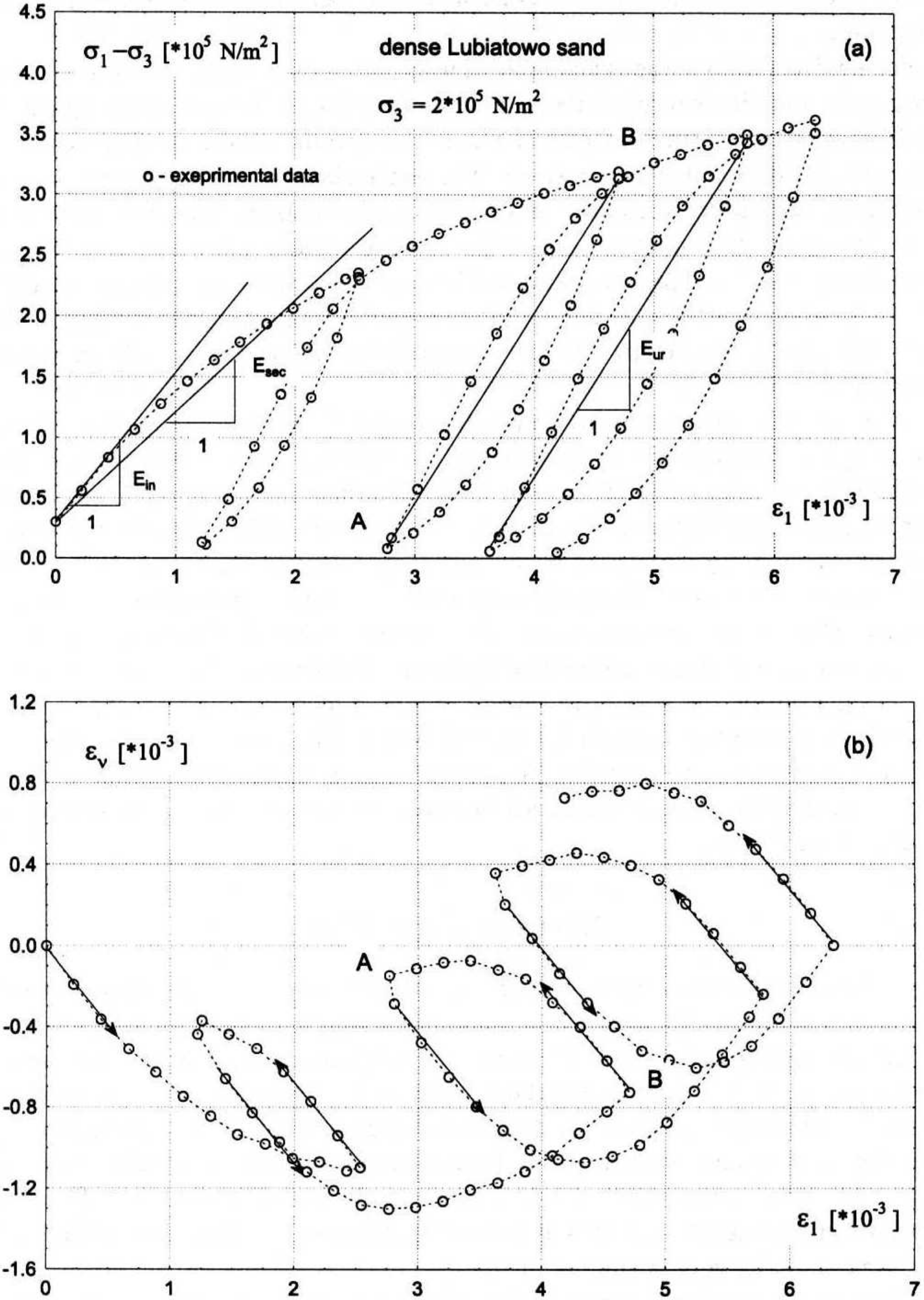

Fig. 1. Stress-strain and volume change behaviour of dense Lubiatowo sand in triaxial compression tests with several unloading-reloading cycles

of the device can be found in Świdziński, 2000. The method allows for direct measurement of radial strains on both saturated and dry samples.

For purposes of comparison the experimental data have been plotted in the commonly accepted configuration: stress deviator versus axial strain (Fig. 1a) and volumetric strain versus axial strain (Fig. 1b). The volumetric strain has been calculated from the following formula:

$$\varepsilon_v = \varepsilon_1 + 2\varepsilon_3, \tag{1}$$

where $\varepsilon_1, \varepsilon_3, \varepsilon_v$ denote axial, radial and volumetric strains, respectively. For the interpretation of test results a standard convention has been assumed where compressive stresses and strains are taken as positive.

In the diagram representatives of various Young's modulus definitions have been shown schematically.

Historically, the most common approximation of elastic response of the particulate material was an initial tangent modulus. Usually, the modulus is defined as tangent to the stress-strain curve at the origin or initial slope of the curve for triaxial compression (Janbu, 1963; Lade and Nelson, 1987; Bałachowski et al., 1991). This definition is based on the assumption that during the first stage of loading in triaxial compression conditions, the material produces elastic strains. In fact, an initial sector of loading curve may be treated as linear, however, even for a small range of strains, both reversible and irreversible deformations take place.

In Fig. 2 are shown the results of a triaxial test on the same dense Lubiatowo sand as in Fig. 1. In the test a sample was subjected to a single cycle of loading and unloading during which the loading reversal took place after relatively small strains of the order of 1.5×10^{-3}. The diagrams contain both total axial and total radial strains measured during the experiment, which were plotted versus deviatoric stress. It can easily be noticed that for such relatively low stress and strains levels the axial deformations of the sample are significantly higher than deformations in radial direction. However, for both strain components, despite the almost linear loading curve, irreversible strains remain after unloading to zero stress level. These residual plastic deformations are equal to approximately 35% of the total strains that developed in a sample at the stress reversal. It clearly suggests that even for a very small range of strains generating in a sample during the initial phase of loading, both reversible and irreversible deformations develop in the material tested. For the following definition of initial tangent modulus:

$$E_{in} = \lim_{\varepsilon_1 \to 0} \frac{\sigma_1 - \sigma_3}{\varepsilon_1}, \tag{2}$$

the value of Young's modulus for the stress-strain characteristics shown in Fig. 2 will be $E_{in} = 0.91 \times 10^8$ N/m^2.

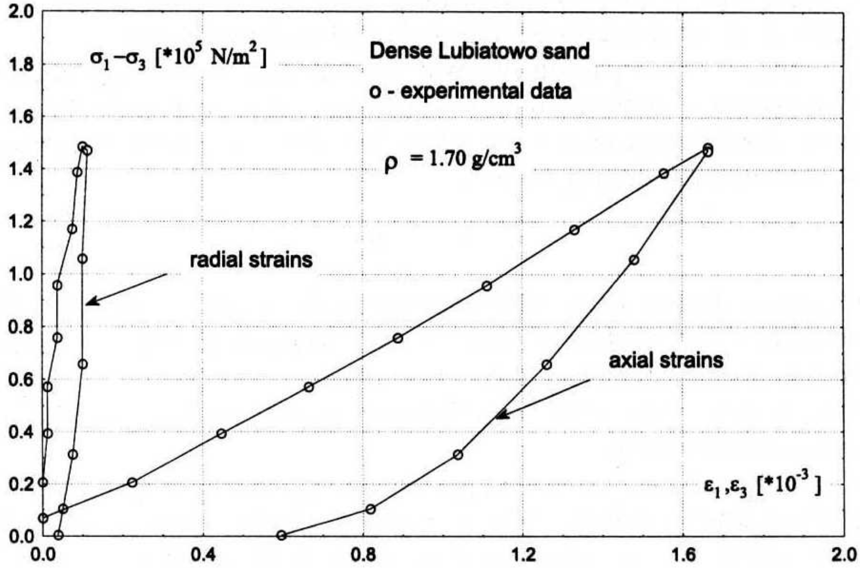

Fig. 2. Development of axial and radial strains in the triaxial compression test on dense Lubiatowo sand for level of small strains

Bałachowski et al., 1991 determined elastic properties for two types of sand directly from the initial slope of triaxial compression curve. They assumed that the material tested was within the elastic range when the deformations do not exceed 4×10^{-3} and 8×10^{-3} of axial strain for two sands tested, respectively. In their opinion such axial strains are in the compression range of soil behaviour before maximum compression volumetric strain is reached. The values of Young's modulus for the same Lubiatowo sand varied from 0.5×10^8 N/m^2 to 1×10^8 N/m^2. These values are several times lower than the respective elastic moduli determined by other methods. This fact can easily be explained. Development of plastic strains parallel with the elastic response in the initial phase of unloading makes the loading curve more inclined to the strains axis, which consequently decreases the value of the initial tangent modulus. In the light of the above considerations such an approach cannot be accepted and initial tangent modulus can not be treated as the elastic property of non-cohesive soil.

Additionally, the same authors analyse dependence of Young's modulus on the angle of internal friction of sand tested. The angle of internal friction is the strength characteristic of the material. The material will also exhibit an elastic response for stress levels for which no slips of particles are observed. Thus the assumption relating elastic modulus with the angle of internal friction cannot be accepted.

Another inconsistency of such an approach in which Young's modulus is identified as an initial tangent modulus is related to the fact that lateral (radial) strain

ε_3 is excluded from the definition of elastic constant (Eq. 2) which means that only partial information about the material's behaviour is taken into account. Fig. 3 shows the results presented in Fig. 1 (first two cycles of loading and unloading) but in another configuration in which changes of both axial and radial strains are shown. It can be seen that for higher levels of stress deviator the development of radial strains is much larger than in the case shown in Fig. 2. Therefore, in order to determine elastic constants both strains should be taken into account. The calculation of the initial tangent modulus from modified Eq. 2 by including the radial strains in the strain deviator for the data presented in Fig. 2 yields the value $E_{in} = 0.83 \times 10^8$ N/m^2 which is different from that obtained by the relationship (2). It must be noted that for higher stress levels this difference would be larger.

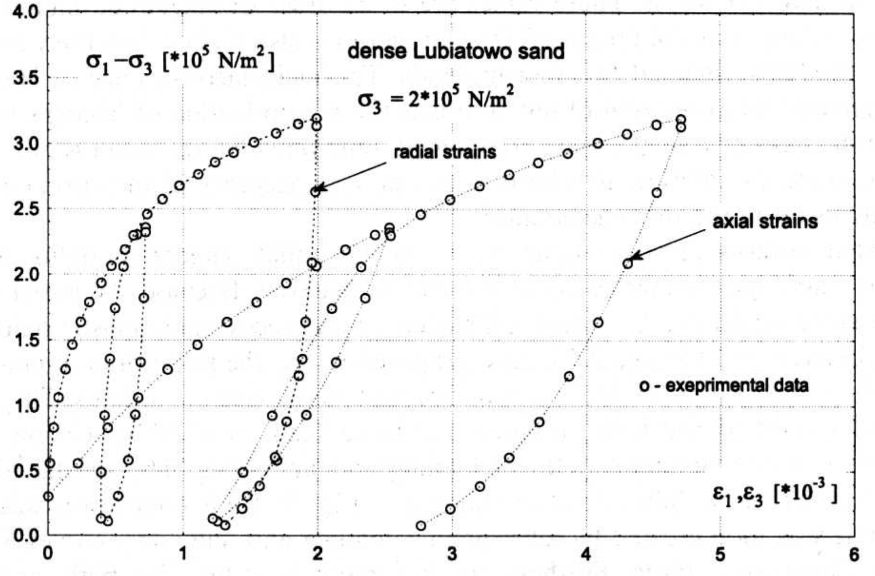

Fig. 3. Another interpretation of test results on dense Lubiatowo sand – the development of axial and radial strains during the first two cycles of loading and unloading

The problem of limits for determination of initial Young's modulus was investigated by Hicher, 1996, who has experimentally analysed the elastic behaviour of particulate materials subjected to very small stress and strain levels. He and his co-workers have developed a triaxial apparatus adapted to the measurement of small deformations. The local measurement of axial and radial strains is made by proximity transducers that are installed in the middle part of the samples. High sensitivity of the transducers enables measurement of axial and radial strains between 10^{-6} to 10^{-2}.

The samples of various particulate materials (such as sands, gravel and glass balls) were subjected to loading and unloading at different stress or strain levels within both compression and extension ranges. Young's modulus was determined from the initial linear stage of loading curve that expressed the relation between deviatoric stress and axial strains. After analysis of the results of many tests carried out at different confining pressures Hicher found that particulate materials exhibit reversible behaviour for strains lower than 1 to 3×10^{-5} depending on the material tested. Above that limit value irreversible strains occurred and the stress-strain curve became non-linear. Poisson's ratio was calculated as the ratio between measured radial and axial strains and was found to be linear within the range from 10^{-6} to 5×10^{-5}. His results show that the reversible response of particulate materials corresponds to very low level of strains that cannot be achieved and measured in conventional laboratory devices. The values of Young's modulus for dry sand varied from 3.1×10^8 N/m^2 to 5.6×10^8 N/m^2 depending on mean effective stress. These values are of an order of magnitude higher than corresponding values of the initial tangent modulus also higher than the values of elastic modulus obtained by other methods. The main inconsistency of Hicher's experimental interpretation of test results is the determination of Young's modulus on the basis of a single strain component, only when lateral strain is not taken into account. In addition, it is hard to accept such accuracy of measurements as was discussed in the previous section.

Other authors present similar views on the initial tangent modulus as improper elastic characteristics of the particulate material. Because the initial slope of the stress-strain curve is often influenced by non-recoverable plastic deformations, Duncan and Chang, 1970, have proposed to use the slope of an unloading-reloading cycle from a triaxial compression test as the elastic modulus. They performed several triaxial tests on dense and loose sands in which specimens were subjected to one or more cycles of loading and unloading. The results of one such test for dense Silica sand are shown in Fig. 4. Their conclusion was that the hysteresis loop created by subsequent reloading and unloading curves is very nearly linear and elastic. Furthermore, the modulus values for both cycles of unloading-reloading are the same, even though they occur at different strains and stress levels. On the basis of these observations they postulated that stress-strain behaviour of soil on unloading and reloading might be approximated with a high degree of accuracy by a linear sector that corresponds to elastic response of the soil.

However, when analysing the results of a similar test shown in Fig. 1 it can be seen that for dense Lubiatowo sand the reloading-unloading curves are only partially linear. This linear sector occurs after any of the stress reversals. Further change of stresses causes significantly non-linear behaviour of sand that can not be approximated by the same liner sector. In addition, the hysteresis loops produced during reloading and unloading are essentially larger than in the case of Duncan and Chang test results. Other shortcoming of the Duncan and Chang approach is the taking into account of only one of the principle strains and neglecting the

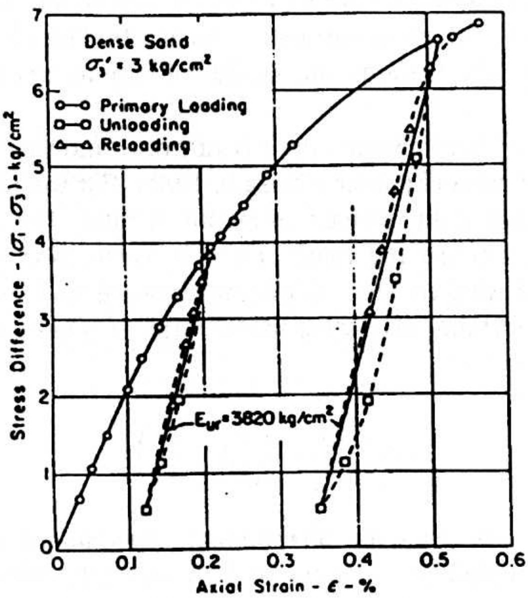

Fig. 4. Unloading and reloading of silica sand under drained triaxial test conditions, (after Duncan and Chang, 1970)

lateral deformations. The available experimental data from their experiments did not allow the including of these deformations into the analysis of the results.

The analysis of experimental data suggests that the modulus determined from the reloading-unloading curve is usually several times higher than initial tangent modulus.

A somewhat different view of isolation of elastic response of particulate materials is presented by Sawicki, 1994 and Sawicki and Świdziński, 1998. Their experimental observations and theoretical considerations concerned the behaviour of various particulate materials subjected to a one-dimensional stress state in an oedometer with additional measurement of lateral stresses. They have provided a lot of experimental evidence confirmed by theoretical analysis for the assumption that only the first stage of unloading can be treated as being purely elastic. This approach will be discussed further in the paper.

The above considerations lead to the conclusion that initial tangent modulus can not be identified with elastic property of the particulate materials including non-cohesive soils, at least with respect to the stress-strain curves obtained in conventional triaxial apparatuses. A better representative of elastic response may be the initial tangent modulus determined for a small range of strains. However, the values obtained are higher than those of Young's modulus corresponding to the reloading-unloading curve. At this stage of the analysis it is difficult to compare specific moduli determined by various methods due to the lack of appropriate test results referring to the same particulate materials. The above conclusions are based on the general observations and studies of various test results on different materials.

It has been found that Poisson's ratio is isotropic and practically constant for a soil at a given void ratio (Lade and Nelson, 1987). For the increasing void ratio of a soil one can observe an increase of Poisson's ratio. The most common way of experimental determining the value of ν is its calculation directly from stress paths in triaxial test with constant confining pressure and measurement of the lateral or volumetric strains, according to the formula (Duncan and Chang, 1970; Lade and Nelson, 1987):

$$\nu = -\frac{\dot{\varepsilon}_3}{\dot{\varepsilon}_1} = \frac{1}{2}\left(1 - \frac{\dot{\varepsilon}_\nu}{\dot{\varepsilon}_1}\right), \tag{3}$$

where the term $\dot{\varepsilon}_\nu/\dot{\varepsilon}_1$ is the slope of the volumetric change curve. Duncan and Chang, 1970 have analysed the values of Poisson's ratio obtained for different phases of triaxial loading and unloading for which this slope was calculated. For example, the last phase of loading curve yielded the values of $\nu = 0.65$. This value is in contradiction to the fundamental restrictions imposed on that coefficient by the theory of elasticity ($\nu \leq 0.5$). A similar tendency of accepting the values of Poisson's ratio greater than 0.5 can also be found in other authors (e.g. see Lambe and Whitman, 1970). Other values of Poisson's ratio obtained by Duncan and Chang are also not too reliable. For example the values of this coefficient for reloading-unloading curve ranged from 0.15 to 0.4 and were larger for unloading than for reloading thus such a method can not correctly describe the value of Poisson's ratio.

Lade and Nelson, 1987, suggest calculating Poisson's ratio from the volumetric curve immediately after stress reversal. The calculation of Poisson's ratio of Lubiatowo sand on the basis of the results presented in Fig. 1b yielded value $\nu = 0.14$.

3. Models for Determination of Young's Modulus

On the basis of experimental observations it is commonly assumed that the elastic properties of particulate materials are functions of the state of the soil and the state of stress acting on it (see e.g. Duncan and Chang, 1967; Seed et al., 1985; Lade and Nelson, 1987; Hardin and Blandford, 1989; Hicher, 1996). The state of the soil is usually expressed in terms of soil density or void ratio whereas the state of stress in terms of confining stress σ_3 or mean effective pressure p which is related with principal stresses by the following formula:

$$p = \frac{\sigma_1 + 2\sigma_3}{3}. \tag{4}$$

Some authors describe these dependencies on stress level and initial density by barotropy and pycnotropy terms, respectively (Bałachowski et al., 1991).

Based on experimental or theoretical observations a number of elastic models with stress dependent moduli have been proposed. The simplest and most acceptable form for calculation of Young's modulus, confirmed by a large number of experiments and theoretical considerations (see e.g. Janbu, 1963; Makhlouf and Stewart, 1963; Ko and Scott, 1967a, 1967b; Hardin and Drnevich, 1972; Krizek et al., 1974; Mindlin and Deresiewicz, 1953; Duffy and Mindlin, 1957) can be expressed as a power function of the initially isotropic, effective confining pressure σ_3:

$$E_j = K_j p_a \left(\frac{\sigma_3'}{p_a}\right)^n, \tag{5}$$

where p_a is atmospheric pressure expressed in the same units as E_j and σ_3', K_j is a modulus number, and n is an exponent determining the rate of variation of E_j with σ_3. The parameters K_j and ν are dimensionless and have to be determined experimentally. Sometimes confining stress in Eq. 5 is replaced by mean effective pressure given by Eq. 4, (Hardin and Drnevich, 1972; Hicher, 1996).

Originally, modulus number K_j was defined as an initial slope of the stress-strain curve for triaxial compression so that the Young's modulus E_j corresponded to initial tangent modulus E_{in}. However, as said in the previous section, during virgin loading both plastic and elastic deformations develop in the sample, even for the very early phase of loading corresponding to small strains. Thus the modulus number K_{in} has been replaced by the slope of an unloading-reloading cycle, denoted as K_{ur} and the Young's modulus in equation 5 refers to E_{ur}, respectively (Duncan and Chang, 1970). The modulus number corresponds in this case to the secant sector connecting points A and B within the hysteresis loop in Fig. 1a.

Janbu, 1963 has experimentally found that for various types of sands the modulus numbers varied from 50 to 500. The analysis of a large number of the experiments (Wong and Duncan, 1974) showed that the modulus number K_{ur} determined from the reloading-unloading curve was from 1 to 3 times higher than modulus number K_{in}.

On the basis of the same tests Janbu, 1963 has determined the range of the power n, which varied from 0.35 and 0.55 for sands and silty sands. Some authors suggest that for soils one single value $n = 0.5$ can be assumed (Hardin and Drnevich, 1972; Seed et al., 1985).

The deformation characteristics of soil may also be expressed through the shear modulus G, which is related to the deviatoric stress component. The shear modulus is usually determined from wave propagation velocities and from small amplitude cyclic simple shear tests and is related to Young's modulus E and Poisson's ratio ν by the well known relationship:

$$G = \frac{E}{2(1+\nu)}. \tag{6}$$

Wroth and Houlsby, 1985 proposed for sands a general expression for shear modulus that has a similar form as in Eq. 4 (Viggiani and Atkinson, 1995):

$$\frac{G}{p_r} = A\left(\frac{p}{p_r}\right)^n, \tag{7}$$

where the dimensionless parameters A and n depend primarily on the nature of the soil and the current strain, and p_r is a reference pressure. However, these small strain tests produce higher modulus values than static tests with larger changes of stress and strain (Lade and Nelson, 1987).

Opposite to Young's or shear moduli, the experimental data indicate that the influence of mean effective pressure on Poisson's ratio is rather small and can be negligible (El Horsi, 1984).

Although the dependence of Young's or shear moduli on mean effective pressure or confining stress has been widely accepted in soil mechanics this problem seems not to be sufficiently clear. Let us consider the reloading-unloading curves shown in Fig. 1a. Let us also assume that according to Duncan and Chang Young's modulus is determined from reloading-unloading curve (sector AB of second hysteresis loop). The value of Young's modulus is constant along the whole stress path corresponding to the reloading-unloading curve. However, along this stress path an essential change of mean effective pressure takes place from almost 310 kPa to reference stress level which is in contradiction to the relationship given by formula (5). Other authors also note this inconsistency of Young's modulus dependence on the mean effective stress, which follows from the analysis of triaxial test results. Lade and Nelson, 1987 state that the expression relating elastic modulus to the mean normal stress is unable to capture the variation of Young's modulus correctly. Some authors clearly do not confirm such dependence (cf. Sawicki, 1994; Sawicki and Świdziński, 1998).

Zytynski et al., 1978 have found that the model for variation of Young's modulus with effective confining pressure results in violation of the principle of conservation of energy. Thus, depending on the direction of a closed stress loop, the model will generate or dissipate energy, which is inconsistent with the elastic behaviour. Lade and Nelson, 1987 have proposed an isotropic model for the elastic behaviour of soils that is based on the theoretical development guaranteeing a lack of energy generation or dissipation for any closed-loop stress or strain path. In the model it is assumed that Poisson's ratio is constant and Young's modulus is a function of mean normal stress as well as deviatoric stress. The considerations of elastic work in accordance with the principle of conservation of energy lead to the differential equation for Young's modulus the solution of which can be given in the form of the following power law:

$$E = Mp_a \left[\left(\frac{I_1}{p_a} \right)^2 + R\frac{J_2'}{p_a^2} \right]^\lambda, \tag{8}$$

in which p_a is the atmospheric pressure expressed in the same units as E. I_1 is the first invariant of the stress tensor and reflects the dependence of E on mean normal stress whereas J_2' is the second invariant of the deviatoric stress tensor corresponding to the deviatoric changes. The modulus number M and the exponent λ are constant, dimensionless numbers which have to be determined experimentally from any type of tests with a measurement of all stresses and strains (triaxial compression tests including unloading-reloading cycles, cubical triaxial tests, etc.). Parameter R is a function of Poisson's ratio and can be calculated from the following formula:

$$R = 6\frac{1+\nu}{1-2\nu}. \tag{9}$$

According to Lade and Nelson's suggestions the Poisson's ratio can be calculated in terms of Eq. 3 from the slopes of the volumetric change curves with reloading parts initiated at or near the hydrostatic axis. Young's modulus can be determined from reloading-unloading curves obtained for a set of tests with different stress paths and confining pressures. The experimental data are then presented in a log-log diagram in which the vertical axis is related to the E/p_a and the horizontal axis refers to the expression $[(I_1/p_a)^2 + R(J_2'/p_a^2)]$. The intercept of the best-fitting straight line with vertical axis is the value of M, and l is the slope of the line.

The results presented by the authors suggest that the method proposed deserves attention however, it suffers from some shortcomings and inconsistencies. The first concerns a large number of experiments to be done in order to determine the parameters of the analysed material. The second shortcoming is related to the interpretation of the stress-strain curve in order to determine Young's modulus from the reloading-unloading branch. For this purpose test results are plotted in the deviatoric stress versus axial strain representation, whereas the lateral deformation of the sample is excluded from the analysis. The next one considers the determination of Poisson's ratio from the volumetric curve. It has already been shown that the slope of volumetric strain-axial strain curve may not represent this coefficient correctly. In addition, some authors do not confirm the dependence of deviatoric stress on Young's modulus. Analysing series of experiments performed on Hostun Sand subjected to the small range of stresses Hicher, 1996 concludes that the influence of deviatoric stress on elastic modulus does not have to be taken into account, which is in contrast to the main assumptions accepted in the Lade and Nelson proposal.

Although presented data from a triaxial test performed on dense Lubiatowo sand (Fig. 1) unable the finding of the full form of equation 5 however, the elastic constants can be determined according to Duncan and Chang, 1970 and Lade and Nelson suggestions. Young's modulus calculated as a slope of unloading-reloading curve is equal to $E_{ur} = 1.63 \times 10^8$ N/m^2.

A different concept of the determination of elastic constants is presented by Sawicki, 1994 (see also Sawicki and Świdziński, 1998). The method is based on a new interpretation of oedometric tests, with the additional measurement of lateral stresses. Typical experimental results corresponding to the dense Lubiatowo sand subjected to one cycle of loading and unloading are shown in Fig. 5. During the virgin loading one follows the straight line OA in the stress space (Fig. 5b). This is the well-known K_0 – line from $\sigma_x = K_0\sigma_z$. During the unloading one follows a different path in the stress space, denoted as ABC. After unloading, i.e. when the vertical stress is removed, there is a residual stress in the soil (point C in Fig. 5b).

In the case considered the unloading path has been approximated by two linear sectors, denoted as AB and BC in Fig. 5. Such a bilinear approximation of experimental data plays a key role in this approach. Analysis of extensive experimental data supported by some theoretical consideration on the mechanical behaviour of the wedge (Sawicki, 1996) allowed for the conclusion that the first stage of unloading can be treated as an elastic response of the material (sector AB in Fig. 5). This elastic response has linear form and does not depend on the maximum stress level which is in contrast to the commonly accepted opinion.

It can be shown, on the basis of some theoretical considerations, that during the first stage of unloading the elastic and plastic parts of lateral deformation are equal to zero. Such an assumption allows for the simple determination of Young's modulus and Poisson's ratio from the following formulae:

$$\nu = \frac{1}{1+a} \tag{10}$$

$$E = E^* \left[1 - \frac{2}{a(1+a)}\right] \tag{11}$$

where a denotes the slope of the unloading path AB in the stress space:

$$a = \frac{\sigma_z^A - \sigma_z^B}{\sigma_x^A - \sigma_x^B} \tag{12}$$

and E^* is the slope of the unloading sector AB in the σ_z, ε_z space:

$$E^* = \frac{\sigma_z^A - \sigma_z^B}{\varepsilon_z^A - \varepsilon_z^B}. \tag{13}$$

Analysis of the large number of tests on different particulate materials, in which the samples were subjected to several cycles of loading and unloading,

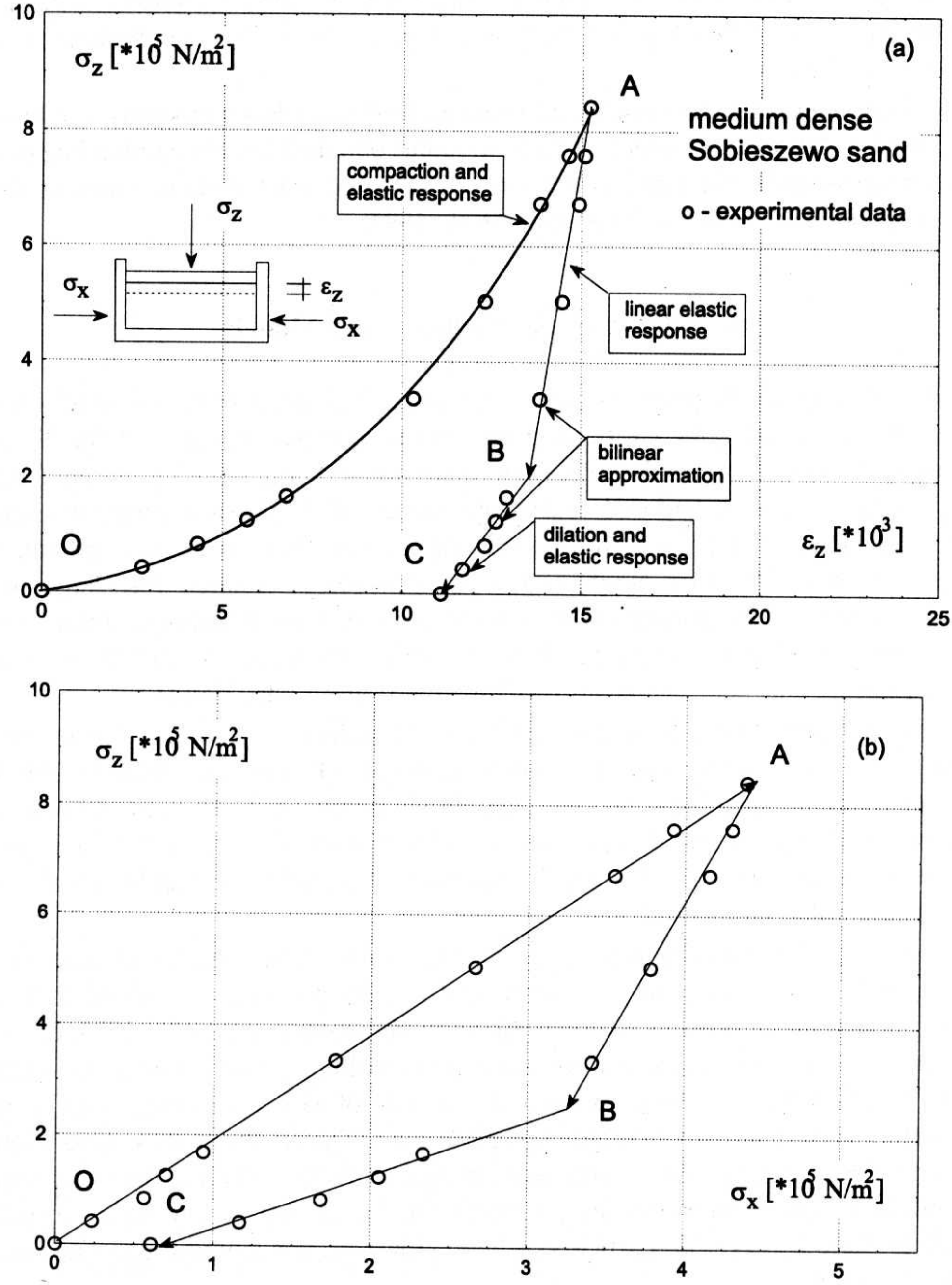

Fig. 5. Typical experimental results of oedometric test with additional measurement of lateral stresses on dense Lubiatowo sand

has shown that the initial and current state of the material do not significantly influence the values of elastic moduli. In the case of dense Lubiatowo sand the following values were obtained: Poisson's ratio $\nu = 0.17$, Young's modulus $E = 3.35 \times 10^8$ N/m^2.

In some recent papers (Pezo and Hudson, 1994) the elastic modulus is defined by so-called "resilient modulus", i.e. the modulus obtained from the unloading part of the stress-strain diagram "as the ratio of repeated axial deviator stress to the recoverable axial strain" (Mohammad et al., 1994).

4. Validity of the Methods and Models

In determining elastic properties the range of applied stress levels for which they can serve to describe the elastic response of the material should be defined. Although the elastic properties constitute an intrinsic feature of the material the high level of stresses imposed causes the change of its physical properties and consequently material itself. It is commonly known that under high pressures (higher than 1 MPa $= 10^6$ N/m^2) such a phenomenon as crushing (breakage and fragmentation) of the particles takes place which causes the material to differ from its initial state. It was found that effects of particle breakage on soil behaviour at high pressure was very significant (cf. Yamamuro and Lade, 1996).

Fig. 6 presents a conceptual interpretation of hydrostatic and one-dimensional compression for freshly deposited cohesionless soils (Pestana and Whittle, 1995). It can easily be seen that for the range of so-called low level stresses there is almost no change of initial void ratio. Limiting compression curve (LCC) begins to be non-linear above 1 MPa which suggests that particle crushing begins above this stress level.

An extensive experimental study on the stress-strain, volume change and strength behaviour on dense Cambria sand at high pressure in drained triaxial compression and extension tests was made by Yamamuro and Lade, 1996. In the experiments the samples of sand were tested at confining pressures from the range of 0.05 to 52 MPa. In order to evaluate the amount of particle crushing, after completion of each test a sieve analysis was performed. The results of these tests show that particle crushing is the single most important factor affecting the behaviour of granular soil at high pressures. It has been found that at low mean normal stresses there is very little particle crushing. However, this amount increases rapidly at 4 MPa.

Therefore, in the methods of determination of elastic constants of particulate materials based on the conventional laboratory devices the upper limit of confining pressure should be restricted to 1 MPa.

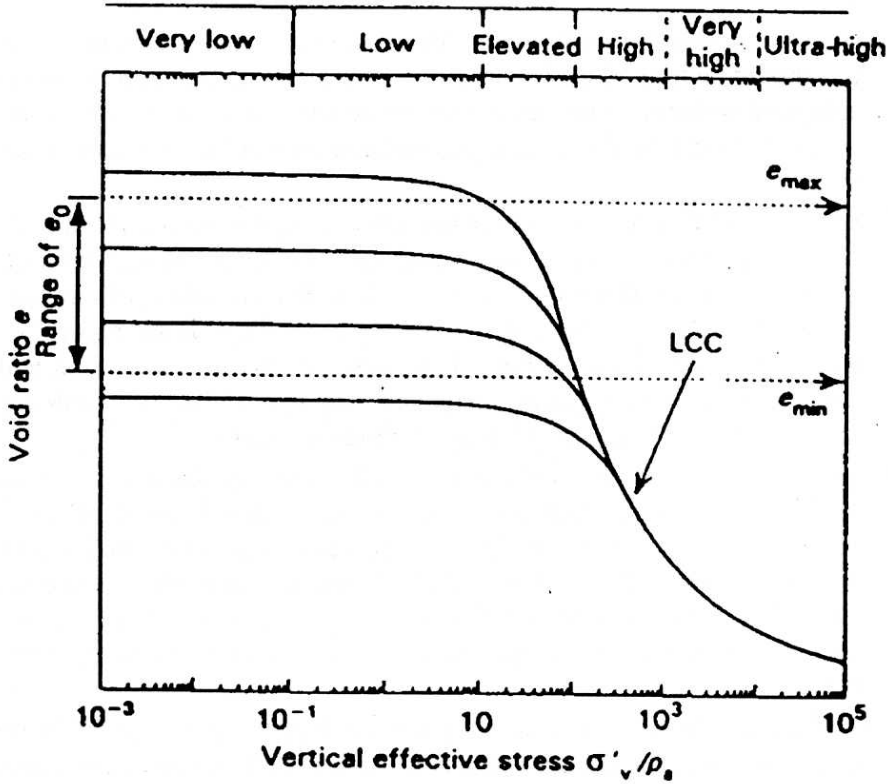

Fig. 6. Conceptual interpretation of hydrostatic and one-dimensional compression for freshly deposited cohesionless soils (e_{max} and e_{min} are index properties defined by ASTN D4253-93 and D4254-91), (after Pestana and Whittle, 1995)

5. Conclusions

In the paper, various methods of determining elastic constants of particulate materials were analysed and discussed. Despite of much effort made in the attempt to isolate the elastic response experimentally and variety of direct and indirect methods making use of different types of laboratory apparatuses and equipment there is still no general reliable experimental procedure for extracting the elastic deformations. Various proposals based on either small strains' measurements or relatively larger deformations involved in the analysis usually suffer from some shortcomings an inconsistencies and that sometimes apparently violate the foundations of the theory of elasticity. At this stage, it is rather difficult to indicate the most appropriate and reliable method that will yield the most representative values characterising the real elastic response. However, some general conclusions can be drawn and comparisons of methods made. These conclusions are as follows:

1. An assumption of isotropic behaviour of elastic response can be justified, but should be treated as simplification of the real material response. This assumption reduces a number of the elastic coefficients necessary to describe elastic response of the particulate material to two independent elastic constants.
2. Due to elastic and plastic deformations developing during the initial phase of loading in the conventional triaxial test, the initial tangent modulus cannot be identified with elastic response. Usually, the values of initial tangent moduli are several times lower than corresponding elastic moduli determined in terms of other methods. It is caused by plastic strains that make the slope of loading curve more inclined to the horizontal strain axis resulting in the decrease of the initial tangent modulus value.
3. Initial tangent modulus obtained from the tests in which small strains are measured can be considered to be a better approximation of real elastic behaviour. However, it should be noted that, in general, small strain tests, both static and dynamic produce higher modulus values than tests with larger changes of strain. In addition, the reliability of accuracy of measurements in such tests is difficult to accept, especially in the case of particulate materials tested.
4. For conventional tests in which relatively larger strains are measured the elastic response is often determined from reloading-unloading curve. The triaxial test performed on dense sand showed that the unloading-reloading curve has a curvilinear character with apparent hysteresis loop, therefore such methodology does not correspond exactly with the real elastic response of the material. A much better solution is to determine the Young's modulus value from the first stage of unloading (see, Sawicki, 1994; Sawicki and Świdziński, 1998).
5. Poisson's ratio cannot be determined from an arbitrary part of volumetric strain – axial strain curve. More reliable seems to be the Lade and Nelson proposal to calculate this elastic constant from a slope of the curve immediately after stress reversal or application of the method proposed by Sawicki, 1994.
6. Basic shortcoming of almost all experimental methods of Young's modulus determination is neglecting the lateral strains. More careful and complete analysis according to the theory of elasticity would require including all strain components.
7. The dependence of Young's modulus on the stress level is not sufficiently proved and requires further investigations.

8. In the experimental methods for determination of elastic constants for particulate materials upper limit of stress level should be considered. This stress level is related to the crushing of the particles changing the initial state of the material and subsequently its elastic properties. Such limits should not extend the value of confining pressure of 1 MPa.

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