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# Cam-clay Approach to Modelling Pre-failure Behaviour of Sand against Experimental Data

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**Abstract**

The aim of this paper is to compare pre-failure deformations of sand, obtained from a Cam-clay type elastic-plastic model, with experimental data. The experiments were performed in a modern triaxial apparatus, enabling local measurement of both vertical and lateral strains. The experimental programme is briefly described, then a slightly modified, Cam-clay type, elastic-plastic model of sand is presented. Predictions of this model, for different stress paths, are compared with respective experimental results, then extensive discussion of basic problems connected with elasto-plastic modelling of sand is presented. The aim of the discussion is to display some shortcomings of this type of modelling.

**Key words:** plasticity, experiment, Cam-clay, theoretical prediction

## 1. Introduction

The theory of plasticity is the heart of contemporary soil mechanics if measured by a number of various constitutive models, respective journals, publications or conferences. The role of soil plasticity is of basic importance in geotechnical engineering, as it provides rational tools enabling the estimation of both the bearing capacity and deformations of earth structures, which are fundamental engineering needs regarding the design and construction of various objects. The opinions regarding quality and usefulness of soil plasticity differ greatly. For example, Zienkiewicz et al (1999) reckon that the great variety of existing soil models are able to deal with most of the observed features of the mechanical behaviour of these materials. More critical opinions regarding the constitutive relations, which are the central point of theoretical soil mechanics, have been formulated by Bolton (2000, 2001) and Kolymbas (2000). Some international workshops have also revealed that a large number of constitutive models of soils have not positively passed the basic experimental tests regarding the stress-strain response, cf. Saada and Bianchini (1989). Note that these are experiments which ultimately decide as to the value of a particular theory.

It seems that the only reliable theory, which constitutes the backbone of soil mechanics, is the theory of limit states which enables efficient prediction of limit loads, sufficient for practical reasons. Mathematical structure of the classical theory of limit states is fairly simple and convincing from the physical point of view. This theory and its simplified variants have already found their proper place in basic textbooks, as good standards. Many more problems exist in connection with reliable prediction of strains in stages preceding failure. There exist tens, or perhaps even hundreds, of various models, but none of them has been accepted as being of reliable standard, as these models rarely give predictions which are conformable with experimental data, and their mathematical structure does not always conform with the physical intuition, which is important for engineers. There therefore exists a need to elaborate a simple engineering model that would enable estimation of strains preceding soil failure. The first step of such research should be examination of already existing elastic-plastic models of soils, where the basic criterions should be: simplicity, conformity with physical intuition and with experimental data, self-consistence and conformity with general laws of physics.

In the Institute of Hydro-Engineering, we have initiated a programme of experimental investigations of deformations of granular media in stages preceding failure. The experiments were carried out in a computer-controlled hydraulic triaxial testing system from GDS Instruments Ltd. (UK) which had been additionally equipped with special gauges for local measurement of both vertical and radial deformations. Such measurement is more reliable than traditional external measurement of strains which is subjected to a number of uncontrolled errors, therefore the local strain gauges are strongly recommended in many publications, see Tatsuoka and Shibuya (1992), Jamiolkowski et al (2001). More realistic experimental data should also be helpful for developing better models of soils.

The above investigations provide some data which can be used for calibration and verification of existing models of soils. For comparative analysis we have chosen the Cambridge-type approach, as it has become a kind of standard in soil mechanics, starting with the famous book by Schofield and Wroth (1968), through several academic textbooks (Atkinson 1993, Atkinson and Bransby 1978, Wood 1990) and other monographs (Zaman, Gioda and Booker 1999, Zienkiewicz et al 1999), up to recent papers regarding this matter (Jefferies and Shuttle 2002). The Cambridge philosophy has established a kind of paradigm in contemporary soil mechanics, hence it would seem natural to begin with the examination of the Cambridge-type approach to modelling the pre-failure deformations of soils.

In this paper, a slightly modified version of the original Cam-clay model has been derived and then compared with experimental data. This comparison forms a basis for extensive discussion of fundamental problems related to elasto-plastic modelling of sand behaviour.

## 2. Experimental Programme

The triaxial testing system GDS Instruments is described by Menzies (1988), and its properties regarding the measurement of strains were investigated by Świdziński and Mierczyński (2002). Their investigations support the conclusions of such other experimentalists as, for example, Tatsuoka and Shibuya (1992) that local measurement of strains is more reliable than traditional external measurement, partcularly for small strains, not exceeding the value of $10^{-2}$. The accuracy of measurement was $10^{-5}$. We have investigated the behaviour of various granular materials, but in this paper we shall refer only to the results corresponding to initially loose *Skarpa* sand. The basic characteristics of this sand are the following: mean grain diameter 0.42 mm; minimum void ratio $e_{\min} = 0.432$; maximum void ratio $e_{\max} = 0.677$; coefficient of uniformity $C_u = 2.5$; specific gravity $G = 2.65$; angle of internal friction of loose sand $\varphi = 33°$; angle of internal friction of initially dense sand $\varphi = 41°$. The sand is built of medium subrounded grains.

Cylindrical soil samples were approximately 80 mm high with an average diameter of 38 mm. All specimens were prepared by air-dry pluviation from different heights, which varied depending on the required density. Special attention was paid to prepare loose specimens. In this case, the funnel with sand was raised very slowly keeping its open end just above the specimen surface. After having prepared a sample, the local strain gauges were assembled then the triaxial cell was installed.


**Fig. 1.** Basic stress paths

Fig. 1 shows the basic stress paths applied in the experiments, presented in the space of stress invariants, defined as follows:

$$p = \frac{1}{3}(\sigma_1 + 2\sigma_3), \tag{1}$$

$$q = \sigma_1 - \sigma_3, \tag{2}$$

where $\sigma_1$= vertical stress, $\sigma_3$= horizontal stress. Particular stress paths will be discussed, in more detail, in subsequent sections. The vertical and horizontal displacements were recorded, and then respective strains calculated ($\varepsilon_1, \varepsilon_3$). For traditional reasons, the experimental results were also presented in the form of relationships between the stress invariants and the respective strain invariants, defined as follows:

$$\varepsilon_v = \varepsilon_1 + 2\varepsilon_3, \tag{3}$$

$$\varepsilon_q = \frac{2}{3}(\varepsilon_1 - \varepsilon_3), \tag{4}$$

where $\varepsilon_v$ = volumetric strain, $\varepsilon_q$ = deviatoric strain.

In order to deal with experimental data, the professional program *Statistica* have been used, which enables their presentation in various ways, see Sawicki and Świdziński (2002a). For the sake of convenience, some special stress and strain units have been introduced to analyse the experimental data, namely: stress unit $10^5$ N/m$^2$ and strain unit $10^{-3}$. For example, $p = 2$ means that the real value of mean stress is $2 \times 10^5$ N/m$^2$, and $\varepsilon_q = 3$ denotes the real value of this strain $3 \times 10^{-3}$.

## 3. Theoretical Model

### 3.1. General Relations

Elasto-plastic modelling of soil behaviour has had a long tradition, and most of the existing models have been formulated within this framework. It is impossible in this paper to review these models since their already large number is still growing. The general structure of elasto-plasticity is fairly simple, and is presented in commonly available publications, see Scott (1985), Wood (1990), Zaman et al (1999), Zienkiewicz et al (1999). In this Section, the already famous *modified Cam-clay* model will be outlined, with some minor changes which follow from a slightly different than original definition of volumetric strain and from different formulae describing soil's elastic response. Cambridge-type model has been chosen for comparative analysis for a few important reasons, namely:

a) Cam-clay and its variants are widely known and are often used as standard options in FE codes;
b) Cam-clay approach forms a very broad framework for elasto-plastic modelling of soils, and is therefore convenient for discussing some basic problems related to this kind of modelling;
c) a structure of Cam-clay is simple and elegant, which still attracts research workers, cf. Jefferies and Shuttle (2002), and is also useful for pedagogical purposes, see Wood (1990, 1999).

A methodology presented in the book of Wood (1990) will be followed in this Section. He mentions the four ingredients of an elastic-plastic model, namely: elastic properties, yield surface, plastic potential and hardening rule. Assume that the elastic properties of sand are known (respective equations will be presented later), and concentrate attention on determination of the plastic strains. The yield surface is assumed in a well-known form, as an ellipse in the space of triaxial stress invariants:

$$f = q^2 - M^2[p(p_0 - p)] = 0, \tag{5}$$

where $p_0$ = the stress controlling the size of an ellipse, $M$ = parameter controlling its shape, see Fig. 2.

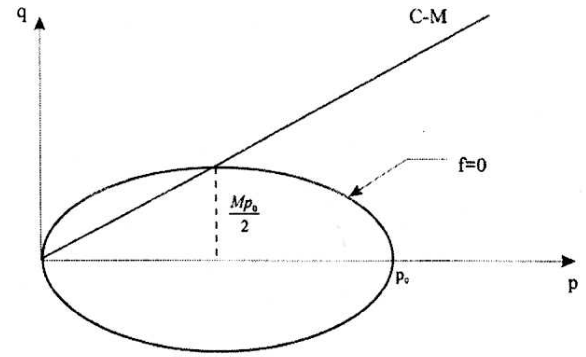



**Fig. 2.** Elliptical yield surface

In the case of dry granular media, coefficient $M$ is related to the Coulomb-Mohr failure criterion, see Atkinson (1993):

$$M = \frac{6 \sin \varphi}{3 - \sin \varphi}. \tag{6}$$

The notion of yield surface has a fundamental meaning in the elasto-plastic modelling of material behaviour, as it bounds a region in the stress space where the soil's behaviour is elastic, as it also helps to distinguish whether subsequent stress increment denotes loading or unloading.

The yield condition (5) is identified with the plastic potential, thus the increments of plastic volumetric and deviatoric strains are given by the following relations:

$$d\varepsilon_v^p = d\lambda \frac{\partial f}{\partial p}, \tag{7}$$

$$d\varepsilon_q^p = d\lambda \frac{\partial f}{\partial q}, \tag{8}$$

where $d\lambda$ = scalar multiplier and superscript $()^p$ denotes the plastic part of respective strain.

The stress $p_0$ controlling the size of yield surface is linked with the increments of plastic strains by the following hardening rule:

$$dp_0 = \frac{\partial p_0}{\partial \varepsilon_v^p} d\varepsilon_v^p + \frac{\partial p_0}{\partial \varepsilon_q^p} d\varepsilon_q^p. \tag{9}$$

The condition that during the loading the stress state remains on the yield surface is

$$df = \frac{\partial f}{\partial p} dp + \frac{\partial f}{\partial q} dq + \frac{\partial f}{\partial p_0} dp_0 = 0. \tag{10}$$

Eqs. (5), (7)–(10) are sufficient for determination of incremental relations for the plastic strains that develop during loading.

### 3.2. Specific Relations

Before deriving the incremental relations for plastic strains, let us consider different definitions of volumetric strain, and their influence on the relations between the increments of total volumetric strain $d\varepsilon_v$ and specific volume $dv$. Let $V_0$ denote the volume of soil before deformation, and $V$ the volume containing the same grains after deformation. For small deformations, the volumetric strain can be defined as:

$$\varepsilon_v = \frac{V_0 - V}{V_0}, \tag{11}$$

where the soil mechanics sign convention is adapted (plus sign means compression). It can be shown (Sawicki and Świdziński 2002b) that

$$d\varepsilon_v = -\frac{1}{v_0} dv, \tag{12}$$

where $v_0$ = specific volume before deformation. In publications devoted to Cambridge models, cf. Atkinson (1993), Wood (1990), a different relation is recommended, namely:

$$d\varepsilon_v = -\frac{1}{v} dv, \tag{13}$$

following from a different definition of volumetric strain. Integration of Eq. (13) and simple algebraic manipulations lead to the following expression:

$$\varepsilon_v = -\ln \frac{V}{V_0}. \tag{14}$$

For small strains, Eqs. (11) and (14) give practically the same results, but we shall adapt definition (11) as it leads to simpler constitutive equations.

The hardening parameter $p_0$ is assumed to be dependent on the plastic strain increments (Eq. 9), and a specific form of this relationship should be proposed. Again, we shall follow the Cambridge approach by assuming that $p_0$ depends on the plastic volumetric strain which develops during isotropic compression.

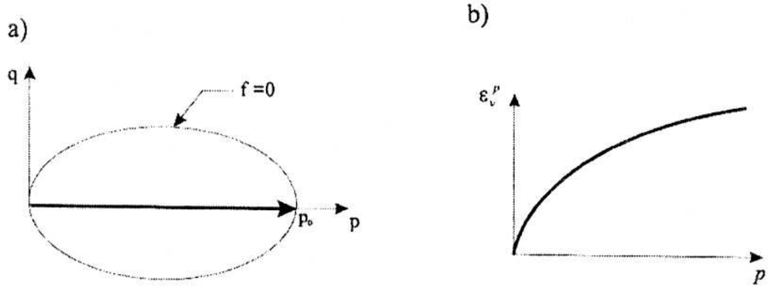



**Fig. 3.** Yield surface generated during isotropic compression a); associated development of plastic volumetric strains b)

Fig. 3 shows schematic illustration of the plastic volumetric strain that develops during isotropic compression. The following formula nicely approximates the experimental results, see Sawicki and Świdziński (2002a):

$$\varepsilon_v^p = A_v^p \sqrt{p_0}, \tag{15}$$

where $A_v^p$ is constant for a given initial relative density of sand. Note that Eq. (15) is a formal alternative to the original logarithmic representation of empirical data. Differentiation of Eq. (15) leads to the following formula:

$$\frac{\partial p_0}{\partial \varepsilon_v^p} = \frac{2\sqrt{p_0}}{A_v^p}. \tag{16}$$

Note that for isotropic material $\partial p_0 / \partial \varepsilon_q^p = 0$, so Eq. (9) takes the following form:

$$dp_0 = \frac{2\sqrt{p_0}}{A_v^p} d\varepsilon_v^p = d\lambda \frac{2M^2(2p - p_0)\sqrt{p_0}}{A_v^p} = \xi d\lambda. \tag{17}$$

Experimental data of Sawicki and Świdziński (2002a) show that during isotropic compression, some deviatoric strains are also generated, which indicates that investigated samples display an anisotropic character. Their value is about 10% of volumetric strains, and it is a matter of subjective judgement whether such deviatoric strains can be neglected in the first approximation. The problem of accuracy of experimental data and their interpretation will be discussed in subsequent sections. Assuming that during isotropic compression only the volumetric strains develop, we obtain from Eqs. (7), (8) and (10) the following relation:

$$dp_0 = \left(2 - \frac{p_0}{p}\right) dp + \frac{2q}{M^2 p} dq. \tag{18}$$

Comparison of Eqs. (17) and (8) leads to determination of a scalar multiplier:

$$d\lambda = \frac{1}{\xi}\left[\left(2 - \frac{p_0}{p}\right)dp + \frac{2q}{M^2 p}dq\right]. \tag{19}$$

Therefore, the problem of determination of the plastic strain increments has been solved. Eqs. (5), (7), (8) and (19) lead to the following explicit formulae:

$$\left\{ \begin{array}{c} d\varepsilon_v^p \\ d\varepsilon_q^p \end{array} \right\} = \frac{A_v^p}{2M\sqrt{p(M^2+\eta^2)}}\left[ \begin{array}{cc} M^2-\eta^2 & 2\eta \\ 2\eta & \frac{4\eta^2}{M^2-\eta^2} \end{array} \right]\left\{ \begin{array}{c} dp \\ dq \end{array} \right\}, \tag{20}$$

where $\eta = q/p$.

The multiplier on the RHS of Eq. (20) is different from that in Eq. (5.12) of Wood (1990) which is a consequence of the different definition of volumetric strain and different shape of the function approximating experimental data. The elastic strain increments can be approximated by the following relation, see Sawicki and Świdziński (2002a):

$$\left\{ \begin{array}{c} d\varepsilon_v^e \\ d\varepsilon_q^e \end{array} \right\} = \frac{1}{\sqrt{p}}\left[ \begin{array}{cc} M_{vv} & M_{vq} \\ M_{qv} & M_{qq} \end{array} \right]\left\{ \begin{array}{c} dp \\ dq \end{array} \right\}, \tag{21}$$

where $M_{vv}$ etc. are constant coefficients and superscript $()^e$ denotes the elastic part of the respective strain. Eq. (21) also differs from the classical relations describing the soil's elastic response. Discussion of this problem is presented in Sawicki and Świdziński (2002c).

## 4. Predictions Against Experimental Data

This Section is devoted to comparison of experimental data with theoretical predictions obtained from a simple elastic-plastic model described in the previous Section. A comparative study will be performed for the data corresponding to initially loose *Skarpa* sand (average initial relative density $D_r \cong 0.3$, see Sawicki and Świdziński (2002a)), for which $A_v^p = 1.49$ and $M = 1.33$, which follows from Eq. (6). The matrix of elastic compliances, appearing in Eq. (21), has the following form (recall the stress and strain units):

$$\mathbf{M} = \left[ \begin{array}{cc} 2.14 & -1.74 \\ -0.232 & 1.55 \end{array} \right]. \tag{22}$$

Theoretical predictions will be obtained by integration of Eqs. (20) and (21) for given loading paths.

### 4.1. Shearing at Constant Mean Stress

Shearing at constant mean stress is represented by stress path AC in Fig. 1. First, the sample is isotropically compressed (path 0A), then the stress deviator increases whilst the mean stress is kept constant ($dp = 0$). Elastic strains can be calculated immediately ($q = 0$ is the reference state):

$$\varepsilon_v^e = -\frac{1.74}{\sqrt{p}}q, \qquad \varepsilon_q^e = \frac{1.55}{\sqrt{p}}q. \tag{23}$$

The above equations can be rewritten in the following form, convenient for comparison with experimental data:

$$\frac{\varepsilon_v^e}{\sqrt{p}} = -1.74\eta \ , \qquad \frac{\varepsilon_q^e}{\sqrt{p}} = 1.55\eta. \tag{24}$$

The plastic volumetric strains are given by the following differential equation:

$$\frac{d\varepsilon_v^p}{dq} = \frac{A_v^p}{M\sqrt{p}}\frac{q}{\sqrt{M^2p^2+q^2}}, \tag{25}$$

the solution of which (with the zero initial condition) can be found analytically:

$$\frac{\varepsilon_v^p}{\sqrt{p}} = A_v^p\left(\sqrt{1+\eta^2/M^2}-1\right). \tag{26}$$

The total volumetric strain divided by $\sqrt{p}$ is the sum of Eqs. (24a) and (26). A differential equation describing the plastic deviatoric strains is the following:

$$\frac{d\varepsilon_q^p}{dq} = \frac{2A_v^p\sqrt{p}}{M}\frac{q^2}{(M^2p^2-q^2)\sqrt{M^2p^2+q^2}}, \tag{27}$$

or in an alternative form:

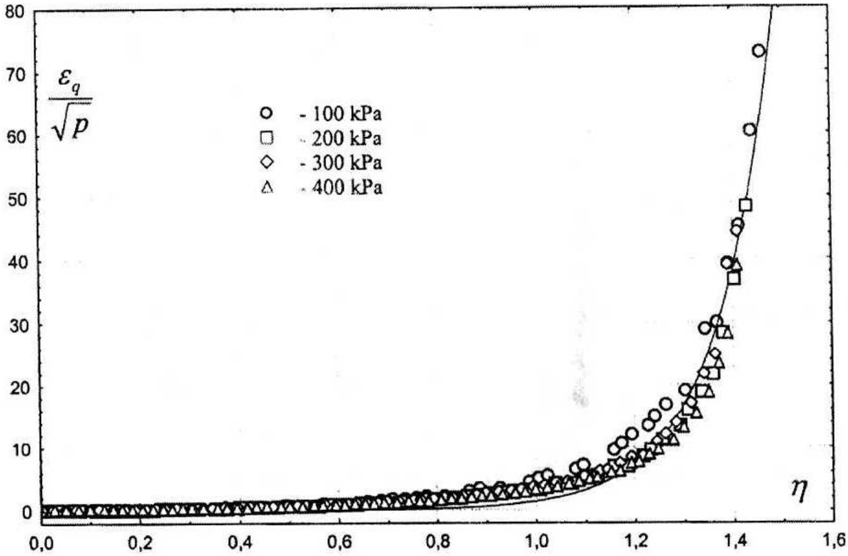
$$\frac{d\varepsilon_q^p}{d\eta} = \frac{2A_v^p\sqrt{p}}{M}\frac{\eta^2}{(M^2-\eta^2)\sqrt{M^2+\eta^2}}. \tag{28}$$

It was rather difficult to find a simple analytical solution of this equation, therefore we have decided to integrate it numerically.

Fig. 4 shows collated experimental results corresponding to shearing at constant mean stress of initially loose *Skarpa* sand. These results have been obtained from four experiments performed with different values of the mean stress, namely 1, 2, 3 and 4 (in unit $10^5$ N/m$^2$). Fig. 5 shows a comparison of theoretical predictions with experimental data in a scale different from that in Fig. 4.

The prediction of volumetric strain differs greatly from experimental results. It shows steadily increasing dilation whilst the experiments show initial compaction followed by dilation. The deviatoric strains display similar qualitative behaviour, but quantitative agreement is worse.

a)




b)

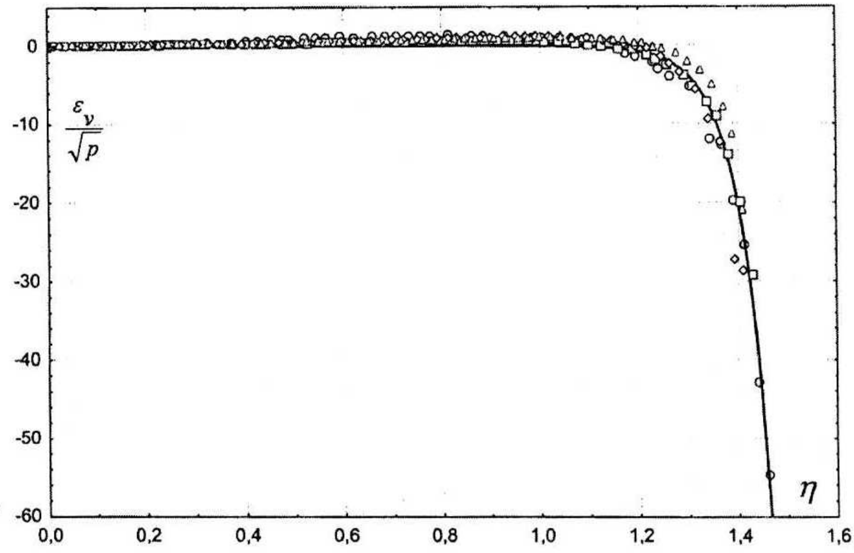



**Fig. 4.** Shearing at constant mean stress of initially loose *Skarpa* sand. Collated experimental results

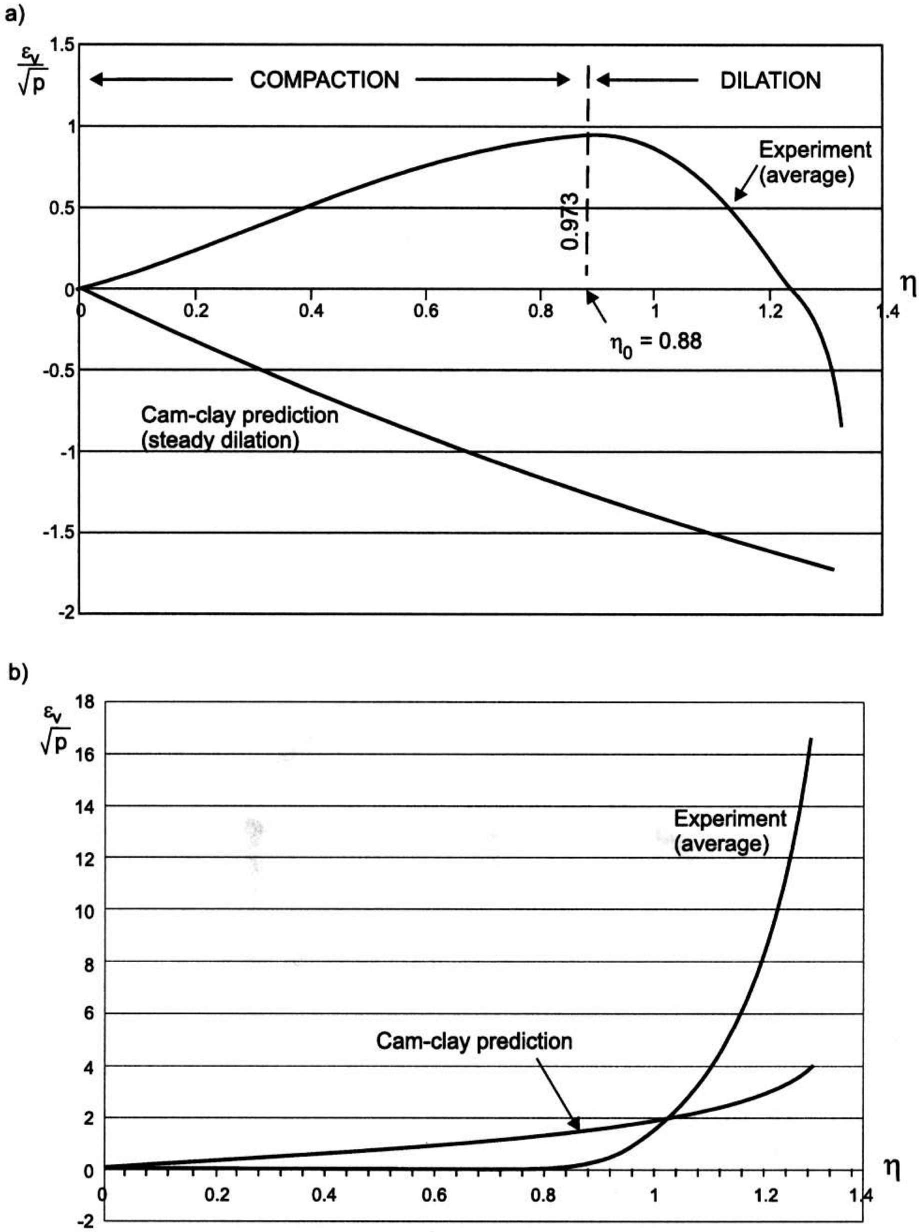



**Fig. 5.** Experimental data against theoretical predictions, cf. Fig. 4

### 4.2. Anisotropic Compression

Anisotropic compression is represented by stress path OD (Fig. 1), along which $\eta = \text{const}$ ($dq = \eta dp$). In this case, Eqs. (20) and (21) can be easily integrated, giving the following expressions for the total strains:

$$\varepsilon_v = \left[\frac{A_v^p\sqrt{M^2+\eta^2}}{M} + 2\left(M_{vv} + \eta M_{vq}\right)\right]\sqrt{p} = R\sqrt{p}, \tag{29}$$

$$\varepsilon_q = \left[\frac{2A_v^p\eta\sqrt{M^2+\eta^2}}{M\left(M^2-\eta^2\right)} + 2\left(M_{qv} + \eta M_{qq}\right)\right]\sqrt{p} = S\sqrt{p}. \tag{30}$$

Coefficients $R$ and $S$, appearing in Eqs. (29) and (30), have been calculated for average input data (recall: $A_v^p = 1.49$, $M = 1.33$), and for values of the stress ratio $\eta$ applied in the experiments, and then compared with respective experimental results, see Table 1. The experimental results have been approximated by the same functions as those in Eqs. (29) and (30).

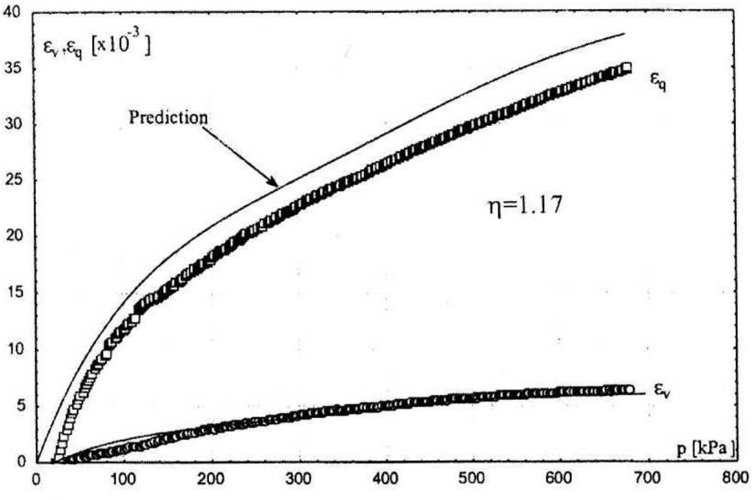



**Fig. 6.** Anisotropic compression of loose *Skarpa* sand ($\eta = 1.17$). Theoretical prediction against experimental data

Before analysing these data, note that experimental results correspond to four different samples of initially loose *Skarpa* sand. These samples were prepared using the same technique as that during preparation of other samples, including those investigated under hydrostatic compression (see Section 2). As already mentioned, those samples had displayed some anisotropic features, so it is very

**Table 1.** Anisotropic compression of loose *Skarpa* sand. Theoretical predictions against experimental data. Coefficients $R$ and $S$ are defined in Eqs. (29) and (30)

| No. | $D_r$ | $\eta$ | $R$ pred. | $R$ exp. | $S$ pred. | $S$ exp. |
|---|---|---|---|---|---|---|
| 1 | 0.33 | 0.4 | 4.45 | 7 | 1.55 | 0 |
| 2 | 0.32 | 0.75 | 3.38 | 5.5 | 3.99 | 1.77 |
| 3 | 0.34 | 0.875 | 3.02 | 6.5 | 5.36 | 4.08 |
| 4 | 0.282 | 1.17 | 2.19 | 2.5 | 14.74 | 13.1 |

probable that other samples, subjected to different stress paths, were also characterised by some degree of anisotropy. The results of experiment No. 1 from Table 2 support this observation, as in the special case of anisotropic compression ($\eta = 0.4$), the deviatoric strain is equal to zero. In the case of an initially isotropic sample subjected to anisotropic compression, there should have been some deviatoric strain. In the first theoretical approximation, the deviatoric strains are neglected, but it should be remembered that ideal isotropy is a rare property of soils. Very special methods are necessary in order to prepare isotropic samples, and it seems that this important practical problem has not yet been solved. We should therefore treat investigated samples as 'almost isotropic', which obviously influences criteria regarding conformity between experimental data and theoretical predictions.

There is a qualitative agreement between theory and experiment in the case of anisotropic compression, due to the similar character of the stress-strain curves. Moreover, experimental results have been approximated by a simple function ($y = a\sqrt{x}$), which is the same as that obtained theoretically. For practical reasons, it is important to use the most simple mathematical functions. Quantitative agreement is worse, but in one case (No. 4, see Fig. 6) it may be regarded as quite good.

### 4.3. Unloading

According to classical interpretation, the current yield surface bounds the region of soil's elastic behaviour. Each stress increment, directed inwards of the yield surface, means elastic unloading. Therefore, the stress path AB shown in Fig. 7 should produce only elastic strains, and the stress cycle ABA should give zero net strain.

The experiment performed on initially loose *Skarpa* sand shows that during the stress cycle ABA, both volumetric and deviatoric plastic strains develop, which means that classical definition of unloading is not realistic. Fig. 8 shows respective strains that develop during this stress cycle, after initial isotropic compression (path OA).

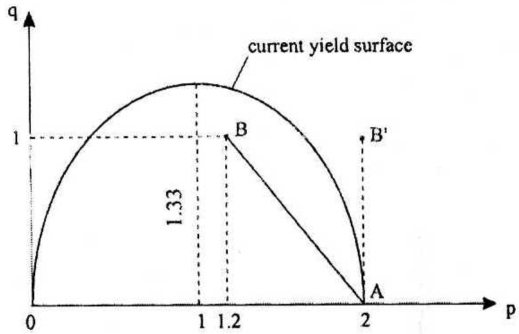



**Fig. 7.** Does the stress path AB really correspond to unloading?

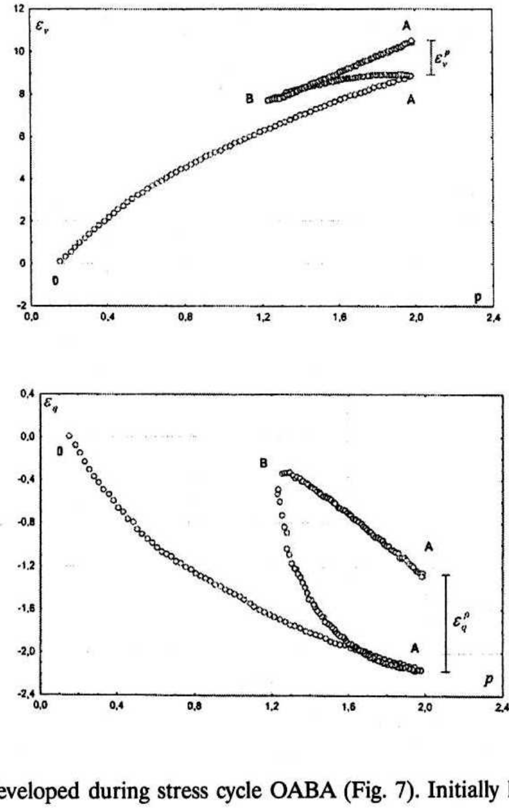


**Fig. 8.** Strains developed during stress cycle OABA (Fig. 7). Initially loose *Skarpa* sand

### 4.4. Conventional Stress Path

Conventional triaxial test corresponds to the stress path OAE in Fig. 1. First, the sample is isotropically pre-compressed (path OA) up to stress state $\sigma_1 = \sigma_3 = \sigma_0$. Then, the horizontal stress is kept constant whilst the vertical stress increases. The stress-strain state at point A is assumed as a reference level, as we shall analyse the sample behaviour along path AE, where $\sigma_1 = \sigma_0 + x$, $\sigma_3 = \sigma_0 =$ const, $x =$ new stress variable. Therefore, along AE, we have:

$$p = \sigma_0 + x/3, \qquad q = x. \tag{31}$$

Elastic strains that develop along path AE can be obtained easily by integration of Eq. (21). For the input data one obtains:

$$\varepsilon_v^e = 3.544\left(\sqrt{3\sigma_0} - \sqrt{3\sigma_0 + x}\right), \tag{32}$$

$$\varepsilon_q^e = 5.102\left(\sqrt{3\sigma_0 + x} - \sqrt{3\sigma_0}\right). \tag{33}$$

The plastic strains have been computed numerically from Eqs. (3.20). The total volumetric and deviatoric strains for $\sigma_0 = 1$, are shown in Fig. 9, where the experimental results are also presented. Predicted volumetric strains differ greatly from experimental ones, as in the case of pure shearing. Deviatoric strains have a similar qualitative character, but they differ quantitatively.

## 5. Discussion

### 5.1. On Accuracy of Predictions

The aim of each theoretical model is to provide predictions which are conformable with empirical observations. There are two main aspects of such conformity, namely qualitative and quantitative agreement between theoretical predictions and experimental data. It seems that in the process of modelling, which is generally very complex, striving after qualitative agreement is of the first importance. It is fairly easy to notice whether theoretical prediction is qualitatively conformable with experimental data or not. Consider, for example, Figs. 5 and 9. In both cases, experimentally determined volumetric strains increase during the first stage of loading (compaction) and then the stage of dilation follows. Theoretical prediction of volumetric strains shows continuous dilation from the beginning of loading, which is different from the real behaviour of soil. Hence there is no qualitative agreement between theory and experiment regarding volumetric strains.

In the case of deviatoric strains, the model prediction is qualitatively similar to experimentally observed behaviour since these strains increase exponentially in both cases, but there are quantitative differences. In the special case of anisotropic compression, shown in Fig. 6, there is good conformity, both qualitative

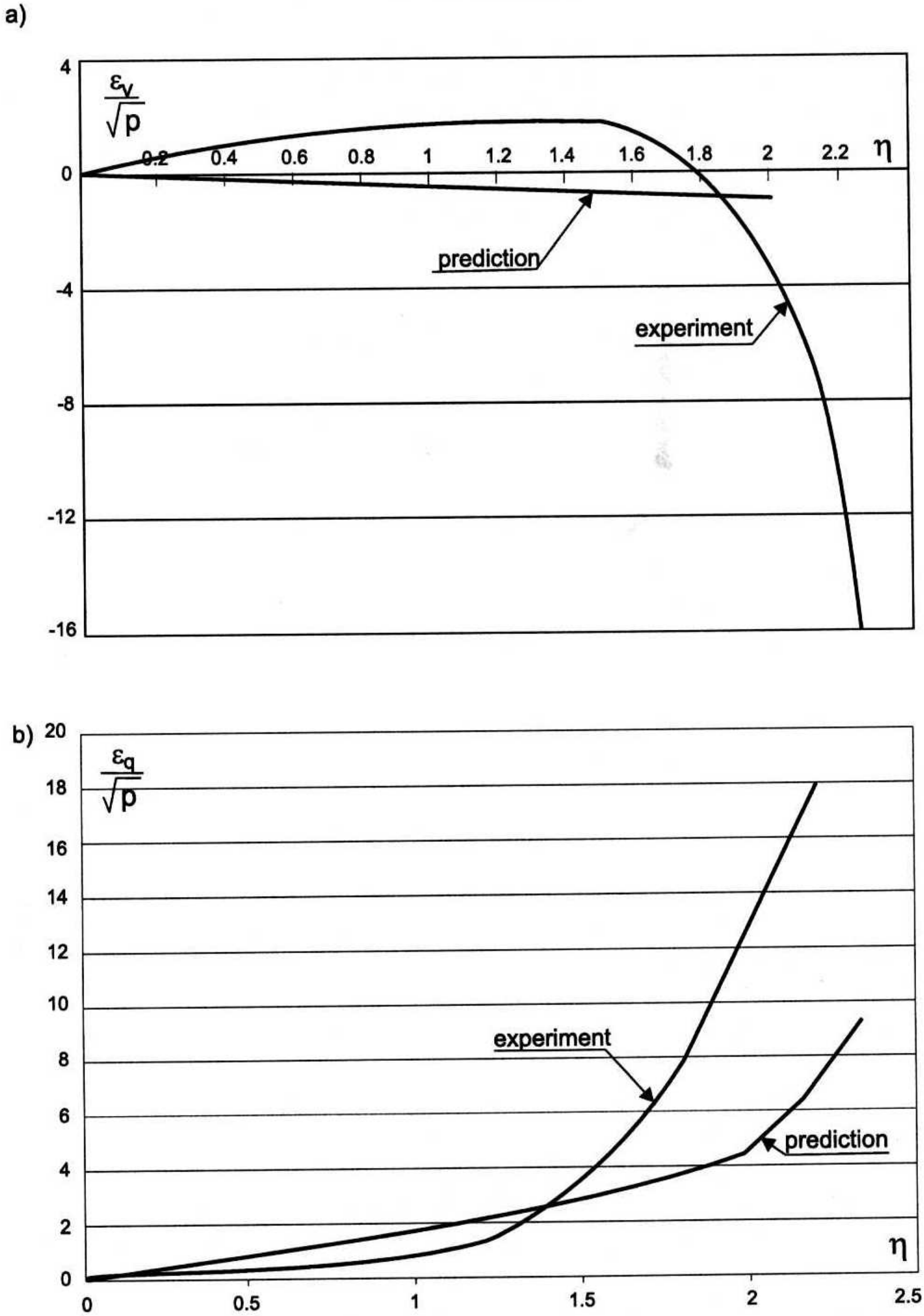



**Fig. 9.** Conventional stress path ($\sigma_0 = 1$). Volumetric a) and deviatoric b) strains. Theoretical prediction against experimental data

and quantitative, regarding volumetric strains, as well as fair agreement regarding deviatoric strains. In other cases of anisotropic compression (cf. Table 1), quantitative agreement between theory and experiment is worse, whilst there is still good qualitative conformity.

Criteria deciding as to qualitative and quantitative conformity of theoretical predictions with experimental data are scarcely discussed in soil mechanics literature, although from time to time some special workshops are organized, see Saada and Bianchini (1989). Many authors have noticed that rather simple and elegant models do not always give realistic predictions, but despite that they have still been taught as standards. For example, Cam-clay model is recently regarded as a 'student's model', just for it's simplicity and elegance.

Probably most research workers have no problems with qualitative judgement of theoretical predictions, hence a more important problem is to establish useful quantitative criteria. Consider, for example, a set of isotropic compression tests performed on initially loose *Skarpa* sand, Sawicki and Świdziński (2002a). There were 11 samples investigated, characterized by the average initial relative density $D_r = 0.225$, and average deviation of 7.7% from this value. Note that it is impossible to prepare identical samples from the same set of grains, so such deviations are unavoidable. The volumetric and deviatoric strains were approximated by the following formulae:

$$\varepsilon_v = A_v\sqrt{p}\ , \quad \varepsilon_q = A_q\sqrt{p}, \tag{34}$$

where the average values of respective coefficient were $A_v = 5.77$ and $A_q = -0.723$. The average deviations from these averages were 8.7% and 20.3% respectively.

In this special case of isotropic compression, any theoretical prediction which is located within a fan defined by respective average and average deviations should be regarded as conformable with experimental data from the quantitative stand-point. This is the most simple criterion, since we cannot expect an 'ideal' agreement between theory and experiment in soil mechanics. It should be mentioned that some perfect physical theories, as for example quantum mechanics, have incredible power of prediction, characterized by a relative error of the order of $10^{-10}$, cf. Penrose (1996).

The experimental and theoretical results shown in Fig. 6 differ by some 12% which should be regarded as an acceptable result. However, the differences between theoretical predictions and experimental data presented in Table 1 are as great as some 30% to 125% . How can we interpret such differences? The important problem in soil mechanics is to establish reliable quantitative criteria which would estimate some limits of accuracy of theoretical predictions in comparison with experimental data.

### 5.2. Yield Surface, Loading and Unloading

One fundamental problems in modelling the inelastic behaviour of materials is the definition of loading and unloading, which is a non-trivial problem and therefore needs some comments, cf. Życzkowski (1973). It is intuitively obvious that the loading process takes place when all components of the stress tensor increase. But what about the case when one component increases whilst another decreases (e.g. path AB in Fig. 7)? According to elasto-plasticity, the stress path AB corresponds to elastic unloading as it is located inside the current yield surface. Experimental data shown in Fig. 8 display weakness of that classical definition of unloading as the plastic strains develop along AB.

Along path AB, the stress deviator increases whilst the mean stress decreases, hence it is a kind of deviatoric loading and simultaneous unloading with regard to the mean stress (say isotropic unloading). It suggests that perhaps it would be better to define loading and unloading in a different way, with a clear physical interpretation, also consistent with experimental data.

Consider, for example, the soil's behaviour along path AC in Fig. 1 which corresponds to shearing at constant mean stress, already discussed in Section 4.1. In this case, we obviously deal with loading, and classical elastic-plastic definition of loading is consistent here with physical intuition. Recall that granular soils display barotropic properties which means that the strains depend on the mean stress, i.e. when mean stress increases the deviatoric strains decrease. Therefore, the deviatoric stress which increases along path AB in Fig. 7 should produce larger deviatoric strains than those developed along path AB', simply because the mean stress decreases along AB, and therefore the soil's shearing resistance also decreases. This simple interpretation also shows that the assumption as to elastic unloading along path AB is wrong. Should the students learn such models? Classical elastic-plastic definition of loading and unloading provokes many other questions.

One of them deals with determination of the shape of yield surface. In the original Cam-clay approach, this shape is derived from a single, plastic work dissipation postulate, cf. Schofield and Wroth (1968):

$$q d\varepsilon_q^p + p d\varepsilon_v^p = M p \varepsilon_q^p, \tag{35}$$

which leads to the tear-shaped yield surface, with the corner located on the $p$ axis. In order to obtain a smooth yield surface, Burland (1965) proposed a different shape of the RHS of Eq. (35), namely:

$$RHS(35) = p\left[\left(d\varepsilon_v^p\right)^2 + \left(M d\varepsilon_q^p\right)^2\right]^{0.5}, \tag{36}$$

which leads to the elliptic shape of the yield surface, accepted in this paper.

Such games have been designated as attempts to build 'idealised' models of soils in contrast to 'descriptive' models, which are sometimes treated as just 'curve fitting' excercises, Jefferies and Shuttle (2002). According to such a philosophy, we can play with the shape of yield surface, and therefore also with a definition of loading and unloading. It does not seem physically sensible.

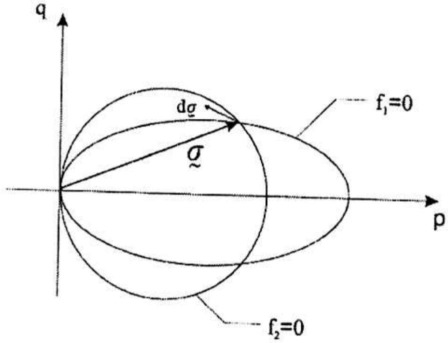



**Fig. 10.** The same stress increment denotes loading or unloading, depending on the shape of yield surface

Assume that, according to two different postulates, the same stress $\boldsymbol{\sigma}$ has generated two different yield surfaces, see Fig. 10. Subsequent stress increment $d\boldsymbol{\sigma}$ denotes either loading or unloading, depending on which surface is dealt with. Therefore, the definition of loading and unloading depends on the assumed postulate regarding the plastic work dissipation. Consequently, the elastic range of soil behaviour also depends on this postulate, which does not sound physically sensible. The fundamental question is whether the process of loading/unloading is objective or depends on the mechanical properties of materials? Another question is whether a single (and arbitrary) postulate can decide as to the extremely complex behaviour of a great assembly of grains, where each grain is of a different size and shape?

Another important issue deals with experimental verification of yield surfaces for soils. Some methods, presented even in textbooks (e.g. Wood 1990), are not convincing for many reasons. For example, such surfaces are searched for using many cycles of loading and unloading, which certainly change material properties. Moreover, the applied 'stress increments' seem somehow to be large, as they are of the order of 200 kPa, etc. All the above mentioned issues indicate that the soil plasticity remains an empirical science rather than a mature subject based on a sound theoretical background. Therefore, further research is necessary in order to describe deformations of granular materials for arbitrary loading paths.

## 6. Conclusions

a) Soil mechanics strongly attracts the attention of theoretical modellers, and already a large number of various plastic models have been proposed. None of these have been accepted by geotechnical engineers as reliable standard, except for the Coulomb-Mohr theory which is useful in engineering applications regarding reliable estimation of limit loads and bearing capacity of soils. It is hard to find a model enabling simple and reliable estimation of strains which develop in granular materials before failure.

b) There are some models of soils which have already found their historical place in geotechnical literature, and which are recommended as standards as, for example, the Cambridge-type approach. However, the authors of these approaches admit that their models are not perfect in comparison with experimental data, but recommend them as elegant standards or *students' models*. Papers dealing with detailed analyses of such models in the light of experimental results are fairly rare in literature. We have therefore decided to check the predictions of the Cam-clay approach against empirical data, obtained from rather sophisticated experiments, performed on dry sand in stages preceding failure.

c) Comparative analysis shows that theoretical predictions obtained from a slightly modified version of the original Cam-clay model are quite far from experimental results, except for qualitative agreement regarding deviatoric strains for some chosen stress paths. It is therefore hard to recommend such an approach as a tool for predicting strains which develop in sand before failure.

d) Comparison of theoretical and experimental results constitutes a basis for more general discussion regarding the usefulness of elasto-plastic modelling of soil behaviour, and some elements of such a critical discussion have been presented in this paper. Discussion shows that there are several problems which need further investigations, including: definition of loading/unloading, yield surface, soil elasticity and anisotropy, reliability of experimental results, 'fuzzy' character of empirical results, etc. Further research on modelling pre-failure behaviour of granular materials is certainly necessary. More attention should be devoted to comparative studies, similar to those presented in this paper, in order to select existing models and elaborate a useful engineering standard.

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