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CONTEMPORARY ASPECTS OF THE THEORY AND APPLICATION OF NONLINEAR ACOUSTICS

L. BJØRNØ

UltraTech Holding
Stendiget 19, DK-2630 Taastrup, Denmark
e-mail: prof.lb@privat.dk

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Dedicated to the memory of my friend and colleague professor Ignacy Malecki

The foundations of nonlinear acoustics may be traced nearly 250 years back in time, but only the last 50 years have shown an increasing number of attempts to exploit the research results in nonlinear acoustics. Based upon the fundamental equations of fluid dynamics, the second-order acoustic equations may be derived which can be reduced to a compound equation describing several of the most important and fast developing areas of research in nonlinear acoustics. The relations between this compound equation and Burgers' equation, Korteweg-DeVries equation, the K–Z–K equation, Westervelt's equation and the general second-order wave equation are discussed in depth. Finally, it is shown how the derivatives of the compound equation can be applied to nonlinear acoustic research related to materials characterisation by use of the B/A-ratio, to underwater acoustics by use of the parametric acoustic array and to focused, high-power ultrasonic fields.

Key words: nonlinear acoustics, absorption, diffraction, dispersion, focused fields, underwater applications, second-order nonlinearity parameter.

1. Introduction

While description and exploitation of acoustical phenomena can be traced more than 2000 years back in time, nonlinear acoustics is a "rather new" discipline being only about 250 years old, starting in 1755 with Euler's formulation of the equations of continuity and momentum for the motion of a fluid [1]. While most of the theoretical basis for nonlinear acoustics was developed during the first 200 years after Euler's first contributions, the last 50 years have in particular brought a substantial contribution to our knowledge about nonlinear acoustic phenomena. Like in many other fields of science the progress in nonlinear acoustics has in particular been supported by the enormous development in computational procedures and facilities permitting a fast solving of nonlinear equations. The last 50 years have been characterised by the emerging of new fields of research, new concepts and new applications of nonlinear acoustics. Among these should be mentioned material characterisation exploiting the concept of second and higher order acoustic nonlinearity parameters, underwater applications of the parametric acoustic array, and medical applications of focused ultrasonic fields. These areas of research in nonlinear acoustics of fluids will be treated more in depth in this paper after their appropriate theoretical bases have been developed.

2. Features of the theoretical basis of nonlinear acoustics

It is a generally accepted fact that the World is nonlinear. In order to be able to deal with scientific and engineering problems it has been necessary to linearize the equations describing the problems. And in most cases with success. However, when it comes to high amplitudes of pressure and particle velocity or to strongly nonlinear materials as for instance two-phase components, it has been necessary to leave the pleasant linear concepts and to consider the real nature of the problems. This has also been the case for acoustics of fluids.

When dealing with nonlinear equations, the principle of superposition of solutions no longer holds, and interactions between waves start to be of importance. This leads to the appearance of a number of new physical phenomena, frequently so significant that they cannot be considered as small corrections to a linear theory.

The fundamental equations of Fluid Dynamics may be written as:

$$\partial\rho/\partial t + \partial/\partial x_i(\rho u_i) = 0, \tag{1}$$

$$\rho(\partial u_i/\partial t + u_j \partial u_i/\partial x_j) = \rho F_i - \partial p/\partial x_i + \partial/\partial x_j[\eta(\partial u_i/\partial x_j + \partial u_j/\partial x_i)] + \partial/\partial x_i[\eta' \partial u_k/\partial x_k], \tag{2}$$

$$\rho c_v DT/Dt + \rho c_v(\gamma - 1)/\beta[\partial u_i/\partial x_i] + \partial q_i/\partial x_i - \Phi - q' = 0, \tag{3}$$

where (1) is the equation of continuity, (2) are the equations of motion (the Navier-Stokes equations), and (3) is an equation of energy. In (1)–(3), p and ρ denote the pressure and the density of the fluid, respectively, x_i denotes the spatial coordinates ($i = 1, 2$ and 3) and t is time. u_i is the particle velocity in the i'th direction, while F_i is an external body force per unit mass. η and η' are the shear and the bulk viscosities of the fluid, respectively, while T denote the absolute temperature. c_v is the specific heat at constant volume, while q_i and q' denote the rate of heat flux vector (comprising heat conduction and radiation) and the internal heat production rate per unit volume, respectively. Φ is the viscous dissipation function.

When a perturbation procedure is applied to the fundamental equations of fluid dynamics, (1)–(3), – of which Eq. (2) is a nonlinear equation – and to an equation of state, maintaining only terms to the second order, the second order acoustic equations may be derived [2]. The perturbation procedure may be written as:

$$\rho = \rho_{(0)} + \rho_{(1)} + \rho_{(2)} + \ldots; \qquad T = T_{(0)} + T_{(1)} + T_{(2)} + \ldots;$$

and

$$u_i = 0 + u_{i(1)} + u_{i(2)} + \ldots;$$

$$p(\rho, T) = p_{(0)}(\rho_{(0)}, T_{(0)}) + [(\partial p/\partial\rho)_T]_{(0)}(\rho - \rho_{(0)}) + [(\partial p/\partial T)_\rho]_{(0)}(T - T_{(0)}) + \ldots.$$

where each term is the *acoustic Mach number*, $M_a = u/c$, smaller than the previous term. If the perturbed quantities are inserted into the fundamental equations of fluid dynamics (1), (2) and (3), and only terms to second order are used, the first-order equations may be shown to lead to the well-known, linear equation of sound, while the equations comprising the second-order terms constitute the full ***second order acoustic equations.***

When the fluid is considered lossless the second-order form of the equation of continuity may be written as:

$$\partial\rho_{(2)}/\partial t + u_{i(1)}\partial\rho_{(1)}/\partial x_i + \rho_{(1)}\partial u_{i(1)}/\partial x_i + \rho_{(0)}\partial u_{i(2)}/\partial x_i = 0 \qquad (4)$$

while the equation of motion, i.e. Euler's equation for a lossless case, may be written as:

$$\rho_{(0)}\partial u_{i(2)}/\partial t + \rho_{(1)}\partial u_{i(1)}/\partial t + \rho_{(0)}u_{j(1)}\partial u_{i(1)}/\partial x_j = -\partial p_{(2)}/\partial x_i. \qquad (5)$$

The equation of state for adiabatic processes will for the lossless case be:

$$p_{(2)} = (\partial p/\partial\rho)_s\rho_{(2)} + (\partial^2 p/\partial\rho^2)_s\rho_{(1)}^2/2. \qquad (6)$$

Combining the equations (4)–(6), the ***inhomogeneous, and nonlinear, second-order wave equation*** will appear as:

$$\partial^2\rho_{(2)}/\partial t^2 - c_0^2\partial^2\rho_{(2)}/\partial x_i^2 = (c_0^2/\rho_{(0)})\{(1 + (\rho_{(0)}/c_0)(\partial c/\partial\rho)_s\}\partial^2\rho_{(1)}^2/\partial x_i^2$$
$$= \beta(c_0^2/\rho_{(0)})\partial^2\rho_{(1)}^2/\partial x_i^2. \qquad (7)$$

This expression shows, that the first-order sound field has become the source function for the second-order contributions.

The full second-order acoustic equations may for the assumption of quasi-plane waves be reduced to a compound equation covering several of the most important and fast developing areas of research in nonlinear acoustics [2, 3]. The ***compound equation*** may for finite-amplitude waves be written in a non-dimensional form as:

$$\partial/\partial y\left\{\partial V/\partial\sigma - V(\partial V/\partial y) - G(\partial^2 V/\partial y^2) - D(\partial/\partial y)\int_{-\infty}^{y}(\partial V/\partial y')e^{-(y-y')/\omega t}dy'\right\}$$
$$= (N/4)\{\partial^2 V/\partial\xi^2 + (1/\xi)(\partial V/\partial\xi)\}, \qquad (8)$$

where $V = u/u_0$, $\sigma = (B/2A + 1)Mkx$ (the dimensionless nonlinearity distance), and $y = kc_0t$. ξ is a dimensionless radial coordinate governing the influence of diffraction on nonlinear wave propagation. u_0 is the particle velocity amplitude at the source, and B/A is the second order nonlinearity ratio expressing the material nonlinearity of the fluid. k is the acoustic wave number and ω is the angular frequency.

The coefficients, G, D and N are determining the type of the nonlinear partial differential equation, governing various types of finite-amplitude wave fields.

For $G \neq 0$, and for $D = N = 0$, the compound equation reduces to:

$$\partial V/\partial\sigma - V(\partial V/\partial y) = \Gamma^{-1}(\partial^2 V/\partial y^2) \tag{9}$$

which is the boundary-value form of ***Burgers' equation*** [2] for a thermo-viscous fluid, including the influence of nonlinearity and attenuation. $G = \Gamma^{-1}$, where Γ is the so-called *Gol'dberg number* [4], describing the relative influence of nonlinearity and dissipation. Gol'dberg showed that for $\Gamma < 1$, a shock formation during propagation of an originally sinusoidal finite-amplitude wave is not likely to take place. Re_a is the *acoustic Reynolds number* expressing the ratio between kinematic and dissipative effects and Γ may be expressed by $\Gamma = (B/A + 2)u_0\rho_0 x/b = (B/A + 2)\mathrm{Re}_a$, where $b = (4/3)\eta + \eta\prime + \kappa(c_v^{-1} - c_p^{-1})$. κ is the coefficient of heat conductivity of the fluid, while c_v and c_p denote the specific heats at constant volume and constant pressure of the fluid, respectively.

The Burgers' equation has, since BURGERS' [5] development of the equation in 1948 for description of certain turbulence problems, been used extensively for description of the propagation of finite-amplitude plane, cylindrical and spherical waves in thermo-viscous fluids [2].

For $D \neq 0$, and for $G = N = 0$, and in the limit of $\omega t \ll 1$, the compound equation reduces to a symbiosis of the ***Korteweg–deVries equation*** involving nonlinarity and dispersion and Burgers' equation involving nonlinarity and dissipation:

$$\partial V/\partial\sigma - V\partial V/\partial y = D\{(\omega t)(\partial^2 V/\partial y^2) - (\omega t)^2(\partial^3 V/\partial y^3)\}. \tag{10}$$

This equation describes the propagation of finite-amplitude waves in relaxing fluids. In the absence of attenuation, Eq. (10) reduces to the general Korteweg–deVries equation describing among others the propagation of solitons. In the absence of dispersion, Eq. (10) reduces to Burgers' equation.

For $G \neq 0$ and $N \neq 0$ and for $D = 0$, the compound equation reduces to a form of the ***Khokhlov-Zabolotskaya–Kuznetzov equation (the K–Z–K Eq.)*** [6, 7] governing the influence of nonlinearity, attenuation and diffraction in the same scale on finite-amplitude wave propagation. The K–Z–K equation – used for description of focused, high-power ultrasonic fields – may be written in a dimensionless form for the acoustic pressure as:

$$\{4(\partial^2/\partial y\partial\sigma) - \nabla\perp^2 - 4\alpha r_0(\partial^3/\partial y^3)\}P = 2(r_0/\zeta)(\partial^2/\partial y^2)P^2, \tag{11}$$

where α is the absorption coefficient [$\alpha = \omega^2 b/(2\rho_0 c^3)$], and $r_0 = \omega a^2/2c_0$ is the Rayleigh distance for a monochromatic source. $\zeta = \rho_0 c_0^3/(\beta\omega p_0)$ is the so-called discontinuity distance, i.e. the distance from the finite-amplitude wave source for the first formation of a discontinuity at a zero-crossing for wave propagation in a lossless fluid. $\beta = 1 + B/2A$. $P = p/p_0$, where p_0 is the pressure amplitude at the source. In the absence of diffraction, i.e. $\nabla\perp^2 P = 0$, the K–Z–K equation reduces to the Burgers' equation. A transformation between pressure p and particle velocity u in the equations may be done by using the linear plane-wave impedance relation: $p = \rho_0 c_0 u$.

Burgers' equation may also be shown to be a reduced form of ***Westervelt's equation*** which has formed the basis for the development of the concept of *parametric acoustic arrays*. Westervelt's equation may for the pressure variation be written as:

$$\{\partial^2/\partial x^2 - (1/c_0^2)\partial^2/\partial t^2\}p - (\delta/c_0^4)\partial^3 p/\partial t^3 = -\{\beta/(\rho_0 c_0^4)\}\partial^2(p)^2/\partial t^2, \quad (12)$$

where $\delta = 2c_0^3\alpha/\omega^2$.

Westervelt's equation may also be derived from Lighthill's equation for dynamical sound generation, or from the second-order wave equation including attenuation. Neglecting attenuation, Westervelt's equation reduces to the lossless form of the ***inhomogeneous second order wave equation*** for the pressure p:

$$\partial^2 p/\partial x^2 - (1/c_0^2)\partial^2 p/\partial t^2 = -\{\beta(\rho_0 c_0^4)^{-1}\partial^2(p^2)/\partial t^2\} \quad (13)$$

which has the same form as Eq. (7) for the density ρ.

3. The second-order acoustical nonlinearity parameter *B/A*

Studies of the second-order acoustical nonlinearity parameter B/A by a number of fluids of industrial, chemical and biological interest have been carried out over the past. In particular the relations between B/A and the molecular structure of the fluids have attracted great attention. For biological fluids, B/A's relation to the intermolecular potentials, the water fraction and the ratio of bound-to-free water, has been studied in some depth [9]. B/A is in particular related to the velocity of sound and its derivatives with respect to pressure and temperature of the fluids and it can be developed from a Taylor series expansion of the equation of state of the fluid for adiabatic changes, retaining only terms to second order [10, 11]:

$$p - p_0 = A\{(\rho - \rho_0)/\rho)\} + (B/2)\{(\rho - \rho_0)/\rho\}^2, \quad (14)$$

where:

$$A = \rho_0(\partial p/\partial\rho)_{0,s} = \rho_0 c_0^2,$$
$$B = \rho_0^2(\partial^2 p/\partial\rho^2)_{0,s} = 2\rho_0^2 c_0^3(\partial c/\partial p)_{0,s}.$$

B/A may now be written as:

$$B/A = 2\rho_0 c_0(\partial c/\partial p)_{0,s} = 2\rho_0 c_0(\partial c/\partial p)_T + (2c_0 T\beta/c_p)(\partial c/\partial T)_p = \{\partial(1/\psi)/\partial p\} - 1, \qquad (15)$$

where β is the volume coefficient of thermal expansion [$= V^{-1}(\partial V/\partial T)_p$] and [$V = 1/\rho$], and ψ is the adiabatic compressibility (the reciprocal of stiffness) for the fluid.

The expressions for B/A have been used extensively for calculations of the value of the second-order acoustical nonlinearity ratio based on ***thermodynamical*** information about the fluids investigated. Of the contribution to B/A arising from the derivatives of the velocity of sound with respect to pressure and temperature, the pressure derivative is the most important as it contributes most and is easiest to determine.

Based on FUBINI's [12] Fourier series solution to the fundamental equations for finite-amplitude wave propagation through a lossless fluid and by introduction of an expression for the attenuation of the fundamental and the second-harmonic waves [9], a ***finite-amplitude method*** has been developed for experimental determination of B/A of fluids. In this method the formation of the second harmonic amplitude $p_2(x)$ as a function of distance, x, from the source is measured, and the ratio of this amplitude and x times the square of the source amplitude p_s is extrapolated back to the finite-amplitude wave source leading to an expression for B/A through:

$$[p_2(x)/xp_s^2]_{\to 0} = [(2 + B/A)\pi f/(2\rho_0 c_0^3)] \exp\{-(\alpha_1 + \alpha_2/2)x\}, \qquad (16)$$

where α_1 and α_2 denote the attenuation coefficients for the fundamental and its second harmonic amplitudes, respectively. This expression has been used extensively for determination of B/A from measurements of the second harmonic amplitude in a broad variety of fluids.

Considerable attempts have been made over the past to exploit B/A as a tissue characterizing parameter due to its proven relation to the molecular and macroscopic structure of tissues [9]. However, the scatter of the B/A values measured using the thermodynamical and the finite-amplitude method have shown, that the inaccuracy of the methods is still too high for using the B/A as a diagnostic tool. Moreover, the *in vitro* character of the methods limits their practical applicability. The accuracy with which B/A can be measured in fluids is about 10% for the finite-amplitude method and about 5% for the thermodynamical method. Among other, less extensively used, methods for determination of B/A should be mentioned optical methods, parametric arrays [19] and cavity resonance.

Recognising that many fluids are mixtures of immiscible components, some systematic studies have been carried out on these fluids. These studies have verified the applicability of an expression like:

$$[B/A]_{\text{eff}} = (1/\zeta_{\text{eff}}^2)\sum_1^n (\zeta_i^2 (B/A)_i x_i), \qquad (17)$$

where the effective adiabatic compressibility $\zeta_{\text{eff}} = \sum_i^n \zeta_i x_i$ and where the sum of the n volume fractions x_i is $\sum_i^n x_i = 1$, for determination of an n-component mixture of mutually immiscible materials when their individual B/A values are known [13].

While the so-called *Ballou's rule*[14], expressing a linear relation between B/A of a liquid and the inverse velocity of sound of the liquid, seems to hold for liquid metals and some other liquids, recent studies of B/A by n-alkane liquids [15] using the Tait equation for describing the *pVT*-relation for the liquid, and studies of B/A in 1-alkanols, ketones and alkyl acetates [16] have shown, that this simple relation between B/A and the velocity of sound is not in general satisfied. In [15] it was shown that the use of the Tait equation was leading to nearly the same accuracy as the use of the thermodynamical method's direct measurement of the sound velocity as a function of pressure and temperature. The chemical families studied in [16] showed that B/A increases with the number of carbon atoms present in the chain and it also increases with the length of the chain, supporting the observation in [17] that the longer the molecule, the more sound propagates within the molecule.

Several tables comprising the compiled values of B/A of fluids measured and calculated using, in particular, the thermodynamical and the finite-amplitude methods can be found in the open literature [2, 9, 15, 16, 18], and only a few values shall be given here to indicate the variability of the second-order acoustical nonlinearity ratio B/A.

Fluid	*B/A*
Distilled Water at 20°C	5.0
Sea water (35 ppm salinity)	5.25
Methanol at 20°C	9.42
Ethanol at 20°C	10.52
Monatomic gas at 20°C	0.67
Diatomic gas at 20°C	0.40
Whole porcine blood at 30°C	6.2
Haemoglobin at 30°C	7.6
Corn oil at 20°C	10.7
Olive oil at 20°C	11.1
Mineral oil at 20°C	11.3
Saturated marine sediments	11.8
Bubbly liquids (vol. conc. 10^{-3})	> 3000 [19]

4. Parametric acoustic arrays

The concept of the ***parametric acoustic array*** was conceived by P.J. WESTERVELT based on his theory of "scattering of sound by sound" [20]. An account of the early developments in parametric acoustic arrays may be found in [21] and more recent results and practical design information may be found in [22].

The parametric acoustic array offers a side-lobe free, narrow beam at low frequencies from a physically small array. Moreover, it offers a very wide bandwidth relative to the central frequency of the ***difference-frequency (secondary) signal.*** But the price is a low conversion efficiency from the primary signals to the secondary signals. The source level of the secondary signal is typically 40 dB less than the primary signal source level for a normally used down-shift ratio of 10 (the downshift ratio is determined by $\chi_{ds} = f_0/f_s$, where $f_0 = (f_1 + f_2)/2$ and f_1 and f_2 are the two primary signal frequencies, and f_s is difference-frequency). However, in spite of this price, the parametric acoustic array has recently found applications in several EU-funded MAST for the study of long-range underwater communication in shallow-water channels [23] and by investigations of the seabed and of sub-bottom profiles [24].

Due to beam-broadening and array length shortening by increasing absorption, when acoustic saturation effects are involved, most use of the parametric array takes place at lower primary source levels not leading to saturation effects [21]. The secondary field of the parametric acoustic array can, for lower source amplitudes, be evaluated in two different ways depending on whether the major nonlinear interactions between the primary waves, and thus the production of the secondary wave, take place within the collimated zone (i.e. the nearfield) of the primary signals or in the spreading beams of their farfield. For more strongly absorbing liquids the interaction will be ***absorption loss limited*** in the primary nearfield, while lower absorption will preserve the acoustic energy for the farfield, thus permitting a ***spreading-loss limited*** array to be formed. The criteria for, in which of the two regions the major interaction between the primaries takes place, may be expressed by:

$$R_p = \alpha_T R_0 \chi_{ds}. \tag{18}$$

Here $\alpha_T = \alpha_1 + \alpha_2$, i.e. the sum of the absorption coefficients at the two primary frequencies, while R_0 is the Rayleigh distance of the transmitting transducer. For $R_p \gg 1$, an absorption limited array described by WESTERVELT's model [8] will result, while the most frequently occurring case, the spreading-loss limited array, will be based on $R_p \ll 1$.

Westervelt's solution to the absorption loss limited array may be written as:

$$p_s(R,\theta) = \omega_s^2 p_1 p_2 S\beta(4\pi\rho_0 c_0^4 R\alpha_T)^{-1}[1 + k_s^2/\alpha_T^2(\sin^4(\theta/2))]^{-1/2}, \tag{19}$$

where p_s is the difference-frequency signal amplitude, while ω_s and k_s denote the angular frequency and the wave number of the secondary wave, respectively. S denotes the cross-sectional area of the collimated beam region, and p_1 and p_2 denote the pressure amplitudes of the primary waves. $\beta = 1 + B/2A$, and R and θ are coordinates in a cylindrical coordinate system. The bracket leads to the half-power beamwidth θ_h expressed by: $\theta_h \cong 2(\alpha_T/2k_s)^{1/2}$. This expression shows that a narrowing of the secondary beam will take place for a decrease in the primary frequencies and for an increase in the secondary frequency.

Spreading-loss limited arrays will show a half-power beamwidth increasing with the source distance R and asymptotically approaching the half-power beamwidth of the product of the two primary beam directivity patterns. For a spreading-loss limited array the secondary signal pressure amplitude may be derived from Westervelt's solution by multiplying this solution with the factor: $K = \varphi_c^2 r_0^2/(8r^2)$, where φ_c is the effective beamwidth of the zone of interaction over which the phase of the carrier wave field may be assumed to be constant, r_0 is the Fresnel distance of the carrier wave and r is the effective source radius [25].

The effective length of the parametric array is governed by $(2\alpha)^{-1}$, where α is the small signal absorption coefficient in nepers/meter. The difference-frequency sound pressure level SL_s (in dB rel. $1\mu\text{Pa}\cdot\text{m}$) may for sea water be expressed by:

$$SL_s = SL_1 + SL_2 + 20\log f_s + Q - 286.5, \tag{20}$$

where f_s is the difference-frequency in kHz, SL_1 and SL_2 denote the sound pressure levels of the two primary signals. The source level coefficient Q may be determined from Fig. 1 in [26]. The numeric 286.5 is arising from 20log$[1000\rho_0 c_0^3/(2^{1/2}\pi\beta)]$.

Recent studies [23] of ***long-range, shallow-water communication*** have shown, that a destructive summation takes place of the secondary signal fields produced parametrically before and after interaction between the sea surface and the parametric array, as the two fields have a phase difference of up to π for a pressure-release surface. Therefore, the surface reflection effect will introduce a reduction in the secondary signal level depending on the characteristics of the parametric source and the geometry it operates under. The geometry influence shows that the amplitude reduction due to surface effects approaches a constant value at grazing angles normally encountered at long distance underwater communication. Also the roughness of the sea surface will have a substantial effect on the reflection, and thus on the long-range propagation, of parametric acoustic signals. The loss of coherence of the primary signals after reflection – and scattering – by a very rough sea surface essentially truncates and terminates the parametric array at the surface. This reduces the application of parametric acoustic arrays for underwater communication due to destructive influences at the interaction between the array and the sea surface.

The SIGMA project [24] comprised the application of a vertically down-looking parametric array for ***bottom and sub-bottom profiling*** and an oblique parametric array for studies of the seabed materials using inverse techniques. The primary frequencies of the vertical array was between 55 and 65 kHz, thus leading to secondary frequencies of 3–9 kHz. The experimental results confirmed the sensitivity of the vertical array to platform stability. Due to the small "footprint" made by the narrow parametric beam, minor seabed slopes or movements of the towed fish carrying the parametric transducer and the receiver, the return signals disappeared. This constitutes a considerable limitation to bottom studies using parametric arrays instead of linear arrays.

By the oblique parametric array, operating at a primary frequency around 75 kHz and using secondary frequencies in the range of 1–12 kHz, the return signals were received using a towed array of hydrophones. A roll of the platform of more than 3° would also here prevent the forward scattered signal from reaching the towed array, leading to an interruption in the return signals.

These recent results show some limitations in the application of the parametric acoustic array. The arrays advantages like their narrow and side-lobe free beams form basis for the limitations in their application, and the interaction process giving rise to the beam qualities may be terminated by the moving rough sea surface. Attempts to improve the conversion efficiency are numerous. Replacing the array nearfield with liquids having a higher β-value and a lower density and in particular a lower velocity of sound have been carried out [25] and a gain of 10–14 dB was found. Also replacing the two high-frequency primary waves with an amplitude modulation of a carrier wave could increase the secondary pressure level with 2.5 dB [25]. The use of higher primary pressure amplitudes, while suppressing cavitation in the nearfield by putting this part of the array under pressure, and still avoiding saturation effects have been done with the gain of some dB's on the secondary amplitude level. However, it may be fair to say, that in spite of its many positive qualities, a real break-through in the underwater application of the parametric acoustic array is still waiting.

5. Focused ultrasonic fields

The *K–Z–K equation* has over the past formed basis for a number of studies of focused ultrasonic fields as for instance in relation to acoustic microscopes, high-intensity focused ultrasonic surgery and occlusion of blood vessels, lithotripsy and cavitation induced tissue destruction. For a source radius of curvature d, the dimensionless coordinate σ may be transformed to $\sigma' = \sigma r_0/d$, which inserted into the K–Z–K equation leads to [27, 28]:

$$[(\partial^2/\partial y \partial \sigma') - (4G)^{-1} \nabla \perp^2 - K_1(\partial^3/\partial y^3)]P = (K_2/2)(\partial^2/\partial y^2)P^2, \tag{21}$$

where $K_1 = \alpha d$, $K_2 = \beta \omega p_0 d/(\rho c_0^3)$ and where the linear focal gain $G = r_0/d$.

This form of the K–Z–K equation may be transformed [28] into a series of coupled partial differential equations in terms of the Fourier components of the pressure, and the equations may be solved using a PC-based procedure. This frequency-domain technique, called ***the spectral method***, has been extensively studied over the past where monochromatic waves or tone bursts were used. However, in many biomedical applications the acoustic waves consist of a small number of cycles or even a single cycle. While the excessive computer time used by calculations involving shock formation due the nonlinearity can be reduced by exploiting the physics in the nonlinear process [28], the situation becomes more difficult if absorption is frequency dependent in a complex way like in tissues and if the beam is strongly focused. The parabolic approximation behind the K–Z–K equation demands that the angular spectrum is narrow, or in other words, that the wave is very close to a plane wave. This is not the case for strongly focused beams or for beams having substantial irregularities in their transverse structure. This weakness limits the applicability of the K–Z–K equation to cases where the diffraction effects are weaker and where the focusing gains are relatively low. In [29] it was shown that the application of the K–Z–K equation for description of focused beams was only correct when the ratio of transducer aperture a to focal distance d, $a/d < 0.5$. This demand is in particular caused by the fact that the parabolic approximation is not able to cover the influence of edge waves. However, as shown in [28] the use of a coordinate transformation originally suggested in [30] made it possible to investigate focusing gains near 200.

In order to overcome some of the problems related to the spectral method, LEE *et al.* [31] developed a ***time-domain algorithm*** for solving the K–Z–K equation. This algorithm used a marching scheme based upon an operator-splitting method and it permits for each step to take individual account of acoustic nonlinearity, attenuation and diffraction. The focused field in an *electrohydraulic lithotripter* was calculated using the time-domain method [32]. Recently, another time-domain approach accounting for full diffraction and arbitrary absorption effects has been developed and used for simulation of the pressure field created in tissues by a highly focused source [33]. Also here an operator-splitting algorithm is used to solve a set of equations accounting for the effects of nonlinearity, attenuation and diffraction.

6. Conclusions

The development in nonlinear acoustics, started nearly 250 years ago, has brought a substantial series of important contributions to the progress of this field of research in acoustics. While the theoretical basis and the numerical tools have advanced very far, the practical applications of the research results in nonlinear acoustics are still lacking behind. Apart from the parametric acoustic array, which hitherto has not found a wider application even in underwater acoustics, no single instrument, more extensively used and solely based on nonlinear acoustics has been developed. Nonlinear acoustics research results have instead found applications by their ability to explain certain observations in relation to, for instance, materials characterisation and use of high-power ultrasound in medicine and biology.

References

[1] BJØRNØ L., *Nonlinear acoustics and its applications: A historical perspective*, Proceedings of the 47-th Open Seminar on Acoustics, Polish Acoustical Society, Vol. 1, 19–30, Solina-Jawor, September, 2000.

[2] BJØRNØ L., *Nonlinear acoustics* [in:] *Acoustics and vibration progress*, Vol. 2, R.W.B. Stephens and H.G. Leventhall [Eds.], Chapman and Hall, London, 101–203, 1976.

[3] RUDENKO O.V., SOLUYAN S.I., KHOKHLOV R.V., *Problems of the theory of nonlinear acoustics* [in:] *Finite-amplitude wave effects in fluids*, L. Bjørnø [Ed.], IPC Science and Technology Press, 92–98, 1974.

[4] GOL'DBERG Z.A., *On the propagation of plane waves of finite amplitude*, Soviet Phys.-Acoust., **2**, 346 (1956).

[5] BURGERS J.M., *A mathematical model illustrating the theory of turbulence*, [in:] *Advances in applied mechanics*, Vol 1., R. von Mises and T. von Kármán [Eds.], Academic Press, New York, 171–199.

[6] ZABOLOTSKAYA E.A., KHOKHLOV R.V., *Quasi-plane waves in the nonlinear acoustics of confined beams*, Soviet Phys.-Acoust., **15**, 35–40 (1969).

[7] KUZNETSOV V.P., *Equations of nonlinear acoustics*, Soviet Phys.-Acoust., **16**, 467–470 (1971).

[8] WESTERVELT P.J., *Parametric acoustic arrays*, J. Acoust. Soc. Amer., **35**, 535–537 (1963).

[9] BJØRNØ L., *Characterization of biological media by means of their nonlinearity*, Ultrasonics, **24**, 254–259 (1986).

[10] BEYER R.T., *Parameter of nonlinearity in fluids*, J. Acoust. Soc. Amer., **32**, 719–721 (1960).

[11] BEYER R.T., *Nonlinear acoustics*, [in:] *Physical Acoustics*, Vol 2., W.P. Mason [Ed.], Academic Press, 231–264, 1965.

[12] FUBINI G.E., *Anomalie nella propagazione di onde acustiche di grande ampiezza*, Alta Frequenza, **4**, 530–581 (1935).

[13] APFEL R.E., *The effective nonlinearity parameter for immiscible liquid mixtures*, J. Acoust. Soc. Amer., **74**, 1866–1870 (1983).

[14] BEYER R.T., *Nonlinear acoustics*, Naval Ship Systems Command, US Department of the Navy, 1974.

[15] HARTMANN B., LEE G.F., BALIZER E., *Calculation of B/A for n-alkane liquids using the Tait equation*, J. Acoust. Soc. Amer., **108**, 65–70 (2000).

[16] BANCHET J., CHEEKE J.D.N., *Measurement of the acoustic nonlinearity parameter B/A in solvents: dependence on chain length and sound velocity*, J. Acoust. Soc. Amer., **108**, 2754–2758 (2000).

[17] MADIGOSKY W.M., ROSENBAUM I., LUCAS R., *Sound velocities and B/A in flourocarbon fluids and in several low density solids*, J. Acoust. Soc. Amer., **69**, 1639–1643 (1981).

[18] EVERBACH E.C., *Parameters of nonlinearity of acoustic media*, [in:] *Encyclopedia of acoustics*, M.J. Crocker [Ed.], John Wiley & Sons, Inc., Vol. 1, 20, 219–226, 1997.

[19] BJØRNØ L., *Acoustic nonlinearity of bubbly liquids*, Applied Scientific Research, **38**, 291–296 (1982).

[20] WESTERVELT P.J., *Scattering of sound by sound*, J. Acoust. Soc. Amer., **29**, 199–203 (1957).

[21] BJØRNØ L., *Parametric acoustic arrys*, [in:] *Aspects of Signal Processing*, G Tacconi [Ed.], Vol. 1, D. Reidel Publishing Comp., 33–59, 1977.

[22] MOFFETT M.B., KONRAD W.L., *Nonlinear sources and receivers*, [in:] *Encyclopedia of acoustics*, M.J. Crocker [Ed.], Vol. 1, 53, 607–617 (1997).

[23] WANG L.S., SMITH B.V., COATES R., *The secondary field of a parametric source following interaction with sea surface*, J. Acoust. Soc. Amer., **105**, 3108–3114 (1999).

[24] SIGMA, Sediment Identification for Geotechnics by Marine Acoustics. EU MAST project No. MAS3-CT97-0100. Final Report 2001.

[25] BJØRNØ L., CHRISTOFFERSEN, B. SCHREIBER M.P., *Some experimental investigations of the parametric acoustic array*, Acustica, **35**, 99–106 (1976).

[26] BERKTAY H.O., *Parametric sources – Design considerations in the generation of low-frequency signals*, [in:] *Ocean seismo-acoustics*, T. Akal & J.M. Berkson [Eds.], Plenum Press, 785–800, 1986.

[27] AANONSEN S.I, *Numerical computation of the nearfield of a finite amplitude sound beam*, Report No. 73, Department of Mathematics, University of Bergen, Norway, 1983.

[28] NEIGHBORS T.H., BJØRNØ L., *Monochromatic focused sound fields in biological media*, J. Lithotripsy and Stone Disease, **2**, 4–16 (1990).

[29] VEFRING E.H., NAZE TJØTTA J., TJØTTA S., *Effects of focusing on the nonlinear interaction between two collinear finite amplitude sound beams*, J. Acoust. Soc. Amer., **89**, 1017–1027 (1991).

[30] HART T.S., *Numerical investigation of nonlinear effects in focused sound beams*, (M.S. Thesis), The University of Texas at Austin, 1987.

[31] LEE Y.S., HAMILTON M.F., *Time-domain modeling of pulsed finite-amplitude sound beams*, J. Acoust. Soc. Amer., **97**, 906–917 (1995).

[32] AVERKIOU M.A., CLEVELAND R.O., *Modeling of an electrohydraulic lithotripter with the KZK equation*, J. Acoust. Soc. Amer., **106**, 102–112 (1999).

[33] TAVAKKOLI J., CATHIGNOL D., SOUCHON R., SAPOZHNIKOV O.A., *Modeling of pulsed finite-amplitude focused sound beams in time domain*, J. Acoust. Soc. Amer., **104**, 2061–2072 (1998).