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Differential Water Capacity of Soil

Krzysztof W. Książyński

Institute of Water Engineering and Water Management, Cracow University of Technology, 24 Warszawska St., 31-155 Cracow, Poland, e-mail wksiazyn@smok.wis.pk.edu.pl

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Abstract

In the pressure form of Richards' seepage equation, the differential water capacity function is used and calculation of its values is necessary to solve the equation. The course of function variability was determined basing on the soil moisture characteristic. To obtain simpler solutions, the water capacity was expressed as a function of moisture. Using van Genuchten and logistic functions, its course for six soils available in the relevant literature was determined. Two sandy soils were used to discuss the water capacity hysteresis and simple formulae were proposed, enabling the coefficient for scanning runs to be determined.

Keywords: seepage, unsaturated flow, water capacity

Notations

c – water capacity (differential water capacity) $[\mathrm{m}^{-1}]$,
c_d – water capacity for drying $[\mathrm{m}^{-1}]$,
c_w – water capacity for wetting $[\mathrm{m}^{-1}]$,
C – parameter determining the curve slope [–],
h_k – capillary head [m],
h_p – suction head at the moisture characteristic point of inflexion [m],
h_s – suction head [m],
H – hydraulic head [m],
k – capillary conductivity [m/s],
m, m_1 – exponents [–],
t – time [s],
x, y, z – orthogonal coordinates [m],
ϑ – saturation rate [–],
θ – volumetric soil moisture content $[\mathrm{m}^3/\mathrm{m}^3]$,
θ_n – soil moisture at full saturation $[\mathrm{m}^3/\mathrm{m}^3]$,

θ_r – irreducible soil moisture content [m^3/m^3],
θ_s – soil moisture content at the reversing point [m^3/m^3].

1. Introduction

To describe water flow in a porous medium, Richards' equation is used (Richards 1931):

$$\overline{\nabla} \cdot (k\,\overline{\nabla} H) = \frac{\partial}{\partial x}\left(k\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial H}{\partial z}\right) = \frac{\partial \theta}{\partial t} \quad (1)$$

giving a precise description of isothermal seepage in both saturation zones. However, it cannot be solved directly, because of two unknown variables which appear in its standard notation: hydraulic head H and moisture θ. These two variables are interrelated by moisture characteristic $\theta(h_s)$ (as $H = z - h_s$). Hence, a solution may be obtained substituting one of these functions for another: $H = H(\theta)$ or $\theta = \theta(H)$. In this way the moisture diffusion equation or the generalized seepage equation (conductivity equation) was obtained. The first is known as the Buckingham equation (Buckingham 1907). An alternative approach consists in replacing the capacity component in Richards' equation (1) by the function of hydraulic head H. This can be done by treating moisture content θ as a composite function:

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial h_s}\frac{\partial h_s}{\partial t} = \frac{\partial \theta}{\partial h_s}\frac{\partial (z-H)}{\partial t} = -\frac{\partial \theta}{\partial h_s}\frac{\partial H}{\partial t} = c\,(h_s)\,\frac{\partial H}{\partial t}, \quad (2)$$

where $c(h_s) = -\partial\theta/\partial h_s$ is capillary capacity (Richards 1931) (differential water capacity – Kovács 1981), being in fact the derivative of moisture characteristic versus suction head h_s. Such a form of the capacity component had been introduced by Richards himself. The parameter c can be generalized for the saturation zone where it is equivalent to the specific capacity (specific storage – Hantush 1964). So the movement equation assumes the following form:

$$\frac{\partial}{\partial x}\left(k\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial H}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial H}{\partial z}\right) = c\,\frac{\partial H}{\partial t}. \quad (3)$$

As hydraulic head H changes in the both zones of saturation, seepage description is possible in both the vadose and the saturation zones, which results in the term generalized seepage equation. Also a term conductivity equation originating from thermodynamic analogy is used. The functional parameter c is not constant at least in the vadose zone. A description of its variability is the subject of this paper.

2. Soil Characteristics

Determination of the nature of variability of differential water capacity requires knowledge of the moisture characteristic, i.e. the relationship of moisture capacity versus suction head $\theta(h_s)$. However, calculation of the coefficient from an empirically established characteristic may be difficult. Individual measurements are burdened with hazard errors and the derivative calculated, may lead to chaotic values of the coefficient. To obtain clear relationships a mathematical function has to be fitted to the characteristic earlier. This will also make it possible to describe the capacity coefficient with a formula.

In the relevant literature a dozen empirical formulae, describing the moisture characteristic with differing precision, can be found. The simplest of these, describing the saturation rate as a power function (Corey, Corey, Brooks 1965) or exponential function (Averjanov, Golovanov, Nikolskij 1974), have an application restricted to low moisture contents, while high moisture contents produce non-physical values. Thus they cannot be used to describe water capacity in the whole range of moisture content. This goal can be attained by more sophisticated formulae enabling the moisture content to be calculated for an arbitrary suction head only.

Functions describing moisture characteristics are expected to have two asymptotes: for $h_s \to 0$ $\theta \to \theta_n$ and for $h_s \to \infty$ $\theta \to \theta_r$. Here the symbol θ_n denotes the maximum moisture content for a main wetting branch equal to the effective porosity n, while θ_r is the minimum moisture content corresponding approximately to the value of residual moisture for the lowest suction head that occurs in nature. To simplify the form of the formulae presented, the moisture content is described in the normalized form:

$$\theta = \vartheta(\theta_n - \theta_r) + \theta_r, \tag{4}$$

where: $\vartheta = \frac{\theta - \theta_r}{\theta_n - \theta_r}$ – saturation rate of soil, typical for a given process, with asymptotes for values $\vartheta = 1$ and $\vartheta = 0$.

This notation makes possible the application of formulae to describe drying or wetting characteristics. Replacing by moisture θ_s in reverse point the value θ_n for secondary draining or θ_r for secondary wetting, one can also describe scanning curves (Kaluarachchi, Parker 1987).

The most famous formula meeting these requirements is the van Genuchten formula (Genuchten 1980) in the form:

$$\vartheta = \frac{1}{\left[1 + \left(C \frac{h_s}{h_k}\right)^{m_1}\right]^m}, \tag{5}$$

where: C, m and m_1 – empirical parameters determining the shape of the characteristic.

The formula is valid for $h_s > 0$. For $m = 1$ a simpler form of the formula is obtained, used in Momii's work (Momii et al. 1988) and giving an almost identical curve. For high values of h_s ($h_s \gg h_k$) formula (5) reduces to Corey's power form.

As a generalization of exponential formulae, the following formula for a logistic curve never used before in the pertinent literature, was adopted in this paper:

$$\vartheta = \frac{1}{\left\{1 + \exp\left[C\,\frac{h_s - h_p}{h_k}\right]\right\}^m}. \tag{6}$$

It affords a graph similar to the van Genuchten formula and can be used for each suction head. Unfortunately, for some soils m values are so low that numerical calculations are difficult. For sandy soils m values are close to 1, which makes the application of simpler formula, used in particle physics for description of potential barrier, possible:

$$\vartheta = \frac{1}{1 + \exp\left[C\frac{h_s - h_p}{h_k}\right]}, \tag{7}$$

The formula gives an antisymmetrical curve with the inflexion point for $\vartheta = 0.5$ and it has easy interpretable parameters: $h_p = h_s(\vartheta = 0.5)$ is the suction head in the inflexion point and C is a parameter determining the slope of the curve.

Many authors use other functions that enable description of a particular measured moisture characteristic as exactly as possible, but the form of these functions is fitted to the data. To illustrate the formulae presented, data available in the literature on the moisture characteristics for some different soils were used. For two sandy soils wetting and drying characteristics were determined (Figs. 1 and 2) and for four clayey soils wetting characteristics only (Fig. 3). The parameters of the van Genuchten function were estimated from empirical data. For one sandy soil, the logistic curve was also used (Fig. 1). Estimated parameters are shown in Table 1.

The parameters have been fitted by the trial-and-error method to give the smallest sum of squared errors. However, values of limit moisture θ_r and θ_n were established according to the nature of the described process, to avoid deformations in its run. Considering the trouble with capillary height determination, the quotient of C parameter and height h_k was calculated.

3. Differential Water Capacity Functional Parameter

The differential water capacity of soil c is defined as

$$c\,(h_s) = -\frac{\partial \theta}{\partial h_s} = -\,(\theta_n - \theta_r)\,\frac{\partial \vartheta}{\partial h_s}. \tag{8}$$

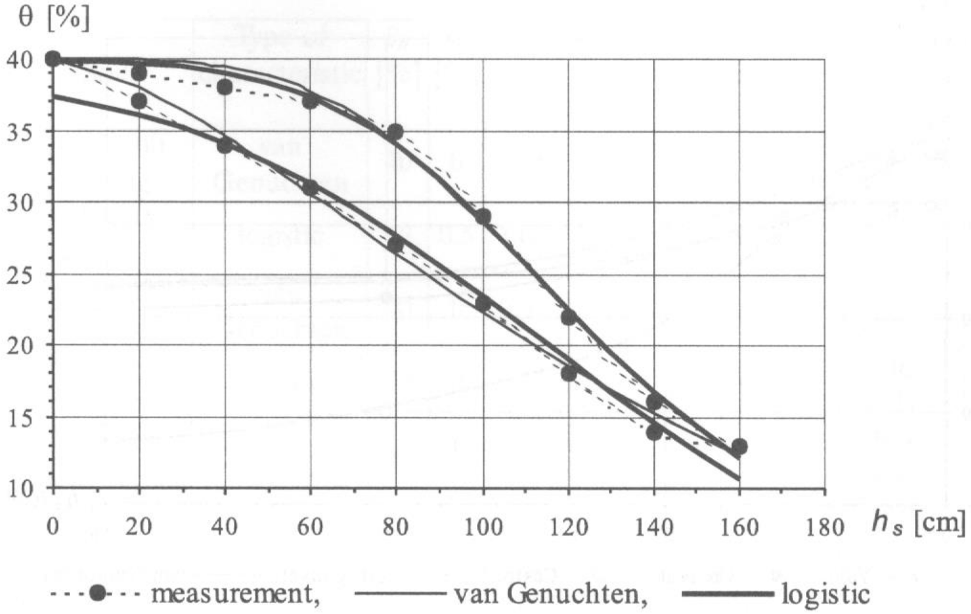

Fig. 1. Moisture characteristic for ABO/100 sand

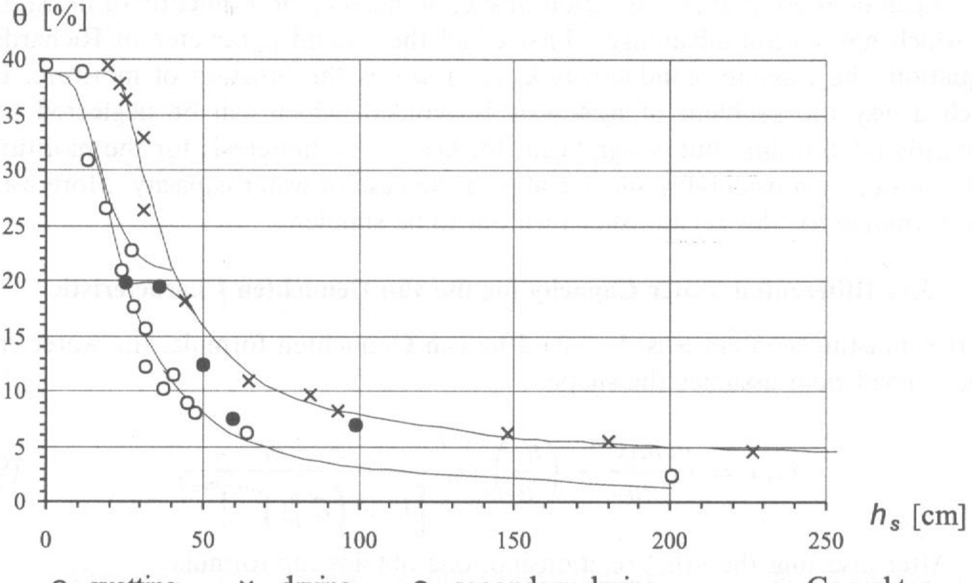

Fig. 2. Moisture characteristic for Rehovot sand

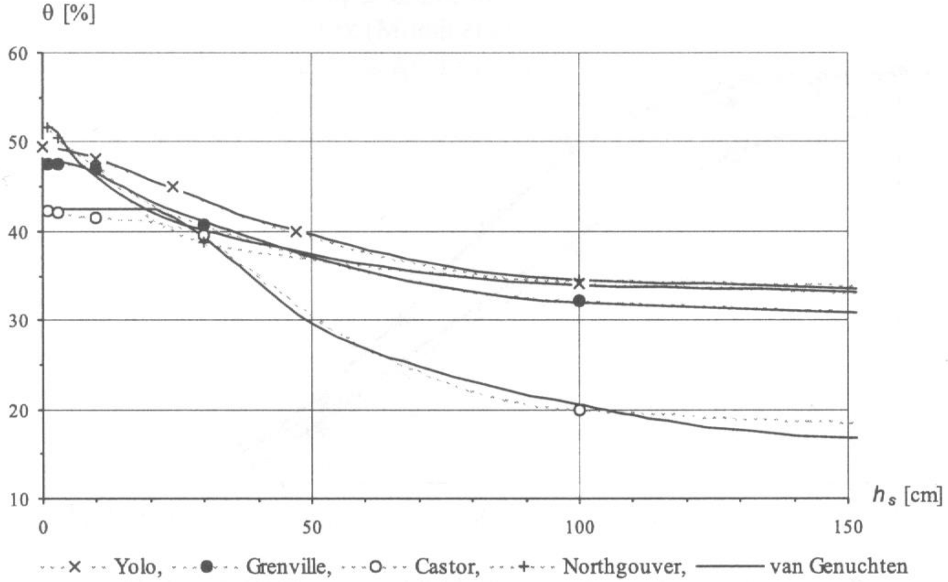

Fig. 3. Moisture characteristics for clayey soils

It can be presented as a function of suction head h_s or a function of moisture θ, which has several advantages. First of all the second parameter of Richards' equation, the capillary conductivity k, is shown as the function of moisture. In such a way the problem of hysteresis is avoided, which can be neglected for the $k(\theta)$ relationship, but is significant for $k(h_s)$. The hysteresis for the moisture relationship is considerably smaller also in the case of water capacity. Moreover, the formulae for this relationship turn out to be simpler.

3.1. Differential Water Capacity for the van Genuchten Characteristic

If the moisture content θ is described by van Genuchten formula, the water capacity coefficient assumes the shape:

$$c\,(h_s) \;=\; \frac{m\,m_1 C^{m_1}}{h_k}\,\left(\frac{h_s}{h_k}\right)^{m_1-1}\,\frac{\theta_n-\theta_r}{\left[1+\left(C\,\frac{h_s}{h_k}\right)^{m_1}\right]^{m+1}}. \tag{9}$$

After inserting the $\theta(h_s)$ relationship, one obtains the formula:

$$c\,(\theta) \;=\; \frac{m\,m_1\,C}{h_k}\,(\theta_n-\theta_r)\,\vartheta^{\frac{m\,m_1+1}{m\,m_1}}\left(1-\vartheta^{\frac{1}{m}}\right)^{1-\frac{1}{m_1}}. \tag{10}$$

Analysis of the $c(\theta)$ function enables us to find its maximum at a point with coordinates:

Table 1. Soil parameters

Soil	Type of characteristic	θ_n [%]	θ_r [%]	m	m_1 h_p [cm]	C/h_k [cm^{-1}]	h_k [cm]	Source of measurement data
ABO/100 wetting	van Genuchten	40	0	33.5	1.52	6.99 10^{-4}	65	Zaradny (1977)
ABO/100 wetting	logistic	40	0.5	1.00	114	0.023	65	Zaradny (1977)
ABO/100 drying	van Genuchten	40	0	0.92	3.77	0.081	65	Zaradny (1977)
ABO/100 drying	logistic	40	0	0.33	89.4	5.02 10^{-2}	65	Zaradny (1977)
Rehovot sand, wetting	van Genuchten	38.7	0.5	0.428	3.67	0.057	11.7	Rubin (1963)
Rehovot sand, drying	van Genuchten	38.7	3.7	0.120	13.6	0.038	11.7	Rubin (1963)
Jolo light clay	van Genuchten	49.5	0	0.110	2.0	0.051	16,1	Philip (1957)
Castor sandy clay	van Genuchten	42.5	4.2	1.97 10^{-2}	32	0.038	26	Staple (1966)
Grenville silt clay	van Genuchten	48	0.9	0.114	2.1	0.056	14.8	Staple (1966)
Northgouver clay	van Genuchten	52	0	0.060	2.3	0.22	3.4?	Staple (1966)

$$\vartheta_p = \left(\frac{m\,m_1 + 1}{m\,m_1 + m_1}\right)^m \quad \text{or} \quad h_p = \frac{h_k}{C}\left(\frac{m_1 - 1}{m\,m_1 + 1}\right)^{\frac{1}{m_1}} \tag{11}$$

and

$$c_{\max} = \frac{m\,m_1\,C}{h_k}\,(\theta_n - \theta_r)\,\frac{(m_1 - 1)^{1-\frac{1}{m_1}}\,(m\,m_1 + 1)^{m+\frac{1}{m_1}}}{(m\,m_1 + m_1)^{m+1}}. \tag{12}$$

In the case of clayey soils one can attain a parameter m_1 of less than 1. This corresponds to the occurrence of such small pores that only adhesive moisture can develop in the soil. Thus the moisture content changes monotonically and after reaching full saturation it remains constant. The capacity coefficient increases as it approaches full saturation and reaches zero at this point. In this case the formula (12) cannot be applied.

3.2. Differential Water Capacity for Logistic Characteristic

For the moisture content described by logistic curve, parameter c can be derived similarly resulting in:

$$c(\theta) = \frac{m\,C}{h_k}\,(\theta_n - \theta_r)\,\vartheta\left(1 - \vartheta^{\frac{1}{m}}\right). \tag{13}$$

Assuming the value of parameter $m = 1$ one can obtain an even simpler solution:

$$c(\theta) = \frac{C}{h_k}\,(\theta_n - \theta_r)\,\vartheta\,(1 - \vartheta). \tag{14}$$

As can be seen in Fig. 4 (for ABO/100 sand), the water capacity as a function of soil moisture content is simply a section of a parabola. For fine soils the function $c(\theta)$ is not symmetrical but is bell-shaped with a top shifted to a higher moisture content. The analysis of the $c(\theta)$ function makes it possible to find its maximum at a point with coordinates:

$$\vartheta_{\max} = \left(\frac{m}{m+1}\right)^{m} \quad \text{or} \quad h_{\max} = h_k\left(\frac{h_p}{h_k} - \frac{\ln m}{C}\right) \tag{15}$$

and

$$c_{\max} = \frac{C}{h_k}\,(\theta_n - \theta_r)\left(\frac{m}{m+1}\right)^{m+1}. \tag{16}$$

The factor with m in the Eq. (16) attains: for $m = 0.01$ (clays) the value of 0.0095, for $m = 0.1$ 0.072 and for $m = 1$ (sands) 0.25, which gives maxima for saturation rates equal respectively to 0.95, 0.79 and 0.50.

4. Features of Soil Water Capacity

In Fig. 4 the relationship of capacity parameter c versus moisture content θ is presented for various real soils. The more clayey the soil the higher the moisture content for which the maximum value of the capacity c occurs. For clean sands the maximum occurs as early as close to the saturation of $\vartheta = 50\%$. For clays the maximum shifts to about $\vartheta = 100\%$, reaching the limit for soils with very fine pores.

In spite of the fact that the capacity coefficient depends on the moisture content $c(\theta)$, its hysteresis is quite considerable. From the similarity of wetting and drying characteristics results the similarity of the relevant graphs of the coefficient. Extreme moisture contents n and θ_r are achieved for a suction head equal to plus and minus infinite, so the capacity at these points must be equal to zero. The inflexion point of the moisture characteristic usually occurs for a similar moisture

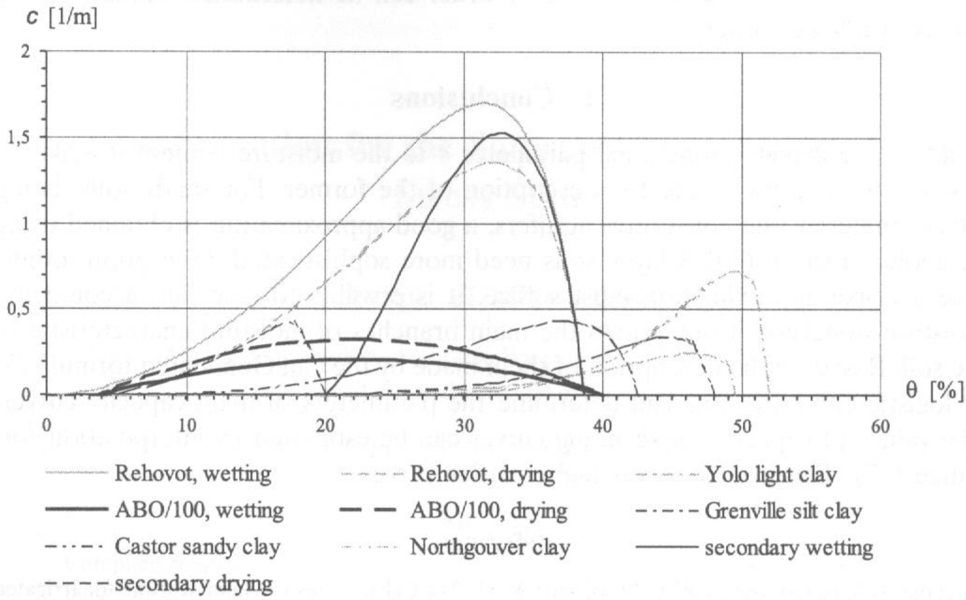

Fig. 4. Water capacity versus soil moisture content

content for both branches, hence the capacity maxima should be close to each other. On the other hand, the values of maxima can differ quite considerably; e.g. for Rehovot sand they are 1.7 and 1.36, for the ABO/100 sand 0.2 and 0.34 [m^{-1}]. So the capacity hysteresis, although being less for the moisture relationship than for the pressure one, cannot be neglected.

The shape of the capacity function for scanning curves is different from the shape for the main wetting and drying branches. In Fig. 4 graphs of the coefficient are presented for two exemplary scanning curves marked in Fig. 2 (secondary wetting and secondary drying curves for Rehovot sand). For the moisture content in the reversing point θ_s (corresponding here for the both curves at circa 20%) the value of capacity decreases to the vicinity of zero which corresponds to the flat fragment for the both scanning curves. During further drying or wetting, the water capacity increases approaching asymptotically the value of capacity for the relevant main branch. An approximate value of capacity c_{w1} for the first order wetting curve and c_{d1} for the first order drying one can be calculated from the interpolation formulae:

$$c_{w1}(\theta, \theta_s) = c_w(\theta)\sqrt[4]{\frac{\theta-\theta_s}{\theta_n-\theta_s}} \quad \text{and} \quad c_{d1}(\theta, \theta_s) = c_d(\theta)\sqrt[4]{\frac{\theta_s-\theta}{\theta_s-\theta_r}}. \qquad (17)$$

The capacity for curves of a higher order can be determined similarly using curves of a lower order.

5. Conclusions

Relating the capacity functional parameter c to the moisture content θ makes it possible to simplify a little the description of the former. For sandy soils, being a basic material that constitutes aquifers, a good approximation is obtained using parabolic function (14). Clayey soils need more sophisticated description, otherwise a worse approximation must suffice. It is possible to take into account the moisture hysteresis if one knows the main branches of moisture characteristic of the soil. Based on the description of them made by the van Genuchten formula (5) or logistic curve (6), one can determine the parameters of main capacity curves. The values of capacity for scanning curves can be estimated by interpolation formulae (17). This also concerns higher order curves.

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