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Certain Solutions of Periodic Long Waves with Non-Linear Dispersion

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Abstract

The paper describes some special solutions of the long water waves theory proposed by Wilde. The wave equation is derived with the help of a variational formulation of the problem with the Lagrangian being the difference between the kinetic and potential energies. In order to look for travelling wave solutions the simple transformation $\theta = x - ct$ is made. The solutions have been found in the same way as in the KdV equation. Solutions for different wave amplitudes are presented in the paper. The special cases of solutions are solitary waves. It is prooved that bounded solutions of an equation can represent periodic or solitary waves and both length and velocity of waves increase when the height of waves increases.

Key words: long waves, Langrangian description

1. Introduction

An important part of the theory of water waves deals with long waves, the length of which is much greater than the water depth. Accurate knowledge of the motion of long waves is essential for extensive application in coastal engineering. Most applications are based on the assumption that the wave slope is small. This assumption is not valid for waves of finite steepness, those having neither sinusoidal nor cnoidal form. In shallow water of uniform depth a wave can propagate without changing, when dispersional and non-linear effect attain equilibrium. It is clear that the shape of wave depends on the governing equation. For instance, solitary waves, which can be obtained from either Bussinesq, or Korteweg-deVries equations are different. In this paper some exact solutions of the long wave theory, proposed by Wilde (2001) are presented.

2. Formulation of the Problem

Let us consider a periodic two-dimensional motion of an infinite layer of fluid. In order to describe the motion, let us introduce such Cartesian co-ordinate system in which vertical and horizontal coordinates denote particles at rest. It is assumed that the fluid for the time $t \leq 0$ is at rest and the corresponding particle co-ordinates are named $X, Y(-H \leq Y \leq 0)$, where H denotes depth of water, and free surface elevation is described as $Y = 0$. The motion of the fluid is described by the mapping of the names into the positions occupied by the points at time t. Thus, the mapping can be given as:

$$
\begin{aligned}
x &= x(X, Y, t) = X + u(X, t), \\
y &= y(X, Y, t) = Y + w(X, t)\frac{Y}{H},
\end{aligned} \tag{1}
$$

where u and w are components of the displacement vector.

The incompressibility condition leads to the following relation:

$$
w(X, t) = \frac{-Hu_X(X, t)}{1 + u_X(X, t)}. \tag{2}
$$

Equation (2) describes also the free surface elevation. Wilde (2000) has derived the differential equation of the problem by means of a variational formulation in Lagrangian variables. The equation reads:

$$
\begin{aligned}
&-\frac{\partial^2 u}{\partial t^2} + \frac{1}{3}H^2\frac{\partial^2}{\partial t \partial X}\left(\frac{\frac{\partial^2 u}{\partial t \partial X}}{\left(1 + \frac{\partial u}{\partial X}\right)^4}\right) + \\
&+\frac{2}{3}H^2\frac{\partial}{\partial X}\left(\frac{\left(\frac{\partial^2 u}{\partial t \partial X}\right)^2}{\left(1 + \frac{\partial u}{\partial X}\right)^5}\right) + gH\frac{\frac{\partial^2 u}{\partial X^2}}{\left(1 + \frac{\partial u}{\partial X}\right)^3} = 0.
\end{aligned} \tag{3}
$$

Let as assume that the solution has the form:

$$
u(X, t) = u(X - ct) = u(\theta), \tag{4}
$$

where $\theta = X - ct$ represent the phase, c – velocity of wave.

Substituting description (4) into equation (3) gives:

$$
\begin{aligned}
&-c^2u''(\theta) + \frac{gHu''(\theta)}{(1 + u'(\theta))^3} + \\
&+\frac{1}{3}H^2\left(\frac{10c^2u''(\theta)^3}{(1 + u'(\theta))^6} - \frac{8c^2u''(\theta)u^{(3)}(\theta)}{(1 + u'(\theta))^5} + \frac{3c^2u^{(4)}(\theta)}{(1 + u'(\theta))^4}\right) = 0.
\end{aligned} \tag{5}
$$

Let us introduce the new function $\eta(\theta)$:

$$\eta(\theta) = -H\frac{u'(\theta)}{1+u'(\theta)}. \tag{6}$$

From equation (6) the following relations can be obtained:

$$u'(\theta) = -\frac{\eta(\theta)}{H+\eta(\theta)}, \tag{7}$$

$$u''(\theta) = -\frac{H\eta^{'}(\theta)}{(H+\eta(\theta))^2}, \tag{8}$$

$$u^{'''}(\theta) = -\frac{H^2\eta^{''}(\theta) + H\eta(\theta)\eta^{''}(\theta) - 2\eta^{'}(\theta)^2}{(H+\eta(\theta))^3}, \tag{9}$$

$$u^{(4)}(\theta) = -\frac{H^2(6\eta^{'}(\theta)^3 - 6(H+\eta(\theta))\eta^{'}(\theta)\eta^{''}(\theta) + (H+\eta(\theta))^2\eta^{'''}(\theta))}{(H+\eta(\theta))^4}. \tag{10}$$

Substituting relations (6–10) into equation (5) one obtains:

$$\begin{aligned}
&\frac{-2c^2\eta^{''}(\theta)(H+\eta(\theta))^3 + 3H^2\left(c^2 - gH\right)\eta^{'}(\theta)}{3H(H+\eta(\theta))^2} - \\
&+\frac{3g\eta(\theta)\left(3H^2 + 3H\eta(\theta) + \eta(\theta)^2\right)\eta^{'}(\theta)}{3H(H+\eta(\theta))^2} + \\
&+\frac{c^2\left(H+\eta(\theta)^2\eta^{(3)}\right)(\theta)}{3H} = 0.
\end{aligned} \tag{11}$$

The equation can be integrated over θ:

$$-\frac{c^2(H+\eta(\theta))^2\eta^{''}(\theta)}{3H} - \frac{Hc^2}{H+\eta(\theta)} - \frac{g\eta(\theta)^2}{2H} - g\eta(\theta) + A = 0. \tag{12}$$

From this equation the following relation results:

$$\eta''(\theta) = \frac{3}{2}\left(-\frac{2H^2}{(H+\eta(\theta))^3} + \frac{H(2A+gH)}{c^2\,(H+\eta(\theta))^2} - \frac{g}{c^2}\right). \tag{13}$$

The above equation can be interpreted as the equation of a non-linear oscillator with non-linear potential energy $U(\eta)$ and unit mass:

$$\eta''(\theta) = -\frac{\partial U(\eta)}{\partial \eta}, \tag{14}$$

where

$$U(\eta) = -\int \frac{3}{2}\left(\frac{2H^2}{(H+\eta)^3} + \frac{H(2A+gH)}{c^2\,(H+\eta)^2} - \frac{g}{c^2}\right)d\eta = \\ = \frac{3g\eta}{2c^2} - \frac{3H^2}{2(H+\eta)^2} + \frac{3(gH^2+2AH)}{2c^2(H+\eta)}. \tag{15}$$

In equation (15) the constant of integration is assumed to be zero.

For a conservative system, the sum of kinetic and potential energy is constant and thus we have:

$$\frac{\eta'(\theta)^2}{2} + U(\eta(\theta)) = E \quad \text{or} \quad \frac{\eta'(\theta)^2}{2} = E - U(\eta(\theta)) \tag{16}$$

where $U(\eta)$ is given by (14).

The typical graph of potential energy is shown in Fig. 1.

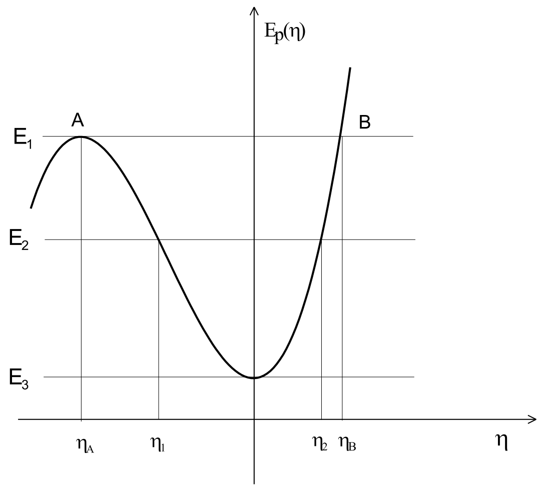

Fig. 1. Potential energy as a function of η

As we are interested in the bounded solution of equation (16), the constant E must satisfy the inequality $E_3 \le E \le E_1$ which is illustrated in Fig. 1. When $E = E_3$, water stays at rest and when $E = E_1$, the wave has a maximum amplitude of $h_{wave} = \eta_B - \eta_A$. In this case, solution of equation (16) represents a soliton.

The difference between the total and potential energies can be read as (Fig. 1):

$$E - U(\eta) = (\eta - \eta_1)(\eta_2 - \eta)\psi(\eta), \quad \eta_1 < \eta_2. \tag{17}$$

Now, let us introduce a new variable $\Omega(\theta)$:

$$\eta(\theta) = \frac{\eta_1 + \eta_2}{2} - \frac{\eta_2 - \eta_1}{2}\cos(\Omega(\theta)) = \alpha - \beta\cos(\Omega(\theta)). \tag{18}$$

It is easy to show that:

$$\eta(0) = \eta_1, \quad \eta(\pi) = \eta_2, \tag{19}$$

$$(\eta - \eta_1)(\eta_2 - \eta) = \beta^2 \sin^2(\Omega(\theta)), \tag{20}$$

$$\eta'(\theta) = \beta^2 \sin^2(\Omega(\theta))\Omega'(\theta)^2 \tag{21}$$

and equation (16) assumes the form:

$$\Omega'(\theta)^2 = 2\psi\,(\alpha - \beta\cos(\Omega(\theta)))\,. \tag{22}$$

In the simplest case, when total energy E is slightly greater than E_3, i.e. $E - E_3 \cong a^2$, $a \ll 1$, the potential energy can be approximated by the quadratic function:

$$U(\eta) = \frac{1}{2}P\eta^2, \tag{23}$$

where $P = \left.\frac{\partial^2 U}{\partial \eta^2}\right|_{\eta=0}$.

In this case, from the physical point of view, the potential energy must have its minimum at $\eta = 0$, i.e.

$$\frac{\partial U(\eta)}{\partial \eta} = 0 \quad \text{and} \tag{24}$$

$$\frac{\partial^2 U(\eta)}{\partial \eta^2} > 0. \tag{25}$$

From equations (15) and (23–25) it results that $A = c^2$, $c^2 < gH$, and the potential energy reads:

$$U(\eta) = \frac{3}{2c^2(H+\eta)^2}\left(gH^3 + c^2H^2 + 2gH^2\eta + 2c^2H\eta + 2gH\eta^2 + g\eta^3\right). \tag{26}$$

When $c^2 = gH$, the waves are non-dispersive and a steady solution of the linearized equation (3) does not exist.

Using relations (23) and (25) after simple manipulations we obtain:

$$P = \frac{3(gH/c^2 - 1)}{H^2}. \tag{27}$$

The solution of the potential function can be found from equation (16). Let us consider now the case of the angle variable described in equation (18). Equation (16) assumes the form

$$\eta'(\theta)^2 = P(\eta(\theta) - a)(a - \eta(\theta)). \tag{28}$$

Using the angle variable Ω:

$$\Omega'(\theta)^2 = P \tag{29}$$

the solution of the equation may be expressed as $\Omega(\theta) = \sqrt{P}\theta$. Now, from equation (18) we have:

$$\eta(\theta) = \eta(X - ct) = a\cos\left(\sqrt{P}(X - ct)\right). \tag{30}$$

Usually $\sqrt{P}$ is denoted as k, i.e.:

$$k = \sqrt{\frac{3(gH/c^2 - 1)}{H^2}} \quad \text{or} \quad c = \sqrt{\frac{gH}{1 + H^2k^2/3}}. \tag{31}$$

Solution (30) is the same as the solution of linearized equation (3).

Let us consider a general case with finite amplitude ($E = U(\eta_2)$) as shown in Fig. 1. The right side of equation (16) has zero at $\eta = \eta_2$ and this equation can be rewritten as:

$$\frac{\eta'(\theta)^2}{2} = U(\eta_2) - U(\eta) = \frac{3g(\eta_2 - \eta)(\eta - \eta_1)(\eta - \eta_0)}{2c^2(H + \eta)^2}, \tag{32}$$

where

$$\eta_1 = \frac{a + \sqrt{\Delta}}{2g(H + \eta_2)^2}, \quad \eta_0 = \frac{a - \sqrt{\Delta}}{2g(H + \eta_2)^2}, \tag{33a}$$

$$a = -H\left[Hc^2 - 2A(H + \eta_2) + g\left(H^2 + 3\eta_2 H + 2\eta_2^2\right)\right], \tag{33b}$$

$$\begin{aligned}\Delta = H^2 \big[&\left(Hc^2 - 2A(H + \eta_2) + g(H + \eta_2)(H + 2\eta_2)^2\right) - \\ &+4g(H + \eta_2)^2\left((2H + \eta_2)c^2 - 2A(H + \eta_2) + g\eta_2(H + \eta_2)\right)\big]\end{aligned}. \tag{33c}$$

We choose the value η_2 in such a way, that the following inequalities are satisfied:

$$\eta_0 \le \eta_1 \le \eta(\theta) \le \eta_2. \tag{34}$$

In the term of the variable $\Omega(\theta)$ equation (32) can be rewritten as:

$$\Omega'(\theta)^2 = \frac{3g\,(\alpha - \beta\cos(\Omega(t)) - \eta_0)}{c^2\,(H + \alpha - \beta\cos(\Omega(t))^2}, \tag{35}$$

where $\alpha = (\eta_1 + \eta_2)/2$, $\beta = (\eta_2 - \eta_1)/2$.

From relation (35) follows:

$$\frac{c\,(H + \alpha - \beta\cos(\Omega))\,d\Omega}{\sqrt{3g\,(\alpha - \beta\cos(\Omega) - \eta_0)}} = d\theta, \tag{36}$$

$$\text{or} \int \frac{c\,(H + \alpha - \beta\cos(\Omega))\,d\Omega}{\sqrt{3g\,(\alpha - \beta\cos(\Omega) - \eta_0)}} = \theta - \theta_0. \tag{37}$$

It is seen from (35–36) that the differential $\Omega'(\theta)$ is always positive, thus the relation (38) is always invertible. Unfortunely, the inverse function cannot be expressed in terms of known functions and thus examples of the solutions have been calculated numerically. Finally, when $\Omega(\theta)$ is specified, $\eta(\theta)$ can be calculated by (18).

For a fixed depth, properties of the solution depend on velocity c. The constant of the integration A results from the consideration, that the mean value of surface elevation is zero. As for any constant A (for given H, η_2 and c) the surface elevation and its mean value are calculated, the procedure of searching for the proper value of A is an optimalization problem.

3. Results of Computation

A shape of waves with different wave heights is shown in Fig. 2. Period of waves was 4 s, and water depth 1 m. When wave height increases, the wavelength also increases, the wave crest is narrower and wave trough wider.

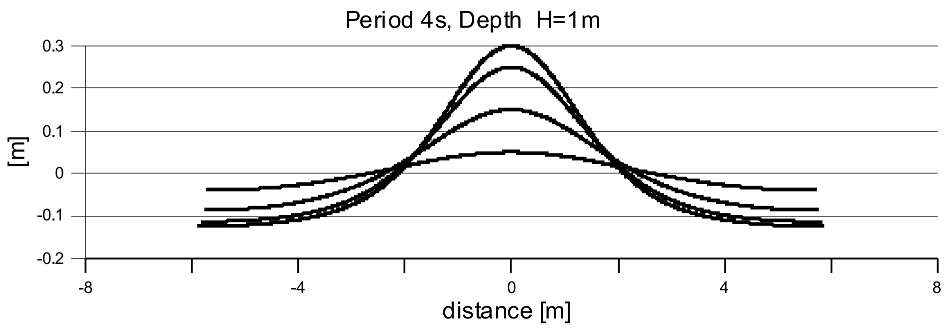

Fig. 2. Wave shapes. Wave period 4 s. Depth of water 1 m

A similar case of waves with period 8 s is depicted in Fig. 3. For greater heights the waves are similar in shape to a solitary wave.

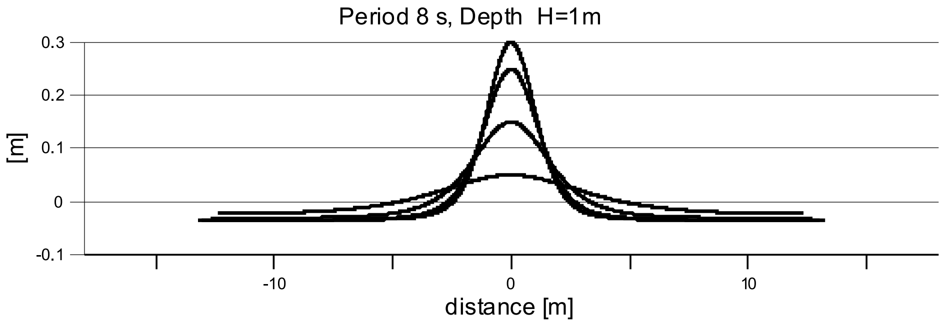

Fig. 3. Wave shapes for different wave heighs. Wave period 8 s. Depth of water 1 m

The wave height – wave velocity relation for waves with period 4 s is shown in Fig. 4. For small height, the velocity is less than in the linear case (linear velocity is ≈ 3 m/s).

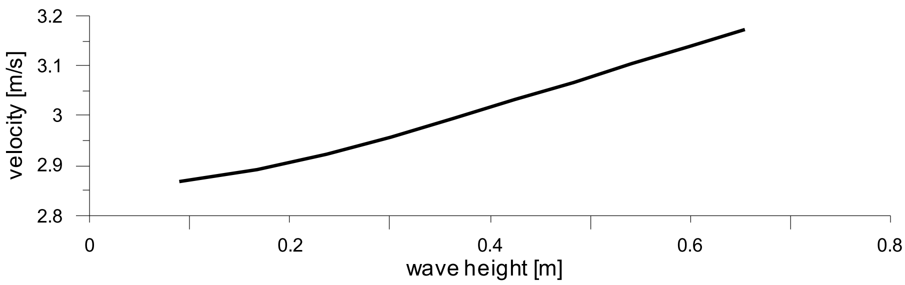

Fig. 4. Velocity for different wave heights. Wave period 4 s. Depth of water 1 m

The wave velocity as a function of wave height for a wave of period 8 s is shown in Fig. 5. In this case the dependency is almost linear.

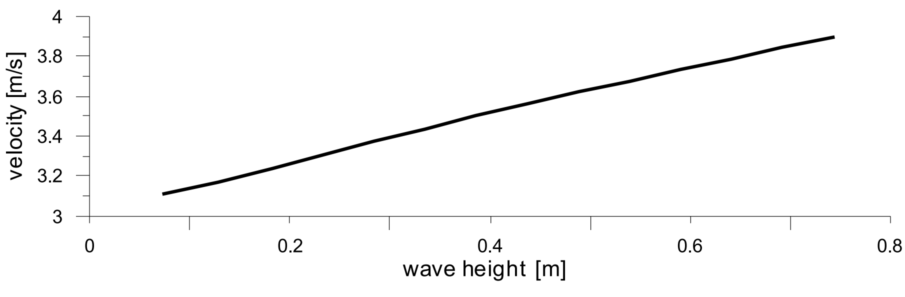

Fig. 5. Velocity for different wave heights. Wave period 4 s. Depth of water 1 m

The Stokes drift for a wave with period 4 s is shown in Fig. 6.

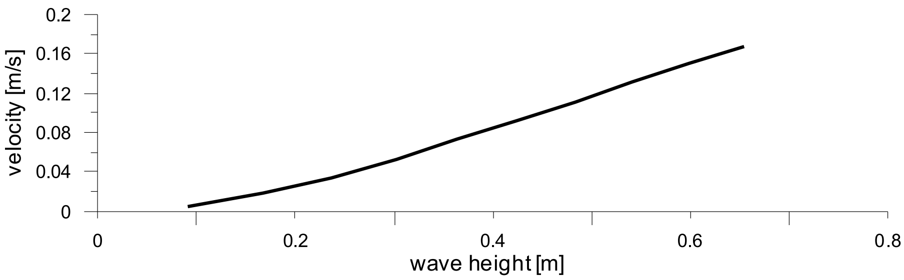

Fig. 6. Velocity of Stokes drift for different wave heights. Wave period 4 s. Depth of water 1 m

4. Conclusions

1. There exists a periodic solution to equation (3).
2. Bounded solutions of equation (3) can represent periodic or solitary waves.
3. Both length and velocity of waves increase when height of waves increases.
4. For solitary waves $\eta_1 = \eta_0 = 0$ and, after (32), it is possible to calculate c and A. In this case $c = \sqrt{g(H + \eta_2)}$. For periodic solutions, velocity of waves is always less than this value.

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