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On Variational Formulations in Water Wave Mechanics

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Abstract

In this paper variational formulations for surface gravitational waves in inviscid incompressible fluids are investigated. The formulations are introduced with the help of the principle of virtual work. The starting point are equations of motion multiplied by a field of virtual displacements and integrated over the region occupied by the fluid. In derivations of the virtual work equation careful attention is paid to mutual relations between Eulerian and Lagrangian descriptions. The integration of the equation with respect to time leads to the expression for the Lagrangian function and then the Hamilton's principle. The case of a potential flow and spatial description provides a generalisation of the Lagrangian given by Luke (1967).

Key words: water wave, variational formulation, spatial and material description

1. Introduction

Variational formulations of equations of fluid dynamics are especially important in developing approximate discrete descriptions of an original task. The formulations enable us to construct numerical models preserving some important features of the original system. With the variational formulations equations of the discrete models can be obtained by direct variation of a set of discrete parameters in the action integral associated with the problem considered.

In the literature on the subject, variational formulations for the dynamics of perfect fluids have been presented in several papers. Herivel (1955) gave Hamilton's principle in two variational formulations corresponding to the Lagrangian and Eulerian variables, respectively. The same problem of formulation of Hamilton's principle for perfect fluids is discussed in Serrin's monograph (1959). Serrin has found that the formulation due to Herivel is satisfactory only for the Lagrangian variables. As concerns the Eulerian description, the Herivel solution should be supplemented with an additional constraint of conservation of the fluid particles identities. The correct version of the variational principle obtained in this way, called the Herivel-Lin principle, takes into account the conservation of

mass, energy and the identity of the particles as side additional constraints in the Lagrangian density function.

As compared to the Hamilton's principle, which is based on the difference of kinetic and potential energies, variational formulations of fluid dynamics are based on the Calculus of Variations and a priori known equations of the fluid motion. In the latter case the fundamental task is associated with a proper description of the Lagrangian density function in the relevant functional, which has a form suited to the problem in question. Bateman (1929) gave an example of such a notable contribution in a variational formulation for motion of an inviscid compressible fluid. He pointed out that in some cases the Lagrangian density function is equal to the fluid pressure. Luke (1967) published a formal variational principle for the classical water wave problem in the Eulerian description. He extended Bateman's variational formulation in such a way, that the new procedure also provides boundary conditions appropriate to the free surface of the fluid. In the Luke's formulation, not only the Lagrangian density function, but also the location of the free surface undergoes variations.

In the last decades an increasing number of papers where Hamiltonian formulations of water wave problems are considered has appeared. Salmon (1988) reviewed applications of the methods of Hamiltonian mechanics to problems in fluid dynamics. A detailed discussion of the Hamiltonian systems, and, more general, by Poisson structures in fluid dynamics may be found in a van Groesen and de Jager monograph (1995) where a bibliography of the subject may also be found.

In this paper we confine our attention to the Lagrangian formulation which is more common and seems to be simpler in applications to discrete descriptions of problems in fluid mechanics than the Hamiltonian approach. The main goal of the paper is to derive the Hamilton principle for perfect fluid, as well as the formulation of the variation of the action integral for potential flow in spatial description. The latter result which takes into account boundary and initial conditions may be considered as a certain generalisation of the formulation given by Luke (1967). With respect to the above, special attention is paid to the boundary and initial conditions of the problems considered. The variational formulations are introduced with the help of a virtual work equation which, integrated with respect to time, gives the action integral of the problem considered. The action integral is supplemented with terms corresponding to the initial and end field of momentum and the work of surface pressures on virtual displacements. The free surface is a surface with known pressures and unknown position. It is a moving surface. Another case is a known moving surface with unknown pressures. A very special case corresponds to the potential flow in spatial description in which the velocity field is a gradient of the velocity potential function.

2. Fundamental Relations

In the presented problem the space is Euclidean and three-dimensional. Thus it is possible to introduce in an actual configuration, a Cartesian co-ordinate system z^i. In the reference configuration, the Cartesian co-ordinates are denoted by Z^α and correspond to names of particles. Einstein's summation convention and the notations of Cartesian tensors are used. In all cases it is assumed that a common Cartesian co-ordinate system is introduced. In the majority of cases it is assumed that the fluid for the time $t \leq 0$ is at rest and the corresponding particle co-ordinates correspond to names. A more general case corresponds to the situation where at the initial time there is a wave on the free surface and a velocity field in the interior of the fluid.

The motion of the fluid is described by the mapping of the names into the positions occupied by the points at time t. Thus the mapping

$$z^i = z^i\left(Z^\alpha, t\right) = \delta^i_\lambda Z^\lambda + w^i\left(Z^\lambda, t\right), Z^\alpha = Z^\alpha\left(z^i, t\right), \tag{1}$$

where δ^i_λ is the Kronecker's delta, $w^i\left(Z^\lambda,t\right)$ are the components of the displacement vector, and for fixed Z^λ the co-ordinates $z^i\left(Z^\lambda,t\right)$ trace the path of the corresponding material point in space. The second relation means that the inverse exists and we assume that it is unique.

The velocity field is defined as the partial time derivative of the equation of motion (1). In the material description it follows

$$v^i = z^i_{,t}\left(Z^\rho, t\right) = w^i_{,t}\left(Z^\rho, t\right), \tag{2}$$

where the subscript $,t$ means the partial derivative with respect to t. This notation will be used in the text in general. For example, the symbol $,i$ denotes the partial derivative with respect to z^i and $,\alpha$ denotes the partial derivative with respect to Z^α. The velocity field in spatial description is given by a simple substitution of the inverse mapping function

$$v^i\left[Z^\alpha\left(z^r, t\right), t\right] = v^i\left(z^r, t\right), \tag{3}$$

where the functions given by the expressions (2) and (3) are different in shape, but give the same values at corresponding points. Thus the same letter denotes them as they represent the same physical quantities in different descriptions.

The acceleration field a^i is defined in the material description as the partial time derivative of velocity of the fluid particle. It follows, that

$$a^i\left(Z^\alpha, t\right) = v^i_{,t}\left(Z^\alpha, t\right). \tag{4}$$

To calculate the acceleration in the spatial description, the chain rule of differentiation has to be used. It leads to the material time derivative in the spatial description

$$a^i\left(z^r,t\right)=\dot{v}^i\left(z^r,t\right)=v^i_{,t}\left(z^r,t\right)+v^i_{,s}\left(z^r,t\right)v^s\left(z^r,t\right). \tag{5}$$

The displacement gradient is defined in the material description by the expression

$$z^i_\alpha\left(Z^\lambda,t\right)=z^i_{,\alpha}\left(Z^\lambda,t\right). \tag{6}$$

It is assumed that the inverse of the displacement gradient Z^α_j exists, is unique and is defined by the following relation

$$z^i_\alpha Z^\alpha_j=\delta^i_j. \tag{7}$$

We assume that the fluid is incompressible. Let us consider a volume element in the reference configuration $dV_R=dZ^1dZ^2dZ^3$ that in the actual configuration goes over to the element $dV=dz^1dz^2dz^3$ by the mapping (1) of names into the positions for given times. From standard analysis it follows that in the mapping the volume elements are related by the expression

$$dV=JdV_R, \tag{8}$$

where J is the Jacobian of the transformation expressed by the determinant of the displacement gradient

$$J\left(Z^\lambda,t\right)=\det\left[z^i_\alpha\left(Z^\lambda,t\right)\right]. \tag{9}$$

The condition that the fluid is incompressible corresponds to the statement that the Jacobian is equal to one for all material points of the fluid and all times.

It follows from the definition of the displacement gradient that the following relation maps the line element from the reference into the actual configuration

$$dz^i=z^i_\alpha dZ^\alpha. \tag{10}$$

The relation between the surface element in the reference configuration dS_R and the corresponding quantity in the actual configuration dS is (Spencer 1980)

$$n_i dS=JZ^\alpha_i N_\alpha dS_R, \tag{11}$$

where n_i, N_α are the components of normal unit vectors in the actual and reference configurations, respectively.

The equation (11) is used to relate surface integrals in the two configurations. The Jacobian is a scalar function and by substitution of the mapping it may be written as a function of the independent variables z^i and time t. The material time derivative of the Jacobian in the spatial and material description reads

$$\dot{J}\left(z^rt\right)=Jv^i_{,i}\left(z^rt\right),\quad \dot{J}\left(Z^\lambda,t\right)=Jv^i_{,\alpha}\left(Z^\lambda,t\right)Z^\alpha_i, \tag{12}$$

where the dot over a symbol denotes the material time derivative.

In the case of spatial description the standard momentum equation of the problem is

$$\rho a_r\left(z^k, t\right) + \left[\rho h\left(z^k, t\right) + p\left(z^k, t\right)\right]_{,r} = 0, \tag{13}$$

where the space co-ordinates and the time are the independent variables and the acceleration and pressure fields are the unknown functions. The potential h of the mass force due to the gravitational field is in the spatial description given by the relation

$$h = -g_i z^i. \tag{14}$$

When the co-ordinates are chosen in such a way that z^3 acts vertically upwards the coefficients g_i are: $g_1 = 0, g_2 = 0, g_3 = -g$, where g is the gravitational acceleration. In the reference configuration the set of differential momentum equations assumes the form

$$\rho a_r\left(Z^\lambda, t\right) + \left[\rho h\left(Z^\lambda, t\right) + p\left(Z^\lambda, t\right)\right]_{,\alpha} Z_r^\alpha = 0. \tag{15}$$

3. The Boundary and Initial Conditions

Let us consider the physical meaning of boundary conditions. In general, the boundary is a moving surface. For example, the boundary may be in the form of an elastic plate that deflects in time. This boundary is a singular surface in this respect that material points on the plate and in the fluid that belong to an infinitesimal neighbourhood at a time t are in general separated by a finite distance after a finite time Δt passed. Another case is a free boundary for which known pressures have to be considered.

For an observer of moving boundaries, at first glance, the motion of individual particles is not important. His natural approach is to fix in the space a co-ordinate system and to describe the family of surfaces with time t as a parameter that gives the position of the surface for the considered times. Such a family of surfaces may be described by the equation

$$f\left(z^r, t\right) = 0, \tag{16}$$

where it is assumed that the function f is continuous and has continuous derivatives.

Of course, the surface may be composed of several parts that are continuous on joints, but with no continuous derivatives. An example may be seen in Fig. 1, where the free surface, moving surface and rigid surfaces forming the fluid boundaries

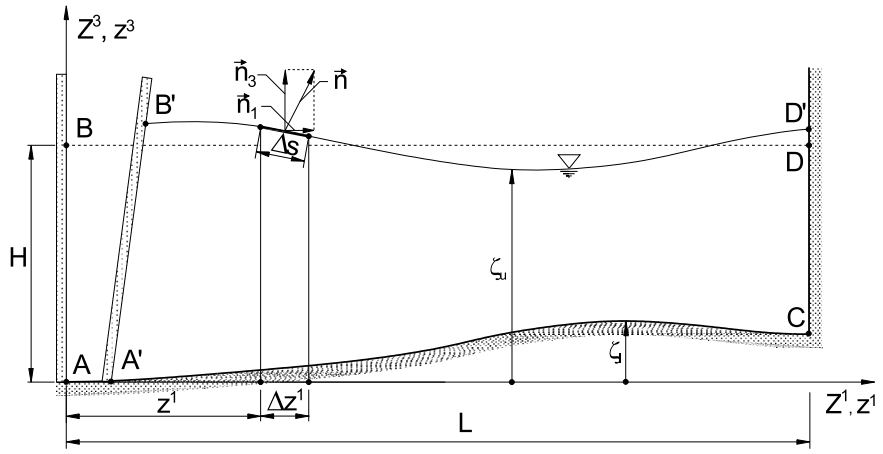

Fig. 1. A bounded fluid domain

are depicted. For instance, the following relations may describe the position of a bottom

$$f\left(z^r,t\right)=z^3-d_1\left(z^1,z^2\right)=0,\quad f\left(z^r,t\right)=z^3-d_2\left(z^1,z^2,t\right)=0, \tag{17}$$

where the function d_1 is known and does not depend upon time (a stationary bottom) and d_2 depends on time in a prescribed way (a moving boundary). The natural description for a free surface is

$$f\left(z^r,t\right)=z^3-\zeta\left(z^1,z^2,t\right)=0, \tag{18}$$

where ζ is an unknown function that has to be calculated. On the free surface the external pressures have to be prescribed.

The standard methods of differential geometry of surfaces lead to the following expression for the coordinates of the unit normal vector **n**

$$n_k\sqrt{\delta^{ij}f_{,i}f_{,j}}=f_{,k}. \tag{19}$$

The speed **u** of the surface is the velocity of points on the surface. The material time derivative of the relation (16) yields

$$\dot{f}^{(s)}\left(z^r,t\right)=f_{,t}+u^kf_{,k}=0, \tag{20}$$

where the superscript (s) indicates: "the points are on the surface". A point of the fluid situated on the boundary has a different velocity **v** and thus

$$\dot{f}^{(f)}\left(z^r,t\right)=f_{,t}+v^kf_{,k}=0, \tag{21}$$

where the superscript (f) indicates: "the fluid points that are on the fluid surface". The three equations (19)–(21) have to be satisfied. From Eq. (20) $f_{,t}$ may be

calculated and substituted into the relation (21). Finally, with the help of Eq. (19) $f_{,k}$ may be expressed in terms of n_k. The final result follows the basic criterion due to Lagrange (Truesdell and Toupin 1960, sec. 74): a necessary and sufficient condition for a surface $f(z^r, t) = 0$ to be the material surface is

$$\dot{f} = \sqrt{\delta^{ij} f_{,i} f_{,j}}(v_n - u_n) = 0. \tag{22}$$

The criterion states that the normal components of velocities of the moving surface and the fluid should be equal to each other. For example, in the case of a free surface defined by the relation (18) the absolute value of the gradient of $f(z^r, t)$ is different from zero and thus the condition of kinematics assumes the following form

$$-v^1\zeta_{,1} - v^2\zeta_{,2} + v^3 - \zeta_{,t} = 0. \tag{23}$$

A simple example of initial and boundary conditions for a plane problem is shown in Fig. 1. At the initial time the fluid is at rest. The free surface is a horizontal plane and the velocities are zero. The left boundary corresponds to a moving, rigid plate that at the initial time is vertical, the velocity and acceleration are zeros. The position of the plate as a function of time is known. The free surface assumes a position variable in time, that may be described by the relation (18). The boundary on the right corresponds to the condition that the horizontal components of displacements are zero for all times. The bottom is a rigid surface that does not change in time and is described by the first relation of (17). The corner points of the fluid at rest correspond to the points of intersection of the above-mentioned bounding surfaces. (In the case illustrated in Fig. 1 and corresponding to a plane problem, all the functions do not depend upon z^2).

In a spatial description the potential energy of the gravitational field may be calculated as the first moment with respect to the horizontal plane (z^1, z^2) by means of the formula

$$E_p = \rho g \iint_{S_p} \frac{1}{2}\left[\zeta_u^2\left(z^1, z^2, t\right) - \zeta_l^2\left(z^1, z^2\right)\right] dz^1 dz^2, \tag{24}$$

where the index u denotes the upper surface (the free surface and the moving rigid plate in the position indicated in Fig. 1) and the index l – the lower surface corresponding to the wetted part of the bottom. It may also include the moving surface in another position. It should be noted that for an incompressible fluid the potential energy in the gravitational field may be calculated if the positions of the bounding surfaces are known.

The standard initial condition corresponds to the case in which the fluid is at rest. In a general case in the material description the following relation may describe the mapping function for very small but positive times

$$z^i(Z^\lambda, t) = \delta^i_\alpha Z^\alpha + v^i(Z^\lambda, 0^+)t + a^i(Z^\lambda, 0^+)t^2/2 + \dots . \tag{25}$$

When for $t \leq 0$ the fluid is at rest, a jump in accelerations is created if a sudden increase in pressures is applied and an acceleration field in the fluid is suddenly created. However it must be mentioned that for an incompressible fluid the velocity of dilatational waves is infinite. This is impossible from the physical point of view, and the real behaviour is more complicated than the solution for an incompressible fluid predicts. In the latter case of compressible fluid the propagation velocity of the dilatational waves is finite and this fact must be taken into account. When waves are generated in a flume it is advisable to assume that at the initial time the velocity and acceleration of the generator-fluid system are zeros.

4. Virtual Work

The starting point of our discussion on expressions for virtual work are the differential equations of motion as given by the expressions (13) and (15) in the spatial and material descriptions, respectively. Now let us define the virtual displacements. At time t the position of the particles is given by the mapping (1). Let us consider any other geometrically possible motion defined by the position vector with co-ordinates

$$\hat{z}^i\left(Z^\lambda, t\right) = z^i\left(Z^\lambda, t\right) + \varepsilon \widetilde{w}^i\left(Z^\lambda, t\right), \tag{26}$$

where $\widetilde{\mathbf{w}}$ is an arbitrary vector field and ε is a real number so small that its squares and higher powers can be neglected. We call $\varepsilon\widetilde{w}^i$ the virtual displacement, or variation, of the position vector and write

$$\delta z^i\left(Z^\lambda, t\right) = \hat{z}^i\left(Z^\lambda, t\right) - z^i\left(Z^\lambda, t\right) = \varepsilon \widetilde{w}^i\left(Z^\lambda, t\right) = \delta w^i\left(Z^\lambda, t\right). \tag{27}$$

This notation corresponds to standard definitions of virtual displacement in applied mechanics and in calculus of variations in mathematics (Weinstock 1952). It should be noted that the infinitesimal virtual displacement field is a vector field (like the infinitesimal differentials dz^i, dZ^α that are components of vectors). The virtual displacements are infinitesimal vectors added to the finite displacements of the fluid particles and thus in a common Cartesian co-ordinate system have equal components in both configurations.

The easiest way to calculate the variation is to expand an expression in power series in ε and to retain the linear term only. The expansion and the variation of the displacement gradient are

$$\hat{z}^i_\alpha\left(Z^\lambda, t\right) = z^i_\alpha\left(Z^\lambda, t\right) + \varepsilon \widetilde{w}^i_{,\alpha}\left(Z^\lambda, t\right), \quad \delta z^i_\alpha\left(Z^\lambda, t\right) = \varepsilon \widetilde{w}^i_\alpha\left(Z^\lambda, t\right). \tag{28}$$

We may change the variables from those corresponding to names to the space coordinates and write

$$\delta z_\alpha^i \left(z^r, t\right) = \varepsilon \widetilde{w}_\alpha^i \left(z^r, t\right). \tag{29}$$

The variation of the Jacobian in the material description is

$$\hat{J}\left(z_\alpha^i + \varepsilon \widetilde{w}_{.\alpha}^i\right) = J|_{\varepsilon=0} + \varepsilon \left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = J + \varepsilon J Z_i^\alpha \widetilde{w}_\alpha^i, \rightarrow \delta J = J Z_i^\alpha \delta z_\alpha^i. \tag{30}$$

The spatial description results from the material description by substitution of the mapping functions. Here the variation of the Jacobian is a scalar and thus we use the chain rule of differentiation to obtain the expression for the variation of the Jacobian in a spatial description. It follows that

$$\delta J\left(z^r, t\right) = J \varepsilon \widetilde{w}_{,i}^i \left(z^r, t\right) = J \delta z_{,i}^i \left(z^r, t\right). \tag{31}$$

The formula is similar to that described by the first expression of Eq. (12) for the material time derivative of the Jacobian. In both cases an infinitesimal neighbourhood is considered. In general we will use the notation of a variation of a quantity χ by the symbol $\delta\chi$ and it is understood that it corresponds to the addition of a quantity $\varepsilon\widetilde{\chi}$ to χ where ε is so small that terms with second and higher powers of ε are neglected and $\widetilde{\chi}$ is arbitrary.

Let us take the scalar product of the equations of motion (15) with the virtual displacement $\delta z^r\left(Z^\lambda, t\right)$ and then integrate throughout the fluid body V_R. We obtain the following expression for the corresponding virtual work

$$\iiint_{V_R} \left[\rho a_r + (\rho h + p)_{,\alpha} Z_r^\alpha\right] \delta z^r J \, dV_R = 0. \tag{32}$$

The expression in the square brackets in the volume integral is zero for any material volume of the fluid and thus the integral is zero for any virtual displacements and any material subregion. We consider an incompressible fluid and thus the Jacobian is one for any material point and any time. Thus in the expression (32) we may omit the Jacobian J. It is retained in view of the general considerations in the material and spatial descriptions, but it must be kept in mind that it is equal to one for the incompressible fluid. With the help of the Euler, Piola and Jacobi identity (Truesdell and Toupin 1960, sec. 18)

$$\left[J Z_r^\alpha \left(Z^\lambda, t\right)\right]_{,\alpha} = 0, \tag{33}$$

we arrive at the following identity

$$\left[(\rho h + p) J Z_{.r}^\alpha \delta z^r\right]_{,\alpha} = (\rho h + p)_{,\alpha} J Z_{.r}^\alpha \delta z^r + (\rho h + p) J Z_{.r}^\alpha \delta z_{.\alpha}^r, \tag{34}$$

where all the functions are in the material description. It follows after substitution into the virtual work equation (32) that

$$\iiint_{V_R} \left[\rho a_r \delta z^r - J^{-1}(\rho h + p)\,\delta J\right] J\,dV_R + \\ + \iiint_{V_R} \left[(\rho h + p)\, J Z^{\alpha}_{,r} \delta z^r\right]_{,\alpha} dV_R = 0. \tag{35}$$

Now we apply the Green's integral transformation theorem to the second integral which allows us to obtain

$$\iiint_{V_R} \left[\rho a_r \delta z^r - \Lambda\left(Z^{\lambda}, t\right)\delta J\right] J\,dV_R + \iint_{S_R} (\rho h + p)\, J Z^{\alpha}_{,r} N_{\alpha} \delta z^r\, dS_R = 0, \tag{36}$$

where the scalar function $\Lambda = J^{-1}(\rho h + p)$ represents the coefficient that stands before the variation of the Jacobian.

A remark is needed. According to the standard calculus of variations, in problems with restrictions imposed through finite or differential equations, the multiplier before the restriction is called the Lagrange's multiplier. In our case of incompressible fluid the scalar function $\Lambda(Z^{\lambda}, t)$ in Eq. (36) that stands before the variation of the Jacobian is simply the Lagrange's multiplier of the problem considered. The equation obtained is the virtual work expression written in the material description. The first term in the volume integral corresponds to the work done by the body forces on virtual displacements. If the virtual displacement is incompressible, then its variation is zero and the second term is zero. The surface integral is written in the reference configuration. There are two terms in the round bracket under the surface integral. The first term corresponds to the potential energy density and the corresponding value of the integral represents the change of potential energy due to the changes in positions of the bounding surface in a virtual motion. The fluid is incompressible and thus the change of the potential energy in the gravitational potential field is only due to the virtual, infinitesimal change of the region of integration that is represented by the surface integral. The second term corresponds to the surface pressure and the integral represents its work on virtual displacement components normal to the surface. For a free surface the position of the boundary is not known, and thus there may be a virtual displacement function. When the boundary is fixed the velocity vector has to be tangent and thus the normal component of velocity has to be zero, and this necessary condition leads to the conclusion that the normal component of virtual displacement has also to be zero. A more detailed discussion of initial and boundary conditions will follow in the next chapter.

Let us change the surface integral in the material description to a surface integral in a spatial description. Upon substitution of the relation (11) it follows

$$\iint_{S_R} (\rho h + p)\, J Z^{\alpha}_{,r} N_{\alpha} \delta z^r dS_R = \iint_{V} (\rho h + p)\, n_r \delta z^r dS, \tag{37}$$

where the right side is in the spatial description and the symbol *equal* means that when calculated, the values will be the same on both sides. The virtual work equation (36) in the spatial description assumes the following form

$$\iiint_{V} \left[\rho a_r \delta z^r - \Lambda\left(z^r, t\right) \delta J\right] dV + \iint_{V} (\rho h + p)\, n_r \delta z^r dS = 0, \tag{38}$$

where Λ is the same scalar function as in (36), but now it is a function of positions in space z^r and time t. It must be remembered that, in the spatial description, the integrals have to be calculated in the actual, changing in time regions.

In the virtual work formulation with the condition of incompressibility introduced by Lagrange's multiplier procedure the unknowns are the mapping function and Lagrange's multiplier. The unknown pressure does not enter directly the description (the internal pressures do not contribute to internal energy and the pressures on the free surface are known).

Let us discuss the solution in the spatial formulation. The application of Green's theorem leads to the following identity

$$\begin{aligned} -\iiint_{V} \Lambda^{\bullet}\left(z^r, t\right) \delta w^i_{,i} dV = \\ = \iiint_{V} \Lambda^{\bullet}\left(z^r, t\right)_{,i} \delta w^i dV - \iint_{S} \Lambda^{\bullet}\left(z^r, t\right) n_i \delta w^i dS, \end{aligned} \tag{39}$$

where $\Lambda^{\bullet}(z^r, t) = J\Lambda = p(z^r, t) + \rho h(z^r, t)$.

Substitution into the virtual work equation (38) yields

$$\iiint_{V} \left[\rho a_i\left(z^r, t\right) + \Lambda^{\bullet}\left(z^r, t\right)_{,i}\right] \delta w^i\left(z^r, t\right) dV = 0. \tag{40}$$

From the basic lemma of variational calculus it follows that the acceleration is a potential field.

If the vector field has a potential Φ then in the spatial description

$$v_i\left(z^r, t\right) = \Phi_{,i}\left(z^r, t\right). \tag{41}$$

When the expression for the acceleration in the spatial description is used and the basic lemma of variational calculus is applied to the virtual work equation (40) the following formula for Lagrange's multiplier results

$$J\Lambda\left(z^r,t\right)=p\left(z^r,t\right)+\rho h\left(z^r,t\right)=-\rho\Phi_{,t}-\frac{1}{2}\rho\delta^{rs}\Phi_{,r}\Phi_{,s}+C\left(t\right), \tag{42}$$

where C is an arbitrary function of time. The well-known expression for pressures in potential flows follows immediately.

If the condition of incompressibility is assumed in the form that $J=1$ at the initial time and the time derivative of the Jacobian is zero for all points and all times, the Laplace equation for the potential flow of an incompressible fluid results.

The analysis in the spatial description may be modified to study the virtual work equation in a material description. In the material description the velocity field for a potential flow is

$$v_r\left(Z^\lambda,t\right)=\Phi_{,\alpha}\left(Z^\lambda,t\right)Z^\alpha_{,r}\left(Z^\lambda,t\right),\quad v_r z^r_{,\beta}\left(Z^\lambda,t\right)=\Phi_{,\beta}\left(Z^\lambda,t\right), \tag{43}$$

where the second relation is obtained by multiplication of the first by the displacements gradient $z^r_{,\beta}$. The differentiation of the second relation of (43) with respect to time and multiplication by $Z^\beta_{,k}$ leads to the expression for the acceleration as a function of the potential in the material description. Substitution into the virtual work equation (36) leads to the following result

$$\iiint\limits_{V_R}\left[\Phi_{,t}-\frac{1}{2}\delta^{rs}\Phi_{,\gamma}Z^\gamma_{,s}\Phi_{,\delta}Z^\delta_{,r}+\Lambda^\bullet\left(Z^\rho,t\right)\right]_{,\beta}\delta w^\beta J\,dV_R=0, \tag{44}$$

where $\delta w^\beta=Z^\beta_{,r}\delta w^r$ is a linear transformation of the infinitesimal virtual displacement.

The virtual displacements have arbitrary values and a linear non-singular transformation does not change this property. The application of Green's theorem and the basic lemma of variational calculus lead to the following expression for the Lagrange multiplier in the material description

$$\Lambda^\bullet\left(Z^\rho,t\right)=\rho h+p=-\rho\Phi_{,t}\left(Z^\rho,t\right)+\frac{1}{2}\rho\delta^{rs}\Phi_{,\gamma}Z^\gamma_{,s}\Phi_{,\delta}Z^\delta_{,r}+C\left(t\right). \tag{45}$$

The expression for the pressures in the material description follows immediately. In the material description of potential flows the incompressibility condition assumes the following form

$$\delta^{rs}\left[\Phi_{,\gamma}\left(Z^\lambda,t\right)Z^\gamma_{,s}\left(Z^\lambda,t\right)\right]_{,\delta}Z^\delta_{,r}\left(Z^\lambda,t\right)=0, \tag{46}$$

which describes the Laplace equation in the material variables.

This differential equation cannot be solved for the potential function when the displacement gradient and its inverse are not known.

As a very special case we may assume that the variation of displacements at time t is equal to the actual velocity multiplied by dt

$$\delta z^i \left(Z^\lambda, t\right) = dt\, v^i \left(Z^\lambda, t\right), \tag{47}$$

which is an infinitesimal quantity.

Let us substitute this virtual displacement into the virtual work equation (36). The velocity field is that which corresponds to the rigorous solution for an incompressible fluid and, the variation of the Jacobian is zero and thus the second term in the integral is zero. The first term in the volume integral corresponds to the time derivative of the kinetic energy and the first term in the surface integral corresponds to the rate of change of the potential energy. In a material description the kinetic energy and rate of change of potential energy are given by the formulas

$$E_k\,(t) = \iiint\limits_{V_R} \frac{1}{2}\rho\delta^{rs} v_r v_s J\, dV_R, \quad \frac{d}{dt}E_p\,(t) = \iint\limits_{S_R} \rho h J N_\alpha Z_r^\alpha v^r\, dS_R. \tag{48}$$

Finally, the equation (36) reduces to the following equation of balance of power

$$\frac{d}{dt}\left[E_k\,(t) + E_p\,(t)\right] + \iint\limits_{S_R} p J N_\alpha Z_r^\alpha v^r\, dS_R = 0. \tag{49}$$

The rate of change of the sum of kinetic and potential energy plus the instantaneous work of pressures (positive when compression) on normal components of velocities on the surface (positive if acting outwards to the region) is equal to zero. The balance of power corresponds to one equation and can be used to verify a solution, but is not sufficient to find any.

5. The Lagrangian and the Hamilton's Principles

Let us multiply Eq. (32) by dt and integrate over the time interval $0 \le t \le t_k$

$$\int\limits_0^{t_k} \left\{ \iiint\limits_{V_R} \left[\rho a_r + (\rho h + p)_{,\alpha} Z_r^\alpha\right] \delta z^r J\, dV_R \right\} dt = 0. \tag{50}$$

We may perform the transformation in space that resulted in the relation (35) and write

$$\int_0^{t_k} \iiint_{V_R} \left[\rho \dot{v}_r \delta z^r + J^{-1}(\rho h + p)\,\delta J\right] J dV_R dt + \\ + \int_0^{t_k} \iint_{S_R} (\rho h + p)\, J N_\alpha Z^\alpha_{,r} \delta z^r dS_R dt = 0. \tag{51}$$

Now we introduce the following integration by parts

$$\int_0^{t_k} \iiint_{V_R} \rho \dot{v}_r \delta z^r J dV_R dt = \\ = \iiint_{V_R} \rho v_r \delta z^r J dV_R \Big|_0^{t_k} - \int_0^{t_k} \iiint_{V_R} \rho \delta^{rs} v_r \delta v_s J dV_R dt. \tag{52}$$

From substitution of the result into Eq. (51) the following relation is obtained

$$\int_0^{t_k} \iiint_{V_R} \left\{ \left[\delta\left(\frac{1}{2}\rho v_r v^r\right) + \Lambda \delta J\right] J dV_R - \iint_{S_R} (\rho h + p)\, N_\alpha J Z^\alpha_{,r} \delta z^r dS_R \right\} dt + \\ - \iiint_{V_R} \rho v_r \delta z^r dV_R \Big|_0^{t_k} = 0, \tag{53}$$

The variational formulation (52) may be written in the following form

$$\int_0^{t_k} \left\{ \delta L\left(Z^\rho, t\right) + \iiint_{V_R} (\Lambda \delta J) J dV_R - \iint_{S_R} p J N_\alpha Z^\alpha_r \delta z^r dS_R \right\} dt + \\ - \iiint_{V_R} \rho v_r \delta z^r J dV_R \Big|_0^{t_k} = 0, \tag{54}$$

where $L = E_k - E_p$ is the Lagrangian, $E_k\left(Z^\lambda, t\right)$ – the kinetic energy and $E_p\left(Z^\lambda, t\right)$ the potential energy of the incompressible fluid body, and variations of the energies are given in the material description by the following expressions

$$\delta E_k(t) = \delta \iiint_{V_R} \frac{1}{2} \delta^{rs} v_r v_s J dV_R, \quad \delta E_p(t) = \iint_{S_R} \rho h J N_\alpha Z^\alpha_r \delta z^r dS_R. \tag{55}$$

When variations in the relation (53) are considered, Lagrange's multiplier is not subject to variations. The integrals have to be calculated in the reference

configurations and thus the boundaries of the regions do not change with time. The last term corresponds to initial and end conditions in time.

It should be stressed that the relation (54) is obtained by transformations of integrals according to mathematics rules and thus the physics are the same as in the relation (50). In view of Eq. (11), the surface integral in (54) may be written in the spatial description as follows

$$\iint_{S_R} p\left(Z^{\lambda}, t\right) J N_{\alpha} Z^{\alpha}_{,r} \delta z^r dS_R = \iint_{S} p\left(z^s, t\right) n_r \delta z^r dS. \tag{56}$$

It represents the work of surface pressures on virtual displacements in the actual positions. The variation of the action integral (54) assumes the following form in the spatial description

$$\int_0^{t_k} \left[\delta L\left(z^r t\right) + \iiint_V \Lambda \delta J dV - \iint_S p n_r \delta z^r dS \right] dt - \iiint_V \rho v_r \delta z^r dV \Bigg|_0^{t_k} = 0, \tag{57}$$

where $L = E_k - E_p$ is the Lagrangian density function in the spatial description. The variations of the kinetic and potential energies are given in the spatial description by the following expressions

$$\delta E_k(t) = \delta \iiint_V \frac{1}{2} \delta^{rs} v_r\left(z^i, t\right) v_s\left(z^i, t\right) dV, \quad \delta E_p(t) = \iint_S \rho h n_r \delta z^r dS. \tag{58}$$

It should be noted that the positions of the boundaries change in time and thus in calculating variations of the integrals the variations in boundaries have to be considered. The two expressions (54) and (57), corresponding to variational formulations in material and spatial descriptions respectively are based on the formulation stated in the relation (50). From the point of physics the known pressures on the free boundaries work on the virtual displacements and there is no kinetic energy in the initial state (the initial velocities are assumed to be zero).

Let us generalise the formulation and supplement the relation (50) by additional terms that take into account kinetic energy at the initial time and by application of known pressures to the bounding surface. Thus we take

$$\int_0^{t_k} \left\{ \iiint_{V_R} \left[\rho \dot{v}_r + (\rho h + p)_{,\alpha} Z^{\alpha}_r\right] \delta z^r J dV_R - \iint_{S_R} p \delta w_{fn} dS_R \right\} dt + \\ - \iiint_{V_R} m_i \delta z^i J dV_R \Bigg|_0^{t_k} = 0, \tag{59}$$

where δw_{fn} is the normal component of the virtual displacement on the surface in the actual configuration.

Repeating manipulations that led to the expression (54) we arrive at the following relation

$$\int_0^{t_k} \left\{ \delta E_k - \delta E_p + \iiint_{V_R} \Lambda \delta J J dV_R + \iint_{S_R} p\left(\delta w_{fn} - J N_\alpha Z_r^\alpha \delta z^r\right) dS_R \right\} dt + \\ + \iiint_V (m_r - \rho v_r) \delta z^r J dV_R \Big|_0^{t_k} = 0. \tag{60}$$

where the meaning of the terms introduced is clear. The functions $m_r\left(Z^\lambda, t\right)$ are the densities of the momentum at the initial and end times, and the expression in the brackets in the surface integral corresponds to the difference between the respective virtual normal components calculated on the basis of surface displacements and the displacements in the fluid body. It is no problem to write down the corresponding to (60) expression in the spatial description as follows

$$\int_0^{t_k} \left\{ \delta E_k - \delta E_p + \iiint_V \Lambda \delta J dV + \iint_S p\left[\delta w_{fn} - n_r \delta z^r\right] dS \right\} dt + \\ + \iiint_V [m_r - \rho v_r] \delta z^r dV \Big|_0^{t_k} = 0. \tag{61}$$

When the additional terms in the relations (60) and (61) vanish, they go over to the corresponding relations (54) and (57). It should be noted that the physical meaning of the additional terms is clear. On the bounding surface the virtual displacements of a point of the surface and the neighbouring fluid particle are not independent. The normal components must be the same, but the tangential may be different. The problem has been discussed in the third chapter. The given initial velocity field multiplied by ρ gives the value of the momentum field m_r at the initial time and cancels the corresponding term in the last expressions. The variational expressions preserve energy and thus the value of m_r at the end has to be equal to the calculated one.

6. The Case of Potential Flow in Spatial Description

If for times $t < 0$ the fluid is at rest and motion starts at zero time due to the motion of the bounding surface with initial displacements equal zero and an irrotational velocity field, then for all times the velocity field is irrotational and has

a potential Φ. Let us start our consideration with discussion of the influence on irrotational motion on the variational formulation given in the relation (61). For the potential motion the variation of the kinetic energy is obtained by means of consideration of a family of "comparison" potentials

$$\hat{\Phi}(z^r, t) = \Phi(z^r, t) + \delta\Phi(z^r, t). \tag{62}$$

The standard procedure of the calculus of variation leads to the variation of the kinetic energy

$$\begin{aligned} \delta E_k = &\iint\limits_S \left[\frac{1}{2}\rho\delta^{ij}\,\Phi_{,i}\left(z^r, t\right)\Phi_{,j}\left(z^r, t\right)\right]\delta w_n dS + \\ &+ \iiint\limits_V \left[\frac{1}{2}\rho\delta^{ij}\,\Phi_{,i}\left(z^r, t\right)\delta\Phi_{,j}\left(z^r, t\right)\right]dV, \end{aligned} \tag{63}$$

where the variation of the velocity potential and the position of the boundary have to be considered. The volume of the fluid body must be constant. Let us apply Green's transformation formula of integrals

$$\iiint\limits_V \delta^{ij}\,\rho\Phi_{,i}\delta\Phi_{,j}dV = -\iiint\limits_V \rho\delta^{ij}\,\Phi_{,ij}\delta\Phi dV + \iint\limits_S \rho v_n\delta\Phi dS, \tag{64}$$

where $v_n = \partial\Phi/\partial n$ is the component of velocity normal for the actual surface. Finally the variation of the kinetic energy term in the action integral is

$$\begin{aligned} \int\limits_0^{t_k} \delta E_k dt = -\int\limits_0^{t_k} \iiint\limits_V \rho\delta^{ij}\,\Phi_{,ij}\delta\Phi dV dt + \int\limits_0^{t_k} \iint\limits_S \frac{1}{2}\rho\delta^{ij}\,\Phi_{,j}\,\Phi_{,i}\delta w_n dS\,dt + \\ + \int\limits_0^{t_k} \iint\limits_S \rho v_n\delta\Phi dS dt, \end{aligned} \tag{65}$$

The term connected with the variation of potential energy assumes the form

$$\int\limits_o^{t_k} \delta \iiint\limits_V \rho h dV dt = \int\limits_0^{t_k} \iint\limits_S \rho h\delta w_n dS dt. \tag{66}$$

To obtain the correct boundary conditions an identity has to be added with a term that corresponds to the partial time derivative of the potential function. The following identity results from the differentiation

$$\frac{d}{dt}\iiint\limits_V \rho\delta\Phi dV = \iiint\limits_V \rho\delta\Phi_{,t}dV + \iint\limits_S \rho u_n\delta\Phi dS, \tag{67}$$

where u_n is the normal component of the speed of the moving surface.

Knowing that

$$\int_0^{t_k} \delta \iiint_V \rho \Phi_{,t} dV dt = \int_0^{t_k} \iiint_V \rho \delta \Phi_{,t} dV dt + \int_0^{t_k} \iint_S \rho \Phi_{,t} \delta w_n dS dt \tag{68}$$

and using the preceding identity one obtains

$$\begin{aligned} &\int_0^{t_k} \delta \iiint_V \rho \Phi_{,t} dV dt = \\ &= -\int_0^{t_k} \iint_S \rho u_n \delta \Phi dS dt + \int_0^{t_k} \iint_S \rho \Phi_{,t} \delta w_n dS dt + \iiint_V \rho \delta \Phi dV \Big|_0^{t_k}. \end{aligned} \tag{69}$$

The addition of the expressions (65), (66) and (68) supplemented by a term corresponding to the boundary work of pressures on virtual displacements leads to the following identity

$$\begin{aligned} &\int_0^{t_k} \left[\delta E_k + \delta \iiint_V (\rho \Phi_t + \rho h) dV + \iint_S p \delta w_n dS \right] dt = \\ &= -\int_0^{t_k} \iiint_V \frac{1}{2} \rho \delta^{ij} \Phi_{,ij} \delta \Phi dV dt + \int_0^{t_k} \iint_S \rho [v_n - u_n] \delta \Phi dS dt + \\ &+ \int_0^{t_k} \iint_S \left[\rho \left(\frac{1}{2} \delta^{ij} \Phi_{,j} \Phi_{,i} + \Phi_{,t} + h \right) + p \right] \delta w_n dS dt + \iiint_V \rho \delta \Phi dV \Big|_0^{t_k} = 0 \end{aligned} \tag{70}$$

The expression on the left hand side of the relation (70) is the variation of the action integral and may also be written in the following form

$$\delta I = \int_0^{t_k} \delta \iiint_V \left[\rho \Phi_{,t} + \frac{1}{2} \rho \delta^{rs} \Phi_{,r} \Phi_{,s} + \rho h \right] dV dt + \int_0^{t_k} \iint_\delta w_n dS dt. \tag{71}$$

Let us assume that the virtual normal displacements of bounding surfaces δw_n and the virtual potentials on the bounding surfaces $\delta \Phi$ are independent. Thus when the basic lemma of the calculus of variations is used it is possible to assume that the variations of potentials at times $t = 0, t = t_k$ within the fluid, and on the boundaries for all times are zero; the variations of normal displacements are

also zero. At the same time, the volume integral must be zero for arbitrary $\delta\Phi$ functions, which means that the potential has to satisfy the Laplace equation. Now let us assume that $\delta\Phi$ is not zero on the bounding surfaces, thus from the condition that the second integral on the right hand side of Eq. (70) has to be zero it follows that the condition of kinematics has to be satisfied. When it is assumed that the normal virtual displacements δw_n are not zero, then the formula for pressures results and in the case of a free surface it corresponds to the dynamic boundary condition. The last term corresponds to the value of volume integrals at times $t = 0$ and $t = t_k$. The standard case corresponds to the initial condition that the potential function is given for the time $t = 0$ and thus $\delta\Phi|_{t=0} = 0$. The variational formulation leads to the solution of the Laplace differential equation with given initial and boundary conditions. If the solution exists and is unique then the potential at $t = t_k$ is obtained from calculation and thus the value of $\delta\Phi$ has to be zero at time $t = t_k$.

A special and important case corresponds to the free boundary described by equation (18). With regard to the relation (23) the integral that corresponds to the condition of kinematics may be written as

$$\int_0^{t_k} \iint_S \rho(v_n - u_n)\delta\Phi dS dt = \int_0^{t_k} \iint_S \rho(-v^1\zeta_1 - v^2\zeta_2 + v^3 - \zeta_t)\delta\Phi dS_p dt, \quad (72)$$

where S_p is the projection of S on the (z^1, z^2) plane and $dS_p = n^3 dS = dz^1 dz^2$. Other parts of the bounding surface are projected onto different planes. For example, for the rigid plane AB in Fig. 1 ($A'B'$ in the actual configuration) the natural way is to choose the vertical plane (z^2, z^3) for the orthogonal projection.

The variational formulation (70) gives the Laplace equation and all the boundary and initial conditions of the problem mentioned. It is straightforward to go from the general relations presented above to the special case discussed by Luke (1967).

7. Concluding Remarks

The analysis performed in the preceding sections is based on rigorous formulations of the principle of virtual work for perfect incompressible fluid. In derivation of the principle the fundamental momentum equations, as well as the boundary and initial conditions, have been taken into account. At each stage of the analysis all terms entering important formulae derived in the paper have clear physical or geometrical meaning. In particular, due to the formulation, we have arrived at proper formulation of Hamilton's principle for the fluid considered. As compared to earlier formulations encountered in the literature on the subject the proposed formulation seems to be free of difficulties in interpretations of all quantities

entering the problems mentioned above. The main results and conclusions are summarised as follows:

- The material description is natural for variational formulations in continuum mechanics. In the paper all the relations are written in both material and spatial descriptions, as the Euler's description is very convenient in problems of fluid mechanics.
- The virtual work equation is obtained as the integral in space of the dot product of the vector form differential equations of the problem and the vector of virtual displacements, and suitable transformations of integrals.
- In the case of potential flows the well-known expressions for the pressures in both descriptions follow from the virtual work equation.
- Integration of the virtual work equation in time leads to the action integral and the Lagrangian functions in both descriptions that correspond to the difference of the kinetic and potential energies of the fluid. The physical background of the relations is discussed and generalisations of the form of the action integral are presented.
- The case of potential flow in a spatial description is discussed in detail. The formulation presented in the paper leads to generalisation of the Lagrangian density function given by Luke (1967).

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