MODELLING OF THE PRESSURE AND DENSITY SENSITIVE BEHAVIOUR OF SAND WITHIN THE FRAMEWORK OF HYPOPLASTICITY

ERICH BAUER

Institute of General Mechanics, Graz University of Technology
Kopernikusgasse 24, A-8020 Graz, Austria
bauer@mech.tu-graz.ac.at

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Abstract: In this paper a hypoplastic model proposed by Gudehus and Bauer is presented which is apt to describe the mechanical behaviour of loose and dense sand within a wide range of pressures and densities using a single set of constants. State changes are assumed to depend on the current void ratio, the Cauchy stress tensor and the stretching tensor. The constitutive Equation is of the rate type and based on non-linear tensor-valued functions. By including a pressure dependent relative density the hypoplastic model describes the influence of pressure and density on the incremental stiffness, the peak friction angle and on the void ratio in a stationary state. The performance of the hypoplastic models is discussed and the results of numerical simulations are compared with experiments.

Keywords: hypoplasticity, element test, void ratio, critical state, granular materials

1. Introduction

The term hypoplasticity was originally introduced by Dafalias (1986) for incremental constitutive relations for which, in contrast to the classic elastoplastic concept, the plastic strain rate is defined without a reference to any plastic potential surface. Concerning this general definition of hypoplasticity many constitutive models discussed in the literature can be included in this class although their physical concepts may be rather different, *e.g.* the models proposed by Darve (1974, 1982), Kolymbas (1977, 1985, 1991), Chambon (1979), Valanis (1982), Wang (1990), Bardet (1990), Wu and Bauer (1992) and others. The present hypoplastic constitutive model is orientated towards the basic concept of the Kolymbas type, where the evolution of the stress state is described by an isotropic and incrementally non-linear tensor-valued function depending on the current state variables and the stretching tensor. The tensor-valued function consists of a part which is linear in the stretching and of a part which is non-linear in the stretching. Since the constitutive Equation is incrementally non-linear an irreversible behaviour is modelled with a single constitutive relation.

It is experimentally evident that the peak friction angle strongly depends on the density, the mean pressure and on the direction of the stress deviator (*e.g.* Matsuoka and Nakai 1974, Bouvard and Stutz 1986). The peak stress ratio is higher for a dense sand under low pressure and decreases with an increase of the mean pressure (*e.g.* Wu 1992). Under deviatoric loading dense sand tends to dilate while an initially very loose sand tends to contract up to the limit state. Within the framework of hypoplasticity, Wu and Bauer (1992) were the first to model such behaviour using a pressure dependent relative void ratio. The efforst by Gudehus (1996), Bauer (1996) and von Wolffersdorff (1996) led to a more comprehensive version with a consistent description of so-called SOM-states, which also allows an easier calibration of the material parameters involved. The hypoplastic model covers a broad range of densities and pressures and is applicable to contractant and dilatant material behaviour using a single set of constants. Stationary states or so-called critical states (Casagrande 1936, Schofield and Wroth 1968, Poulos 1981, Been, Jefferies and Hachey 1991) are included in the constitutive equation for a simultaneous vanishing of the stress rate and volume strain rate which can be achieved by unlimited monotonic shearing. In a recent work by Bauer (2000), general requirements for a consistent embedding of stationary states in this hypoplastic model are presented in detail. The range of possible void ratios for a simple grain skeleton is bounded by the maximum and the minimum void ratio which decreases with the mean pressure. The ratio of the current void ratio to the limit void ratios and to the critical void ratio is mainly responsible for modelling contractancy or dilatancy behaviour, the peak friction angle, strain softening and stationary states. With the hypoplastic model various boundary value problems have been solved and compared with experiments (*e.g.* Hügel 1995, von Wolffersdorff 1997, Bauer and Huang 1997, Herlé and Tejchman 1997). Capacity, limitations and possible extensions of the hypoplastic model have been discussed for instance by Bauer and Wu (1993, 1995), Gudehus (1996), Tejchman and Bauer (1996), Gudehus (1997), Niemunis and Herle (1997) and by Tejchman and Gudehus (1999).

Throughout the paper compressive stress and strain are negative as in the sign convention of continuum mechanics. Bold letters and calligraphic letters denote second order and fourth order tensors, respectively. The following tensor operations are used: $\mathbf{A}\cdot\mathbf{B} = A_{ij}\,B_{jk}$, $\mathbf{A}:\mathbf{B} = A_{ij}\,B_{ij}$, $\mathcal{A}:\mathbf{B} = A_{ijkl}\,B_{kl}$, $\mathrm{tr}\mathbf{A} = A_{ii}$, the summation convention over repeated indices being employed. A superimposed dot indicates a time derivative and the quantity $\|\mathbf{A}\| = \sqrt{A_{ij}A_{ij}}$ denotes the Euclidean norm of $\mathbf{A}$.

2. Pressure dependence of the relative void ratio

In the following it is assumed that a dry and cohesionless simple grain skeleton can be sufficiently characterised by the void ratio e, *i.e.* the ratio of the volume of the voids to the volume of the grains, and the Cauchy stress tensor **T**. State changes are dictated by the current quantities of the state variables (e, **T**) and the rate of deformation **D**, which represents the symmetric part of the gradient of the mean field velocity **v** of the grain skeleton, *i.e.* $\mathbf{D} = [\nabla \mathrm{v} = (\nabla \mathrm{v})^T]/2$. The void ratio mainly changes due to a rearrangement of grains within the grain skeleton caused by compression, extension or shearing of the the granular body and may also be accompanied by deformations, abrasion and the crushing of grains. For granular materials like quartz sand the volume change of the grains is negligible so the rate of the void ratio is proportional to the volume strain rate $\mathrm{tr}\,\mathbf{D}$, *i.e.*:

$$\dot{e} = (1+e)\mathrm{tr}\,\mathbf{D}. \tag{1}$$

Starting from a known initial value e_o the void ratio e after a certain volume strain ε_υ of the granular body can be obtained by integration of Equation (1), which leads to the following relation:

$$e = (1+e_o)\exp\{\varepsilon_\upsilon\} - 1. \tag{2}$$

Herein a negative value to ε_υ means a compaction of the grain skeleton, *i.e.* the void ratio decreases, while for a positive value to ε_υ the void ratio increases. It is experimentally evident that for granular materials there is no unique relation between the void ratio and the pressure. This means that for the same pressure there exists a certain range of possible void ratios which is bounded by a maximum and a minimum void ratio. For simple grain skeletons the limit void ratios are pressure dependent as sketched out in Figure 1 in the so-called phase diagram of grain skeletons (Gudehus 1997). Herein the maximum void ratio e_i and the minimum void ratio e_d decrease with the mean pressure $p = -\mathrm{tr}\,\mathbf{T}/3$.

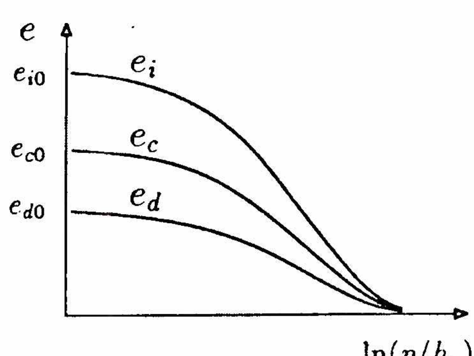


Figure 1. *Pressure dependence of the limit void ratios e_i and e_d and the critical void ratio e_c*

The upper bound e_i can be related to an isotropic compression starting from the loosest possible skeleton with grain contacts, *i.e.* there exists no homogeneous deformation which goes beyond $e = e_i$. With the exponential function (Bauer 1995):

$$e_i = e_{i0} \exp\left[-\left(\frac{3p}{h_s}\right)^n\right], \tag{3}$$

the isotropic compression behaviour of sand can be approximated for a wide range of pressures. Herein e_{i0} denotes the maximum void ratio for granular material under $p \approx 0$, the exponent n is a dimensionless constant and the constant and the constant h_s has the dimension of a stress. Values of the lower bound e_d will be achieved by cyclic shearing with very low amplitudes. By contrast, large monotonic deformations lead to a stationary state or so-called critical state, in which the grain skeleton can continuously be deformed at a stationary critical stress and at a corresponding stationary critical void ratio e_c. It is experimentally evident that the critical void ratio is higher for a lower pressure (*e.g.* Bouvard and Stutz 1986). For the constitutive modelling a similar pressure dependence for e_c and e_d as for e_i was postulated by Gudehus (1996), *i.e.*:

$$\frac{e_i}{e_{i0}} = \frac{e_c}{e_{c0}} = \frac{e_d}{e_{d0}} = \exp\left[-\left(\frac{3p}{h_s}\right)^n\right], \tag{4}$$

wherein e_{c0} and e_{d0} are the corresponding void ratios for a nearly stress free state. Relation (3) defines the range of possible void ratios e depending on the pressure level, *i.e.* $e_i > e_c > e_d$ and therefore $e_i \geq e \geq e_d$ holds for $0 < p < \infty$, as shown in Figure 1.

In order to model the interaction between the void ratio and the stress level it is convenient to relate the current void ratio to the pressure dependent limit void ratios (Wu and Bauer 1992, Wu *et al.* 1996, Bauer 1996, Gudehus 1996). In particular, the relations:

$$r_e = \frac{e_i}{e}, \tag{5}$$

and

$$r_d = \frac{e - e_d}{e_c - e_d}, \tag{6}$$

are suitable quantities, which are incorporated in the present constitutive equation (Section 3). For states which are related to e_d, e_c, or e_i the corresponding quantities for r_e and r_d are independent of the mean pressure, *i.e.* for:

$$e = e_d: \qquad r_e = \frac{e_{i0}}{e_{d0}} > 1; \qquad r_d = 0,$$

$$e = e_c: \qquad r_e = \frac{e_{i0}}{e_{c0}} > 1; \qquad r_d = 1,$$

$$e = e_i: \qquad r_e = 1; \qquad r_d = \frac{e_{i0} - e_{d0}}{e_{c0} - e_{d0}} > 1.$$

3. The hypoplastic model

Stress changes are described by a constitutive equation of the rate type, *i.e.*:

$$\overset{\circ}{\mathbf{T}} = \mathbf{F}(e, \mathbf{T}, \mathbf{D}), \tag{7}$$

where the objective stress rate tensor $\overset{\circ}{\mathbf{T}}$ is represented by an isotropic tensor-valued function $\mathbf{F}$ depending on the current void ratio e, the Cauchy stress tensor $\mathbf{T}$ and the rate of deformation tensor $\mathbf{D}$. The general framework of hypoplasticity requires that function $\mathbf{F}$ in Equation (7) is not differentiable for and only for $\mathbf{D} = \mathbf{0}$ (Wu and Kolymbas 1990). This can be fulfilled by decomposing $\mathbf{F}$ into two parts, *i.e.*:

$$\overset{\circ}{\mathbf{T}} = \mathcal{A}(e, \mathbf{T}) : \mathbf{D} + \mathbf{B}(e, \mathbf{T})\|\mathbf{D}\|. \tag{8}$$

Herein $\mathcal{A}(e, \mathbf{T}) : \mathbf{D}$ is linear in $\mathbf{D}$, while the term $\mathbf{B}(e, \mathbf{T})\|\mathbf{D}\|$ is non-linear in $\mathbf{D}$ with respect to the Euclidean norm of the rate of deformation, *i.e.* $\|\mathbf{D}\| = \sqrt{\mathbf{D} : \mathbf{D}}$. Both functions are positively homogeneous of the first degree in $\mathbf{D}$ and the material behaviour to be described is therefore rate-independent.

In order to visualise the incremental non-linearity of the constitutive Equation (8) Figure 2 sketches, for an axisymmetric problem, the stress rates obtained for a known state ($\mathbf{T}_o$, e_o) and two particular stretching rates $\mathbf{D}_a$ and $\mathbf{D}_b$ with the same magnitudes but opposite principal directions, *i.e.* $\|\mathbf{D}_a\| = \|\mathbf{D}_b\|$ and $\mathbf{D}_a = -\mathbf{D}_b$. The stress rates from the first part of Equation (8) show an incrementally linear behaviour, *i.e.* $\mathcal{A} : \mathbf{D}_a = -\mathcal{A} : \mathbf{D}_b$, while the responsesof the second part of Equation (5), *i.e.* $\mathbf{B}\|\mathbf{D}_a\| = \mathbf{B}\|\mathbf{D}_b\|$, are independent of the stretching. The sums of the corresponding parts have different magnitudes and different directions, *i.e.* $\overset{\circ}{\mathbf{T}}_a = \overset{\circ}{\mathbf{T}}(e_o, \mathbf{T}_o, \mathbf{D}_a) \neq -\overset{\circ}{\mathbf{T}}_b = \overset{\circ}{\mathbf{T}}(e_o, \mathbf{T}_o, \mathbf{D}_b)$. Therefore incremental nonlinearity is accounted for with a single constitutive equation and there is no need to distinguish between loading and unloading explicitly.

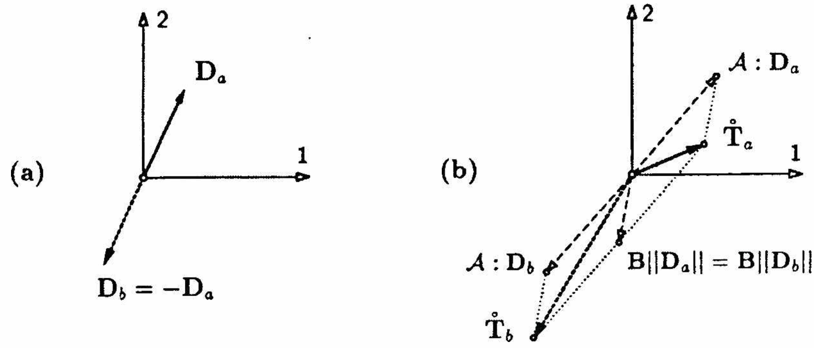


Figure 2. *Responses of the constitutive Equation (9) for the deformation rates* $\mathbf{D}_a$ *and* $\mathbf{D}_b = -\mathbf{D}_a$ *under an axisymmetric stress state: (a) Particular stretching rates* $\mathbf{D}_a$ *and* $\mathbf{D}_b$*; (b) Stress rate response* $\overset{\circ}{\mathbf{T}}_a$ *and* $\overset{\circ}{\mathbf{T}}_b$

In order to make a specific representation of the hypoplastic model more transparent and the calibration easier a factorised representation of (7) was proposed by Gudehus (1996) and Bauer (1996), which reads:

$$\overset{\circ}{\mathbf{T}} = f_s(e,\mathrm{tr}\mathbf{T})\left[\mathcal{L}\left(\hat{\mathbf{T}}\right):\mathbf{D} + f_d(e,\mathrm{tr}\mathbf{T})\mathbf{N}\left(\hat{\mathbf{T}}\right)\|\mathbf{D}\|\right]. \tag{9}$$

Herein $\mathcal{L}(\hat{\mathbf{T}})$ and $\mathbf{N}(\hat{\mathbf{T}})$ are dimensionless tensor functions depending on the normalised tensor $\hat{\mathbf{T}} = \mathbf{T}/\mathrm{tr}\mathbf{T}$. The stiffness factor f_s has the dimension of stress and the dimensionless density factor f_d is related to a pressure dependent relative void ratio. The terms in the factorised form (9) are not restricted to a certain representation but they have to fulfil several conditions for consistency.

3.1 Proposed functions for $\mathcal{L}$ and N

For a consistent description of stationary states the isotropic tensor--valued functions $\mathcal{L}(\hat{\mathbf{T}}):\mathbf{D}$ and $\mathbf{N}(\hat{\mathbf{T}})$ in Equation (9) have to fulfil the condition for a simultaneous vanishing of the stress rate and the volume strain rate independent of the direction of the deviatoric stress as discussed by Bauer (1995, 1996) and von Wolffersdorf (1996). This requirement is fulfiled for the proposed version by Bauer (1995), *i.e.*:

$$\mathcal{L}\left(\hat{\mathbf{T}}\right):\mathbf{D} = \hat{a}^2\,\mathbf{D} + \hat{\mathbf{T}}\,\mathrm{tr}\left(\hat{\mathbf{T}}\mathbf{D}\right), \tag{10}$$

$$\mathbf{N}\left(\hat{\mathbf{T}}\right) = \hat{a}\left(\hat{\mathbf{T}} + \hat{\mathbf{T}}^*\right), \tag{11}$$

which represents a modification of the version proposed by Wu (1992). Herein $\hat{\mathbf{T}}^* = \hat{\mathbf{T}} - (1/3)\,\mathbf{I}$ denotes the deviator of $\hat{\mathbf{T}}$, $\mathbf{I}$ denotes the unit tensor. The dimensionless scalar $\hat{a}$ is related to stationary states and includes at least one constant which can be expressed by the so-called critical friction angle φ_c (Section 3.3).

3.2 Density factor f_d and stiffness factor f_s

In the constitutive Equation (9) the influence of the void ratio e and the mean pressure p is taken into account by the so-called density factor f_d and the so-called stiffness factor f_s. The former is related to the relative void ratio factor r_d in Equation (4) by:

$$f_d = r_d^\alpha = \left(\frac{e - e_d}{e_c - e_d}\right)^\alpha, \tag{12}$$

where $\alpha < 0.5$ is a positive constant and the critical void ratio e_c and the minimum void ratio e_d are pressure dependent as defined in (4).

With the factor f_d the dependence of the peak friction angle and the dilatancy behaviour on the density and pressure are included in the present model as sketched out in Figure 3 for shearings starting from different initial void ratios. For $e_1 < e_c$ Equation (11) yields $f_d < 1$ while for $e_2 > e_c$ a value of $f_d > 1$ is obtained. The peak friction angle (Figure 3a) is higher for a lower void ratio because the influence of the part $f_d\mathbf{N}\|\mathbf{D}\|$ of the constitutive Equation (9) decreases if $f_d < 1$. After the peak strain softening can be observed as a consequence of dilatancy and with advanced

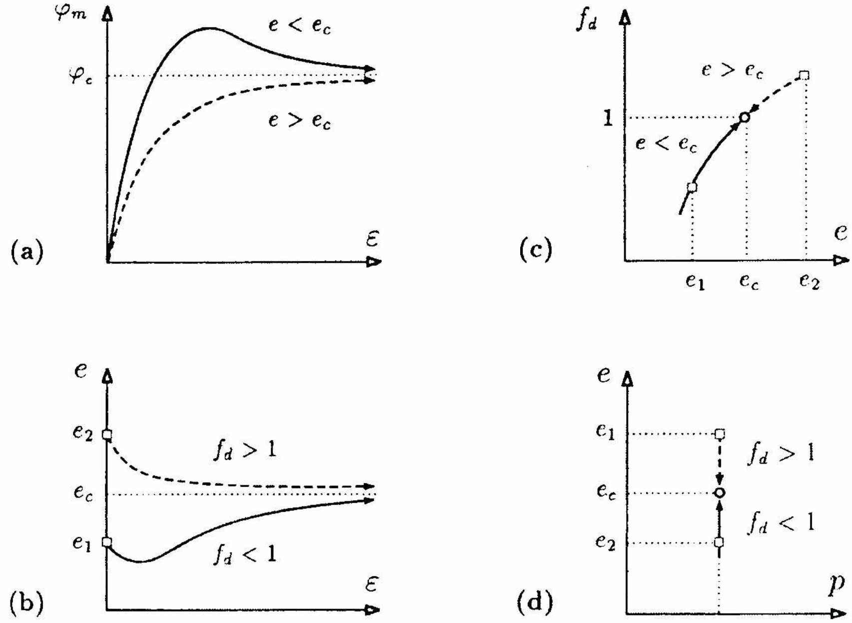


Figure 3. *Homogeneous monotonic shearing under constant mean pressure for different initial void ratios e_1 and e_2: (a) mobilised friction angle φ_m versus shear strain ε; (b) void ratio e versus shear strain ε; (c) void ratio e versus density factor f_d; (d) void ratio e versus mean pressure p*

monotonic shearing the density factor tends towards $f_d = 1$ (Figure 3c). For $e_2 > e_c$ the material shows contractancy up to the critical state (Figure 3b) and for $e = e_c$ the value of the density factor again becomes $f_d = 1$ (Figure 3c). Consequently, in a critical state the value of the density factor is independent of the initial void ratio and the pressure level. This means that for unlimited monotonic shearing a stationary state can be reached asymptotically both for an initially dense and for an initially loose state.

The stiffness factor f_s in Equation (9) can be decomposed into three parts:

$$f_s = f_e f_s^* f_b . \tag{13}$$

Herein f_e is the density dependent part (Bauer 1996), *i.e.*:

$$f_e = r_e^{\beta} = \left(\frac{e_i}{e} \right)^{\beta} , \tag{14}$$

with the constant $\beta \geq 1$. A decrease of the void ratio e results in an increase in f_e, *i.e.* the incremental stiffness increases. The second part, *i.e.*:

$$f_s^* = \frac{1}{\hat{\mathbf{T}} : \hat{\mathbf{T}}} , \tag{15}$$

takes into account a decrease of the incremental stiffness with an increase of $\hat{\mathbf{T}}$ (von Wolffersdorff 1996). For an isotropic state $f_s^* = 3$. A more general form of f_s^*, which also accounts for the dependence on the deviatoric stress direction reads (Bauer 2000):

$$f_s^* = \left(1 + \left\|\hat{\mathbf{T}}^*\right\|\left[c_1 + c_2\cos(3\theta)\right]\right)^{-1}. \tag{16}$$

Herein $\|\hat{\mathbf{T}}^*\|$ denotes the norm of the deviatoric stress ratio and θ denotes the Lode angle. The dimensionless factors c_1 and c_2 in Equation (16) are constants or functions of $\hat{\mathbf{T}}$. It can be noticed that relation (15) is included in (16) for the special expressions $c_1 = [\mathrm{tr}(\hat{\mathbf{T}}^2) - 1] / \|\hat{\mathbf{T}}^*\|$ and $c_2 = 0$.

An increase of the incremental stiffness with the pressure level is taken into account with the pressure dependent part f_b in Equation (13), which can be directly derived from a consistency condition (Gudehus 1996). For isotropic compression starting from the maximum void ratio e_i the rate of the mean pressure calculated from (3), *i.e.*:

$$3\dot{p} = \frac{\dot{e}_i}{e_i}\frac{h_s}{n}\left(\frac{3p}{h_s}\right)^{1-n},$$

and the rate of the mean pressure from (9), *i.e.*:

$$3\dot{p} = f_k h_i \frac{\dot{e}_i}{(1+e_i)},$$

must coincide. Thus, factor f_b can be obtained as:

$$f_b = \frac{h_s}{n\,h_i}\,\frac{1+e_i}{e_i}\left(\frac{3p}{h_s}\right)^{1-n}, \tag{17}$$

with the constant:

$$h_i = 3\hat{a}_o^2 + 1 - \sqrt{3}\hat{a}_o\left(\frac{e_{i0} - e_{d0}}{e_{c0} - e_{d0}}\right)^{\alpha}.$$

Herein $\hat{a}_o$ denotes the value of *hata* for isotropic stress states (Section 3.3).

3.3 Adaptation of $\hat{a}$ to stationary stress states

With the special representation for the tensor functions $\mathcal{L}$ and $\mathbf{N}$ in (10) and (11) the hypoplastic model by Gudehus and Bauer is apt to describe stationary states, *i.e.* states in which a grain aggregate can continuously be deformed at a constant stress and a constant volume under a certain rate of deformation. The corresponding state variables $\mathbf{T}_c$ and $e = e_c$ are called critical stress and critical void ratio, respectively (Casagrande 1936, Schofield and Wroth 1968). With respect to $e = e_c$ the density factor in (12) becomes $f_d = 1$ and for a vanishing stress rate, *i.e.* $\mathbf{T}_c = 0$, the stiffness factor f_s has no influence so that the constitutive Equation (9) reduces to:

$$\hat{a}_c^2 \mathbf{D}_c + \hat{\mathbf{T}}_c \operatorname{tr}\left(\hat{\mathbf{T}}_c \cdot \mathbf{D}_c\right) + \hat{a}_c\left(\hat{\mathbf{T}}_c + \hat{\mathbf{T}}_c^*\right)\|\mathbf{D}_c\| = 0. \tag{18}$$

Dividing relation (18) by $\mathbf{D} \neq 0$ and using the dimensionless tensor $\hat{\mathbf{D}} = \mathbf{D}/\|\mathbf{D}\|$ leads to:

$$\hat{\mathbf{D}}_c = -\frac{1}{\hat{a}_c^2}\hat{\mathbf{T}}_c \operatorname{tr}\left(\hat{\mathbf{T}}_c \cdot \hat{\mathbf{D}}_c\right) - \frac{1}{\hat{a}_c}\left[\hat{\mathbf{T}}_c + \hat{\mathbf{T}}_c^*\right]. \tag{19}$$

For stationary states the volume strain rate must also be vanishing simultaneously with the stress rate, *i.e.*:

$$\operatorname{tr}\hat{\mathbf{D}}_c = -\frac{1}{\hat{a}_c^2}\operatorname{tr}\hat{\mathbf{T}}_c \operatorname{tr}\left(\hat{\mathbf{T}}_c \cdot \hat{\mathbf{D}}_c\right) - \frac{1}{\hat{a}_c}\left[\operatorname{tr}\hat{\mathbf{T}}_c + \operatorname{tr}\hat{\mathbf{T}}_c^*\right] = 0.$$

and with respect to $\operatorname{tr}\hat{\mathbf{T}}_c = \operatorname{tr}\left(\hat{\mathbf{T}}_c/\operatorname{tr}\hat{\mathbf{T}}_c\right) = 1$ and $\operatorname{tr}\hat{\mathbf{T}}_c^* = \operatorname{tr}\left(\mathbf{T}_c/\operatorname{tr}\mathbf{T}_c - \mathbf{1}/3\right) = 1 - 1 = 0$ the equation reduces to:

$$\operatorname{tr}\left(\hat{\mathbf{T}}_c \cdot \hat{\mathbf{D}}_c\right) + \hat{a}_c = 0. \tag{20}$$

Substituting (20) into Equation (19) yields:

$$\hat{\mathbf{D}}_c = -\frac{1}{\hat{a}_c}\hat{\mathbf{T}}_c^*, \tag{21}$$

which reflects coaxiality between $\hat{\mathbf{D}}_c$ and $\hat{\mathbf{T}}_c^*$ for stationary states. Since $\operatorname{tr}\hat{\mathbf{T}}^* = 0$ holds independent of $\hat{\mathbf{T}}^*$, relation (21) also fulfils the requirement for a vanishing volume strain rate, *i.e.*:

$$\operatorname{tr}\hat{\mathbf{D}}_c = -\frac{1}{\hat{a}_c}\operatorname{tr}\hat{\mathbf{T}}_c^* = 0. \tag{22}$$

Inserting (21) into the identity $\|\hat{\mathbf{D}}\|^2 = \operatorname{tr}(\hat{\mathbf{D}}^2) = 1$ leads to an equation for the stationary stress surface (Bauer 1995), *i.e.*:

$$\hat{a}_c = \left\|\hat{\mathbf{T}}_c^*\right\|. \tag{23}$$

With respect to the identity $\hat{\mathbf{T}}_c^* = \mathbf{T}_c/\operatorname{tr}\mathbf{T}_c - (1/3)\mathbf{1}$ it is obvious that the stationary stress surface is a cone with its apex at the origin of the principal stress component space (Figure 4a). Herein the dimensionless factor $\hat{a}_c$ determines the shape of the stationary stress surface and can be expressed as a function of the Lode-angle θ, *i.e.* $\hat{a}_c = \hat{a}_c(\theta)$ (Figure 4b).

In contrast to the elasto-plastic concept, where the limit condition is treated as a separate equation, the limit condition in hypoplasticity is embedded in the constitutive equation by the function $\hat{a}$ and therefore always active. As a consequence,

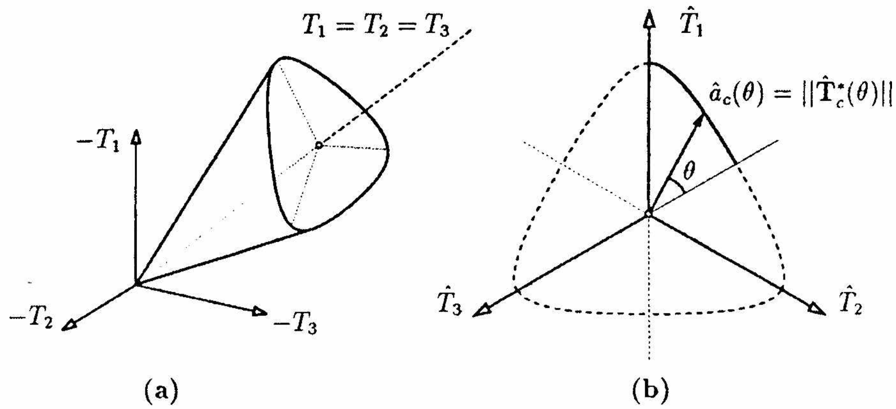


Figure 4. *(a) Stationary stress surface in the space of principal stress components; (b) contour of the stationary stress surface in the π-plane*

the response of the constitutive equation is influenced by the value for $\hat{a}$ also for states which are far from a stationary one so the representation of $\hat{a}$ has to fulfil several requirements (Bauer 2000). For instance the function for $\hat{a}$ must not vanish for any stress and it must not show a jump for stress paths crossing the isotropic state. The latter means that the value of $\hat{a}(\hat{\mathbf{T}}^*)$ may be different to the critical value $\hat{a}_c(\hat{\mathbf{T}}^*_c)$. Thus, $\hat{a} = \hat{a}_c$ only holds for $\hat{\mathbf{T}}^* = \hat{\mathbf{T}}^*_c$. Such behaviour can be modelled by a suitable interpolation function depending on both the Lode angle and the magnitude of the stress deviator. For instance a suitable adaptation of $\hat{a}$ to the limit condition given by Matsuoka and Nakai (1977) reads:

$$\hat{a}(\theta) = \frac{\sin\varphi_c}{3 - \sin\varphi_c}\left[\left\|\hat{\mathbf{T}}^*\right\| - \sqrt{\frac{(8/3) - 3\left\|\hat{\mathbf{T}}^*\right\|^2 + (\sqrt{6}/2)\left\|\hat{\mathbf{T}}^*\right\|^3\cos(3\theta)}{1 + \sqrt{3/2}\left\|\hat{\mathbf{T}}^*\right\|\cos(3\theta)}}\right]. \tag{24}$$

Herein θ denotes the Lode angle, which is defined as:

$$\cos(3\theta) = -\sqrt{6}\frac{\operatorname{tr}\left(\hat{\mathbf{T}}^{*3}\right)}{\left[\operatorname{tr}\left(\hat{\mathbf{T}}^{*2}\right)\right]^{3/2}}. \tag{25}$$

For isotropic stress states, *i.e.* $\hat{\mathbf{T}}^* = 0$, factor $\hat{a}$ reads:

$$\hat{a}_o = \sqrt{\frac{3}{8}}\frac{\sin\varphi_c}{3 - \sin\varphi_c},$$

and for $\hat{\mathbf{T}}^* = \hat{\mathbf{T}}^*_c$ relation (25) represents the limit condition by Matsuoka and Nakai. The only constant in (25) is the critical friction angle φ_c. It can be related to the stationary state in a standard triaxial compression test, in which a specimen is compressed axially at a constant lateral stress $T_2 = T_3$. With respect to the principal stresses $-T_{1c} > -T_{2c} = -T_{3c}$ the critical friction angle is defined as:

$$\sin\varphi_c = \frac{T_{1c} - T_{2c}}{T_{1c} + T_{2c}}. \tag{26}$$

With respect to the normalised representation of stresses, *i.e.* $\hat{T}_{ii} = T_{ii} / (T_{1c} + T_{2c} + T_{3c})$ the limit condition by Matsuoka and Nakai yields for the following boundary conditions:

(i) Isobaric axisymmetric compression ($T_2 = T_3$):

$$\frac{\hat{T}^*_{1c}}{\hat{T}^*_{2c}} = \frac{\hat{T}^*_{1c}}{\hat{T}^*_{3c}} = -2,\quad \hat{T}_{1c} = \frac{1}{3} + \sqrt{\frac{2}{3}}\hat{a}_c,\quad \hat{T}_{2c} = \hat{T}_{3c} = \frac{1}{3} - \sqrt{\frac{1}{6}}\hat{a}_c,\quad \hat{a}_c = \sqrt{\frac{3}{8}}\frac{\sin\varphi_c}{3 - \sin\varphi_c}$$

(ii) Isobaric axisymmetric extension ($T_2 = T_3$):

$$\frac{\hat{T}^*_{1c}}{\hat{T}^*_{2c}} = \frac{\hat{T}^*_{1c}}{\hat{T}^*_{3c}} = -2,\quad \hat{T}_{1c} = \frac{1}{3} - \sqrt{\frac{2}{3}}\hat{a}_c,\quad \hat{T}_{2c} = \hat{T}_{3c} = \frac{1}{3} + \sqrt{\frac{1}{6}}\hat{a}_c,\quad \hat{a}_c = \sqrt{\frac{3}{8}}\frac{\sin\varphi_c}{3 + \sin\varphi_c}$$

(iii) Coaxial plane strain compression ($D_1 = -1,\ D_3 = 0$):

$$\frac{\hat{T}^*_{1c}}{\hat{T}^*_{2c}} = -1,\quad \hat{T}^*_{3c} = 0,\quad \hat{T}_{1c} = \frac{1}{3} + \frac{\hat{a}_c}{\sqrt{2}},\quad \hat{T}_{2c} = \frac{1}{3} - \frac{\hat{a}_c}{\sqrt{2}},\quad \hat{T}_{3c} = \frac{1}{3},\quad \hat{a}_c = \sqrt{\frac{8\sin^2\varphi_c}{9\left(3 + \sin^2\varphi_c\right)}}.$$

(iv) Coaxial plane strain extension ($D_1 = 1,\ D_3 = 0$):

$$\frac{\hat{T}^*_{1c}}{\hat{T}^*_{2c}} = -1,\quad \hat{T}^*_{3c} = 0,\quad \hat{T}_{1c} = \frac{1}{3} - \frac{\hat{a}_c}{\sqrt{2}},\quad \hat{T}_{2c} = \frac{1}{3} + \frac{\hat{a}_c}{\sqrt{2}},\quad \hat{T}_{3c} = \frac{1}{3},\quad \hat{a}_c = \sqrt{\frac{8\sin^2\varphi_c}{9\left(3 + \sin^2\varphi_c\right)}}.$$

(v) Plane shearing under a constant vertical pressure ($T_{22} = T_{22c},\ D_3 = 0$):

- with respect to a fixed co-ordinate system:

$$\hat{T}_{11c} = \hat{T}_{22c} = \hat{T}_{33c} = \frac{1}{3},\quad \hat{T}_{12c} = -\frac{\hat{a}_c}{\sqrt{2}}$$

- with respect to normalised principal stresses:

$$\hat{T}_{1c} = \frac{1}{3} + \frac{\hat{a}_c}{\sqrt{2}},\quad \hat{T}_{2c} = \frac{1}{3} - \frac{\hat{a}_c}{\sqrt{2}},\quad \hat{T}_{3c} = \frac{1}{3},\quad \hat{a}_c = \sqrt{\frac{8\sin^2\varphi_c}{9\left(3 + \sin^2\varphi_c\right)}}.$$

The comparison of (iii) and (v) shows that the normalised principal stresses obtained under plane shearing coincide with the normalised stresses for coaxial plane compression in a stationary state. Substituting this result in Equation (25) yields for plane shearing and biaxial compression a Lode angle of $\theta = \pi/6$, which is independent of factor $\hat{a}_c$. With respect to the principal stresses the friction angle ϕ_b in biaxial compression is related to factor $\hat{a}_c$ and to the critical friction angle φ_c according to:

$$\sin\phi_b = \frac{\hat{T}_{1c} - \hat{T}_{2c}}{\hat{T}_{1c} + \hat{T}_{2c}} = \frac{3\hat{a}_c}{\sqrt{2}} = \sqrt{\frac{4\sin^2\varphi_c}{3+\sin^2\varphi_c}}.$$

3.4 Constitutive constants

The present hypoplastic model includes 8 constants which can be determined from simple index and element tests (*e.g.* Bauer 1996, Herle 1997 and Herle and Gudehus 1999): h_s and n are determined by the compression behaviour, φ_c and e_{c0} are related to the critical state in triaxial compression, α and β depend on the peak friction angle, and e_{i0} and e_{d0} are the limit void ratios. Since the current void ratio e is related to the limit void ratios by the pressure dependent functions f_s and f_d, the constitutive constants are not restricted to a certain initial density. As long as the mechanical behaviour does not change substantially by grain abrasion the parameters of the hypoplastic model remain constant for one granular material and the mechanical behaviour of initially dense or initially loose sand can be described using a single set of constitutive constants.

For the numerical investigations in Section 4 the calibration of the constants is based on data from compression tests and triaxial tests for medium quartz sand. The following values are used:

$$\varphi_c = 30^\circ, \quad n = 0.4, \quad h_s = 190\,\text{MPa}, \quad \alpha = 0.11,$$
$$\beta = 1.05, \quad e_{i0} = 1.2, \quad e_{d0} = 0.51, \quad e_{c0} = 0.82.$$

4. Numerical simulation of element tests

In this section the performance of the present hypoplastic model is demonstrated with numerical simulations of several homogeneous element deformations under drained and undrained conditions.

First oedometric compressions and extensions are considered, which are characterised by zero radial deformation, *i.e.* $D_{22} = D_{33} = 0$. Figure 5 shows the loading and unloading responses of the constitutive model for a specimen with an initial void ratio of $e_o = 0.55$ and Figure 6 the results obtained for $e_o = 0.77$. As can be seen by comparing the numerical results with the experimental data, the stress paths and stress-strain curves for loose and dense sand are well reproduced. For virgin loading the stress paths (Figures 5a and 6a) are almost linear. During unloading the axial stress decreases more rapidly than the lateral one. However, the so-called earth pressure at rest, *i.e.* the stress ratio $K_0 = T_{22}/T_{11}$ for the virgin loading, is not a material constant. K_0 is higher for the initially loose sand than for the denser one. The relationship between K_0 and e_o is shown in Figure 7 together with the experimental data which were obtained using the so-called soft-oedometer device (Kolymbas and Bauer 1993).

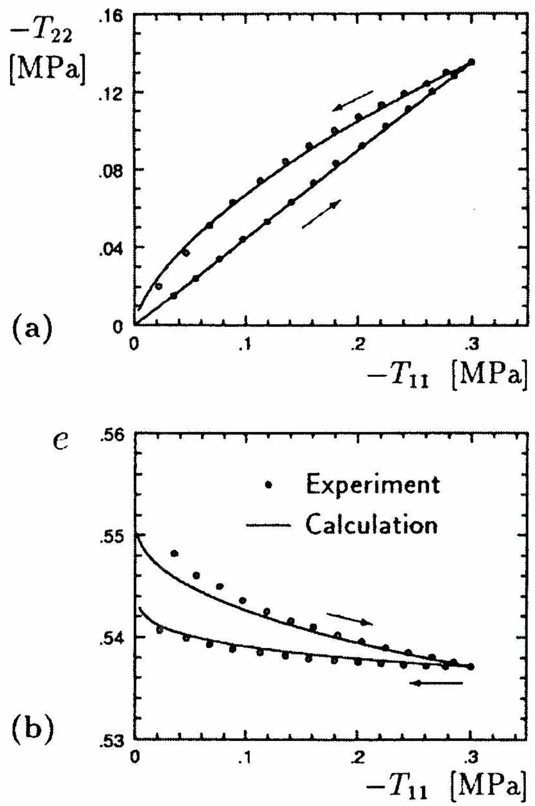


Figure 5. Oedometric compression and extension of dense sand ($e_0 = 0.55$): (a) lateral stress vs. axial stress; (b) void ratio vs. axial stress; experimental data by Bauer (1992)

Figure 8 shows the simulation of triaxial compression tests for various lateral pressures and initial densities starting from an isotropic stress state. For initially dense states and lower mean pressures the peak stress ratio is higher than for loose states and higher mean pressures (Figure 8a). With advanced deformation the stress ratio tends towards a stationary value which is independent of the lateral pressure and the initial void ratio. The volume change is initially contractant and subsequently dilatant (Figure 8b). The dilatancy is more pronounced for a lower pressure and an initially dense state. The void ratio asymptotically reaches a pressure dependent critical value, which is higher for a lower pressure but it is independent of the initial void ratio.

For an initially dense state the dependence of the peak states on the pressure level in the Rendulic-plane is shown in Figure 9. The curve of peak states is not a straight line and lies beyond the critical stress surface. The distance between the curve of peak states and the critical stress surface increases with the mean pressure. However, numerical investigations show that for very high pressures the

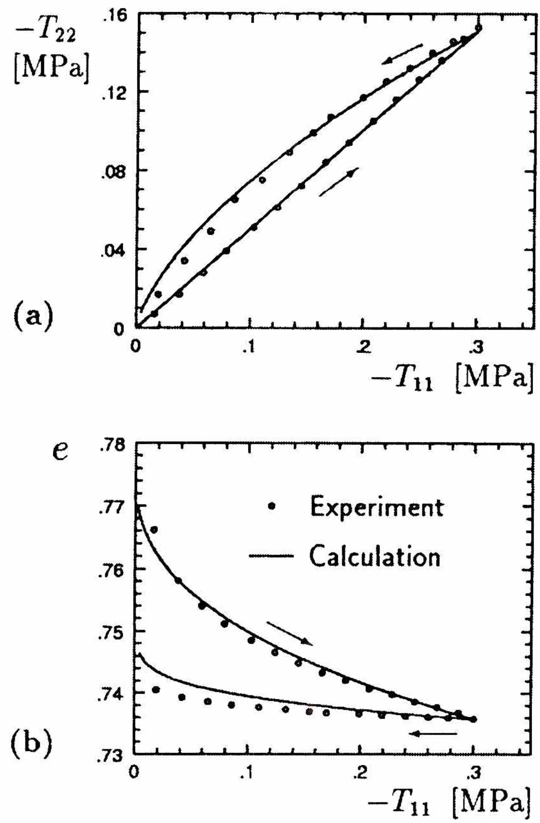


Figure 6. *Oedometric compression and extension on loose sand ($e_0 = 0.77$): (a) radial stress vs. axial stress; (b) void ratio vs. axial stress; experimental data by Bauer (1992)*

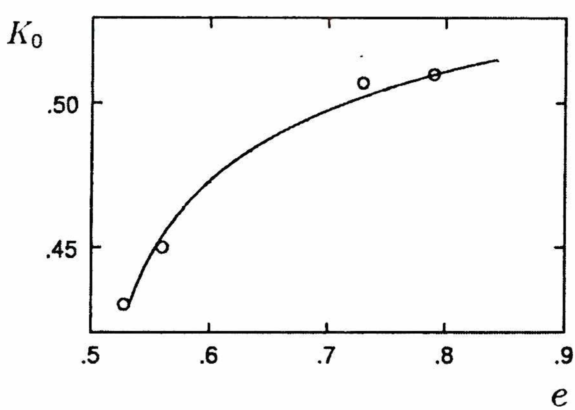


Figure 7. *Dependence of K_0 on the void ratio e; experimental data by Bauer (1992)*

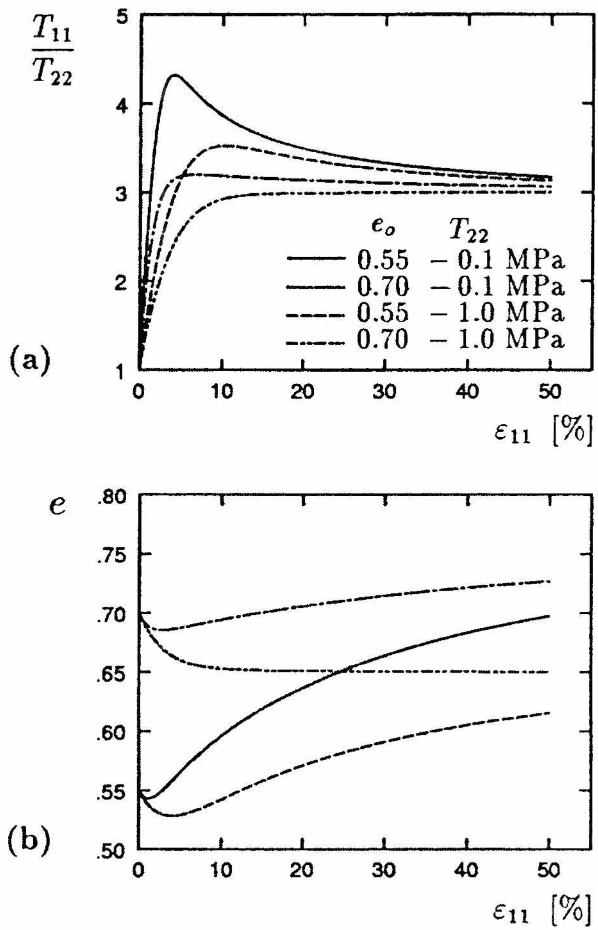


Figure 8. Triaxial compression under drained conditions for various initial densities and lateral pressures: (a) stress ratio vs. axial strain; (b) void ratio vs. axial strain

peak states asymptotically approach the critical stress surface. A comparison of the experimental data of triaxial compression tests with numerical simulations in Figure 10 shows that the behaviour for loose and dense samples can be well approximated with a single set of constitutive constants. The lower limit of the peak friction angle φ_p refers to critical states and initial void ratios $e_o > e_c$. For $e_o \sim e_d$ the maximum value of the peak friction angle is reached, which is higher for a lower consolidation pressure. For higher consolidation pressures the differences between the maximum peak friction angle and the critical value become smaller. This means that the peak friction angle of dense sand under very high pressures is close to the value obtained for loose sand.

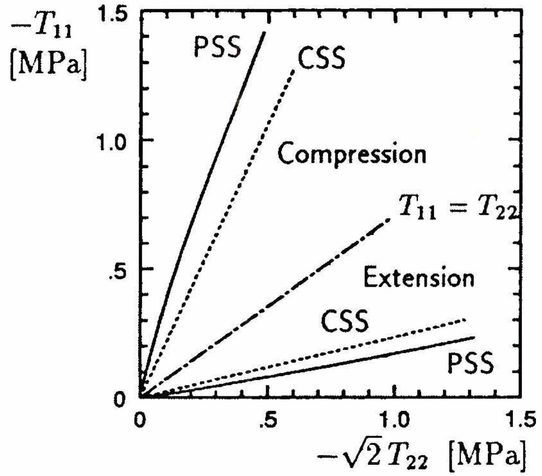


Figure 9. Peak stress states (PSS) and critical stress states (CSS) in the Rendulic-plane

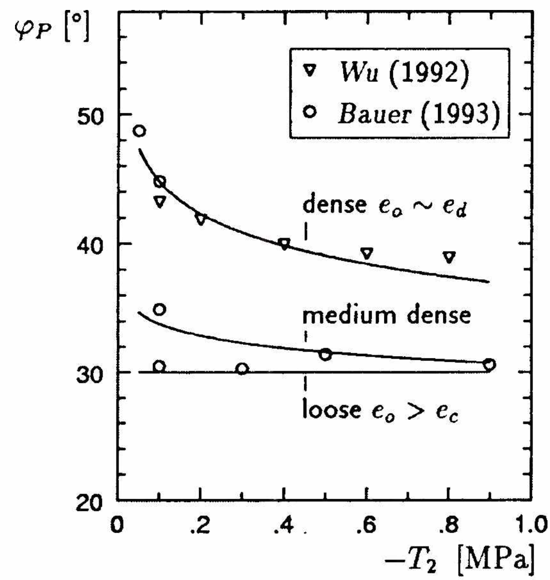
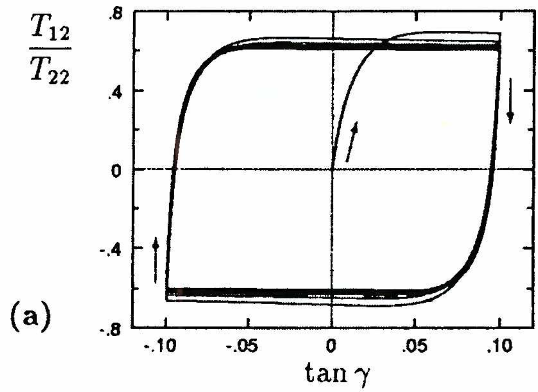


Figure 10. Peak friction angle φ_p versus lateral pressure T_2 for various initial densities: full lines: hypoplastic prediction; o, ▽: experimental data

Cyclic numerical tests for simple shearing with a constant normal stress of $T_{22} = -0.1$ MPa were also performed (Figure 11 and Figure 12). In all the tests an initial void ratio of $e_o = 0.6$ was assumed. With a shear angle of $\tan\gamma = \pm 0.1$ (Figure 11) the volumetric strain is contractant immediately after shearing reversal and becomes dilatant with advanced deformation. With an increasing number of



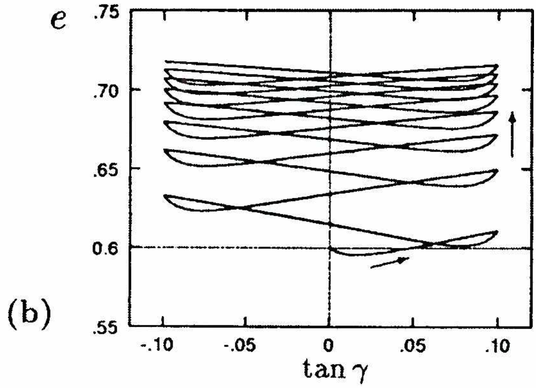


Figure 11. *Simple shearing ($e_o = 0.6$, $T_{22} = -0.1$ MPa) with a large shear amplitude ($\tan\gamma = \pm 0.1$): (a) shear stress vs. shear angle; (b) void ratio vs. shear angle*

cycles the total volumetric strain is dilatant and becomes nearly stationary. The maximum magnitude of the stress ratio decreases in each cycle and gradually approaches a limit value. For a smaller shear angle of $\tan\gamma = \pm 0.01$ (Figure 12) the material becomes denser, and after several cycles a shake-down is reached. A similar behaviour of sand in simple shear experiments was observed by Wood and Budhu (1980).

The results obtained for a water-saturated specimen under cyclic shearing without drainage are shown in Figure 13. Herein an initially isotropic state, *i.e.* $T_{11} = T_{22} = T_{33} = -0.15$ MPa, and a void ratio of $e = 0.65$ was assumed. In accordance with the principle of effective stress, the total stress tensor is assumed to be the sum of the intergranular stress tensor and an isotropic tensor representing the fluid pressure in the voids. With full saturation and incompressibility of the grains and the fluid, *i.e.* $\dot{e} = 0$, the change of the intergranular stress is determined by the constitutive Equation. The change in the fluid pressure can be obtained from the

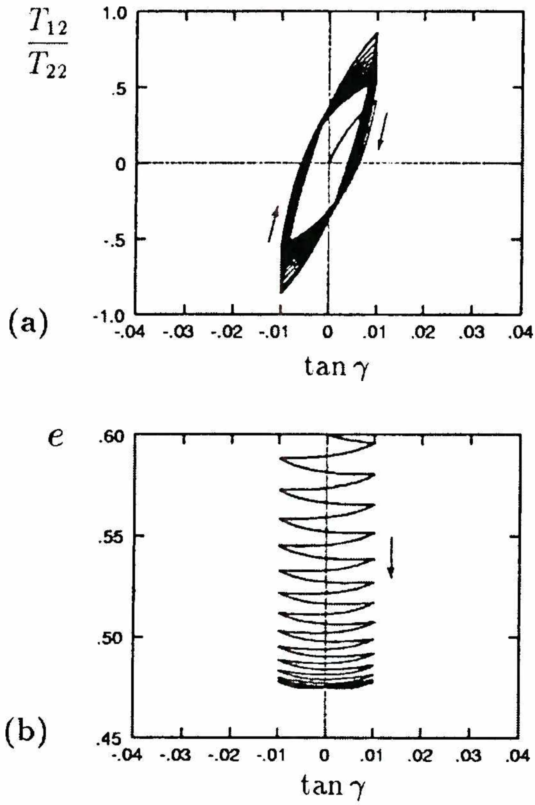


Figure 12. *Simple shearing ($e_o = 0.6$, $T_{22} = -0.1$ MPa) with a small shear amplitude ($\tan\gamma = \pm 0.01$): (a) shear stress vs. shear angle; (b) void ratio vs. shear angle*

static boundary condition. Cyclic shearing leads to the so-called cyclic mobility, which was experimentally observed by several authors, *e.g.* Tatsuoka (1988).

5. Conclusions

A hypoplastic model has been presented to describe the incrementally nonlinear, pressure and density dependent behaviour of granular materials. In order to make the hypoplastic concept more comprehensive a factorised representation of the general form of the constitutive equation has been developed. This factorised form supports the selection of suitable approximation functions describing the limit condition for stationary states and the influence of pressure and density on the incremental stiffness and the peak friction angle. In particular it is shown that with a pressure dependent relative void ratio the model can be applied for a wider range of pressures and densities using only one set of constitutive constants. The numerical simulation of element tests shows that the model appears to be capable of reproducing the salient features of granular material under both drained and undrained conditions.

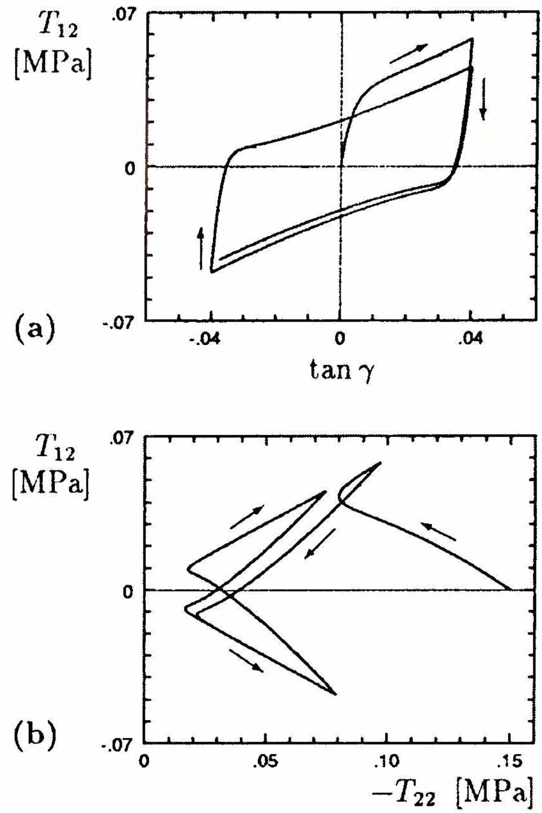


Figure 13. *Simple shearing without drainage for an initial state of $e = 0.65$ and $T_{11} = T_{22} = T_{33} = -0.15$ MPa: (a) shear stress vs. shear angle; (b) effective stress path*

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