*Archives of Hydro-Engineering and Environmental Mechanics*
Vol. 51 (2004), No. 3, pp. 225–241

# Monitoring of Deep Foundation Deflection in Model Experiments by Means of a Magnetic Method: Theoretical Study

**Jerzy Wtorek***, **Zygmunt Kurałowicz****, **Bart Truyen*****

* Gdańsk University of Technology, Faculty of Electronics, Telecommunication and Informatics, ul. Narutowicza 11/12, 80-952 Gdańsk, e-mail: jaolel@biomed.eti.pg.gda.pl

** Gdańsk University of Technology, Faculty of Hydro and Environmental Engineering, ul. Narutowicza 11/12, 80-952 Gdańsk, e-mail: zkur@pg.gda.pl

*** Department of Electronics and Information Processing, ETRO, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussel, Belgium. e-mail: batruyen@etro.vub.ac.be

(Received January 19, 2004; revised May 17, 2004)

### Abstract

In this paper, a new contact-free method for monitoring the displacement of buried foundations, with application to model (laboratory) studies, is considered. This method is based on the measurement of the magnetic induction amplitude created by a magnetic dipole attached to the construction. An array of sensing coils, located distantly, measures the changes in magnetic induction amplitude invoked by displacement of the signal source. The forward and inverse problems are formulated. Reconstruction of the source displacement is performed for synthetic data, both contaminated and uncontaminated by noise. It has been found that the reconstruction of displacements as small as 0.001 m is possible for distances between signal source (transmitting coil – magnetic dipole) and coil array of 0.5 m, even for data contaminated by noise.

**Key words:** pile deflection, measurement, magnetic method, inverse problem

## 1. Introduction

The design of deep foundations requires knowledge of the lateral loading for different types of soil. Determination of the distribution of the load acting on a single pile or group of piles due to a weak layer is one of the more challenging problems in geotechnics. Additional lateral load is caused by external – usually vertical – forces of known value, e.g. high-road embankments located in the vicinity of the pile structure. As a result, laterally loaded piles are deformed or deflected in a way that cannot be easily described and/or measured. Many different methods are available to measure this deflection precisely, such as laser methods (Kurałowicz, Meißner 1994, Kurałowicz 2001). However, the application of most of them is limited only to *transparent* materials. A few attempts to measure pile deflection have been undertaken in laboratory scale studies, using tensometric and string sensor methods. Tensometric methods, however, are sensitive to the soil moisture, and subject to mechanical destruction, while string methods cannot be used when a weak layer is consolidated before invoking lateral load of pile.

A new method, based on the measurement of an induced magnetic field is described. The actual measurement system encompasses a magnetic signal source and a receiver. Two different approaches for measuring the distance between the source and receiver may be conceived. A first approach involves the measurement of the signal delay arising from a change in a propagation distance. However, in practice, it seems impossible to achieve the time-resolution required to discern the relatively small changes in distance between source and receiver. As an alternative, the amplitude decay along the distance from the source measurement may be utilized. Measurement systems based on this concept, however, have to meet several requirements in order to guarantee their accuracy. In particular, the amplitude of the signal source has to be stable during the complete acquisition time, while the character of the amplitude decay should be independent of the properties of the surrounding medium, and known explicitly.

## 2. Method

The method proposed here is based on the measurement of the decay in magnetic flux density along the distance from the source. A schematic representation of the experimental setup modelled in the study is depicted in Fig. 1. The examined pile is located in a glass container, encaged in a metal frame, and filled with sand. A set of transmitters, marked T, are attached to its side. An array of receivers, each marked Rc, is located at a distance $L$ from the pile. The receivers are placed on a plane outside the container. Though presenting an experimental setup, it has to be mentioned that only theoretical considerations are presented in this paper. However, the theoretical model studied here, will reflect the anticipated experimental conditions.

It should be taken into account, however, that the magnetic flux also depends on the magnetic properties of the material in which it propagates. Specifically, the magnetic flux has to satisfy the boundary conditions at the interfaces between materials described by different magnetic permeability values (Jackson 1982). The amplitude of the field changes at each interface, according to the appropriate boundary conditions. Hence, the amplitude of the magnetic field will depend on the source-receiver distance, as well as the parameters describing the materials involved in the measurements. However, many materials, including different types of soil, are appropriately described by the magnetic permeability of the vacuum.

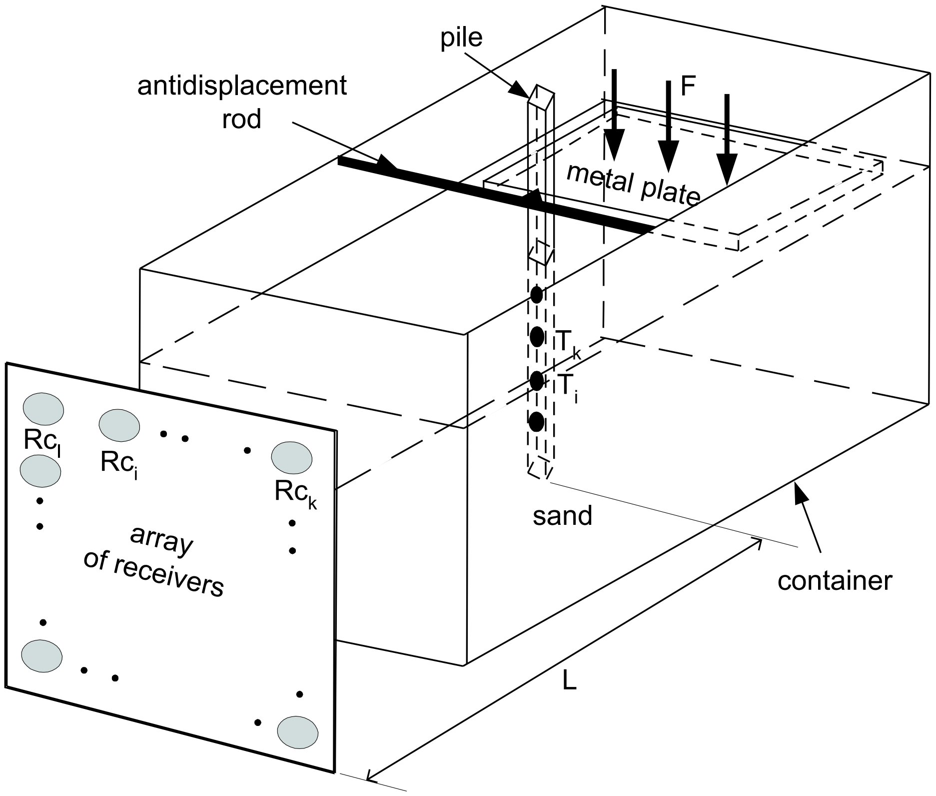


**Fig. 1.** Model of an experimental stand used in the study, $Rc_i$ and $T_i$ represent receiving and transmitting coils, respectively. Transmitting coils $T_i$ are attached to an examined pile while the array of the receiving coils $Rc_i$ is located adistance and outside the container. Lateral load is applied by means of metal plate and hydraulic press. The upper end of the pile is prevented from movement by means of the metal rod (anti-displacement rod)

This is equivalent to the assumption that the material between the source of the magnetic field and the measurement point is homogenous.

### 2.1. Forward Problem

**a) Magnetic source signal**

Let us assume that a coil of radius $a$ is energized by a current $I$ (Fig. 2a). This current creates a magnetic vector potential $\mathbf{A}\left(A_r, A_\phi, A_\theta\right)$ given by the following relationship

$$\mathbf{A} = \frac{\mu I}{4\pi} \oint\limits_{l} \frac{d\mathbf{l}}{r'}, \tag{1}$$

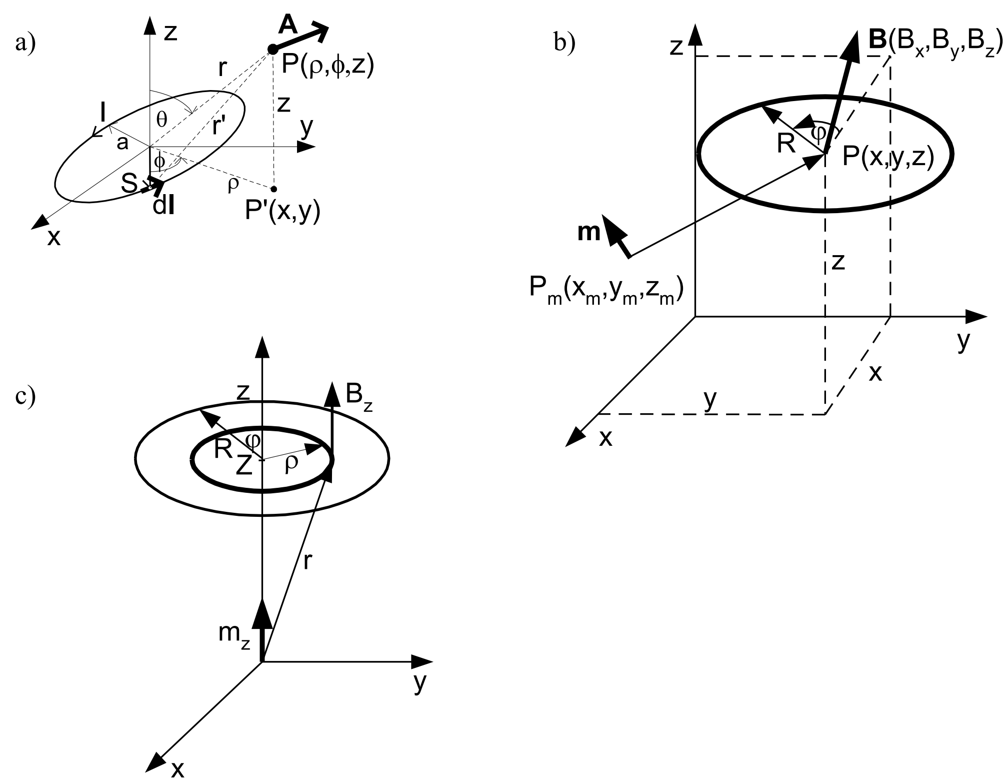


**Fig. 2.** a) Co-ordinate system and notations used in the study; b) The receiving coil and magnetic dipole; c) The model used for calculation of the flux $\Phi$ of the magnetic induction **B**, associated with the magnetic dipole $\mathbf{m}(0, 0, m_z)$, through the receiving coil of the radius $R$ and located on the plane $z = Z$

where $l$ is the length of the loop (coil), $d\mathbf{l}$ is the incremental length, and $r'$ is the distance between the incremental length $d\mathbf{l}$ of the coil located at the point S and the point of observation $\mathrm{P}(\rho, \phi, z)$ (Fig. 2). The magnetic induction $\mathbf{B}\left(B_r, B_\phi, B_\theta\right)$ can be determined based on knowledge of vector potential **A,** and the relation $\mathbf{B} = \nabla \times \mathbf{A}$ (for details see Appendix 1)

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) = \frac{\mu}{4\pi}a^2\pi I\frac{2\cos\theta}{r^3}; \tag{2}$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right) = \frac{\mu}{4\pi}a^2\pi I\frac{\sin\theta}{r^3}, \tag{3}$$

where $\mu$ is the magnetic permeability.

The relations (2) and (3) have been obtained by assuming the observation point to be relatively distant from the coil, or alternatively that the coil radius is small compared with the distance between coil and observation point. The component

$B_\phi$ is equal to zero due to the symmetry of the problem. Hence, the magnetic field components are found to have exactly the same form as the components of the electric field created by an electric dipole (Januszajtis 1982). In this way, a small circular loop can be identified as a magnetic dipole with moment $\mathbf{m} = \pi a^2 I\mathbf{n}$, $\mathbf{n}$ being the unit vector normal to the surface of the coil. The relationships (2) and (3) can be used to evaluate the distance from the magnetic dipole (i.e. transmitting coil) (Weber 1957, Januszajtis 1982). A Cartesian co-ordination system will be used in our study, requiring the magnetic field components to be transformed appropriately. The components of the magnetic induction $\mathbf{B}\left(B_x, B_y, B_z\right)$ at the point $\mathrm{P}(x, y, z)$ for the magnetic dipole $\mathbf{m}$, located at the point $\mathrm{P}(x_m, y_m, z_m)$ (see Fig. 2b) can be obtained from the relations (2, 3) (Januszajtis 1982, Dunajski 1990) in the following form

$$B_x = \frac{\mu}{4\pi r^5}\left[m_x\left(2x'^2 - y'^2 - z'^2\right) + 3m_y x'y' + 3m_z x'z'\right], \tag{4}$$

$$B_y = \frac{\mu}{4\pi r^5}\left[m_y\left(2y'^2 - x'^2 - z'^2\right) + 3m_x x'y' + 3m_z y'z'\right], \tag{5}$$

$$B_z = \frac{\mu}{4\pi r^5}\left[m_z\left(2z'^2 - x'^2 - y'^2\right) + 3m_x x'z' + 3m_y y'z'\right], \tag{6}$$

where $x' = x - x_m$, $y' = y - y_m$, $z' = z - z_m$ and $r^2 = (x - x_m)^2 + (y - y_m)^2 + (z - z_m)^2$.

With the transmitting coil located on the plane $z = 0$, and with the direction of the current $I$ being counter clockwise, the magnetic dipole exhibits only one non-zero component, i.e. $\mathbf{m}\,(0, 0, m_z)$. In this particular case, (4)–(6) reduce to:

$$B_x = \frac{3\mu m_z xz}{4\pi r^5}; \quad B_y = \frac{3\mu m_z yz}{4\pi r^5}; \quad B_z = \frac{\mu m_z\left(2z^2 - x^2 - y^2\right)}{4\pi r^5}. \tag{7}$$

**b) Magnetic flux detector**

The magnetic induction $\mathbf{B}$ can be measured by means of a receiving coil (Fig. 2c). Actually, it is the magnetic flux $\Phi$ flowing through a coil located on the plane $z$=constant that is measured, instead of the magnetic field itself. The relationship between these quantities is given by the integral:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S B_z ds, \tag{8}$$

where $S$ is the area of the receiving coil, with the unit vector of the coil being oriented along the $z$ co-ordinate.

Taking into account relation (7), the magnetic flux $\Phi$ is given by

$$\Phi = \int_S \frac{\mu m_z}{4\pi} \left( \frac{3z^2}{r^5} - \frac{1}{r^3} \right) ds. \tag{9}$$

The notational conventions are illustrated in Fig. 2c.

Despite considering only one particular case of the relation between the magnetic dipole and the receiving coil (see Appendix 2), these results can be generalized to the form

$$U_{iB} = m_z K_i(r), \tag{10}$$

where $K_i\,(r)$ is the proportionality coefficient of $i^{\text{th}}$ coil, the value of which depends on the distance between dipole and the receiving coil. Thus, the voltage change due to a small displacement $\Delta r$ of the transmitting coil is approximated as

$$\Delta U_{iB} = m_z\,(r)\,\frac{\partial K_i\,(r)}{\partial r}\Delta r + K_i\,(r)\,\frac{\partial m_z\,(r)}{\partial r}\Delta r. \tag{11}$$

The second term in (11) is omitted in the considerations that follow, assuming that the deflection of the buried object does not change the value of $m_z$.

### 2.2. The Inverse Problem

Finding the position and parameters of the magnetic dipole requires the solution of an inverse problem (a short introduction to the inverse problem, including a comprehensive example, is given in Appendix 3). The complexity of the considered problem is significantly reduced upon assuming that the moment of the magnetic dipole is known. Thus, the searched parameters are those describing the localization and the direction of the dipole. In a first attempt, it is assumed that the direction of the dipole moment is fixed (cf. description of relation (11)). The problem is then stated as follows: find the displacement value of the dipole given two sets of measurement data. In general, the relationship between the measured value of magnetic flux and the displacement of the dipole is non-linear. The inverse problem can be solved using different approaches, categorized as deterministic or probabilistic. In the presented study, the least squares method has been adopted. Moreover, the change in distance between source and receiver is used as the search parameter, instead of the absolute distance. This differential approach is known to be more resistant to noise, and interferences, than the absolute one. For an array of receiving coils (Fig. 3), a misfit function can be defined as the difference between the measured $\Delta\mathbf{U}^m$ and the calculated values $\Delta\mathbf{U}^c$ (Meju 2001, Kurałowicz, Wtorek 2002, Wtorek 2003)

$$\mathbf{e} = \Delta\mathbf{U}^m - \Delta\mathbf{U}^c, \tag{12}$$

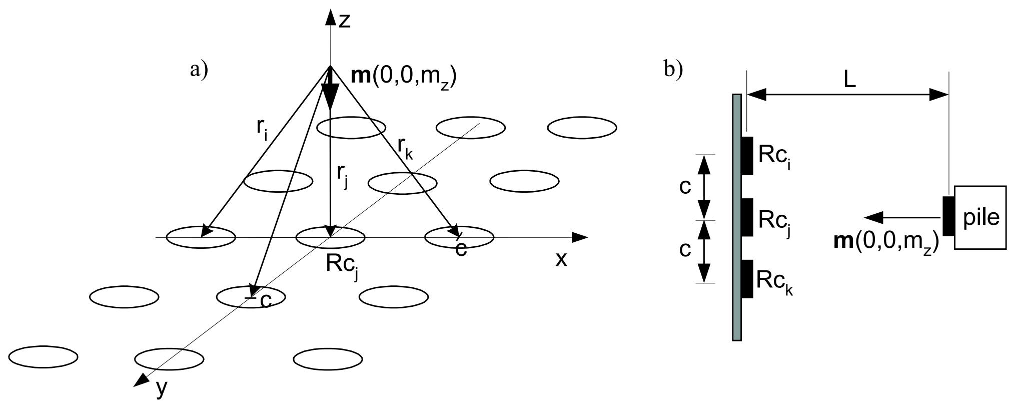


**Fig. 3.** a) The geometrical relation (perspective view) between the array of the receiving coils and the source of magnetic signal; b) The geometrical relation of the transmitter and receivers (lateral view); $Rc_i$ stands for $i^{th}$ receiving coil while **m** for magnetic dipole, $c$ is the distance between the receiving coils (in the study $c = 0.08$ m has been assumed)

where $\Delta\mathbf{U}^m = \mathbf{U}^m (r_0 + \Delta r) - \mathbf{U}^m (r_0)$ is the measured difference, $\Delta\mathbf{U}^c$ is the calculated difference using the adopted model, i.e. relations (4)–(6). The square of the error **e** is thus given by the relation (Meju 2001, Wtorek 2003)

$$\mathbf{e}^T\mathbf{e} = \left(\Delta\mathbf{U}^c - \Delta\mathbf{U}^m\right)^T \left(\Delta\mathbf{U}^c - \Delta\mathbf{U}^m\right). \tag{13}$$

The change in the calculated value can be approximated using a Taylor expansion

$$\Delta\mathbf{U}^c = \mathbf{U}^c (r_0 + \Delta r) - \mathbf{U}^c (r_0) = \frac{\partial \mathbf{U}^c (\mathbf{r})}{\partial \mathbf{r}}_{|r=r_0} \cdot \Delta\mathbf{r} + 0\left(\|\Delta r\|^2\right). \tag{14}$$

Hence, a linearised version of (13) is given by

$$\mathbf{e}^T\mathbf{e} = \left(\mathbf{S}\Delta\mathbf{r} - \Delta\mathbf{U}^m\right)^T \left(\mathbf{S}\Delta\mathbf{r} - \Delta\mathbf{U}^m\right), \tag{15}$$

with **S** being the derivative (Jacobian) presented in the equation (14). The least square solution of (15) is (Meju 2001, Oristalgio, Blok 1995)

$$\Delta\mathbf{r} = \left(\mathbf{S}^T\mathbf{S}\right)^{-1} \mathbf{S}^T \Delta\mathbf{U}^m. \tag{16}$$

In general, inverse problems are ill-conditioned, i.e. a small error in the data may lead to large changes in the estimated vector of displacement $\Delta\mathbf{r}$. The Levenberg-Marquardt modification of the solution (16) has been used in the case of data contaminated by noise (Meju 2001, Wtorek 2003, Brandt 1988, Janczulewicz, Wtorek 2003, Janczulewicz, Wtorek 2003a)

$$\Delta\mathbf{r} = \left(\mathbf{S}^T\mathbf{S} + \lambda \cdot diag\left(\mathbf{S}^T\mathbf{S}\right)\right)^{-1} \mathbf{S}^T \Delta\mathbf{U}^m, \tag{17}$$

where $\lambda$ is called the regularization constant.

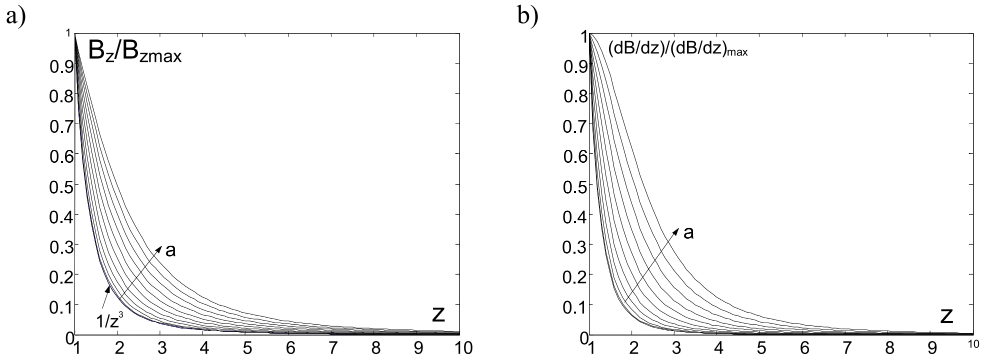


**Fig. 4.** a) Dependence of the magnetic induction $B_z$ (flux density) generated by the transmitting coil on the distance from the coil along its axis; b) Dependence of the magnetic flux density rate on the distance from transmitting coil along its axis. Normalized values are shown. The letter $a$ represents the radius of the transmitting coil

## 3. Results

Values of the magnetic induction $B_z$, and its decay rate $dB_z/dz$, as a function of the distance along its axis from the surface of the source coil, were calculated for different values of the coil radius $a$ (see Fig. 4). The values of $B_z$ and its decay rate $dB_z/dz$ were normalized by the maximal value of $B_z$ and $dB_z/dz$, respectively. The distance from the coil was instead normalized with respect to coil radius $a$. Then, an inverse problem was solved using relation (16) for synthetic data without noise. These synthetic data were calculated, according to relation (6), for the arrangement of transmitting and receiving coils shown in Fig. 3. In particular, an array of fifteen receiving coils, arranged in three rows, each containing five coils, was applied (Fig. 3). The transmitting coil was located on the axis of the middle receiving coil (Fig. 3b). Results of the calculations for three different distances between transmitting coil and array, assuming noiseless data, are presented in Fig. 5a. The deflections presented in Fig. 5b were calculated for the same data, however, with three different levels of noise added. In general, the associated inverse problems are ill-conditioned, and thus are very sensitive to the noise level. To overcome this problem a regularized solution was computed (17). The value of the regularization constant $\lambda$ was assumed to be 0.05. The dependency of the reconstructed value $\Delta r_r$ on the actual displacement of the source $\Delta r_a$ is represented in Fig. 5c for three different noise levels, SNR = 40 dB, SNR = 26 dB, SNR = 20 dB, and assuming a fixed distance between the transmitting coil and the array, $L$ = 0.5 m. Next, the calculations were repeated in Fig. 5d for three different distances, $L$ = 0.3 m, $L$ = 0.4 m, $L$ = 0.5 m, with the data contaminated by noise such that SNR = 26 dB.

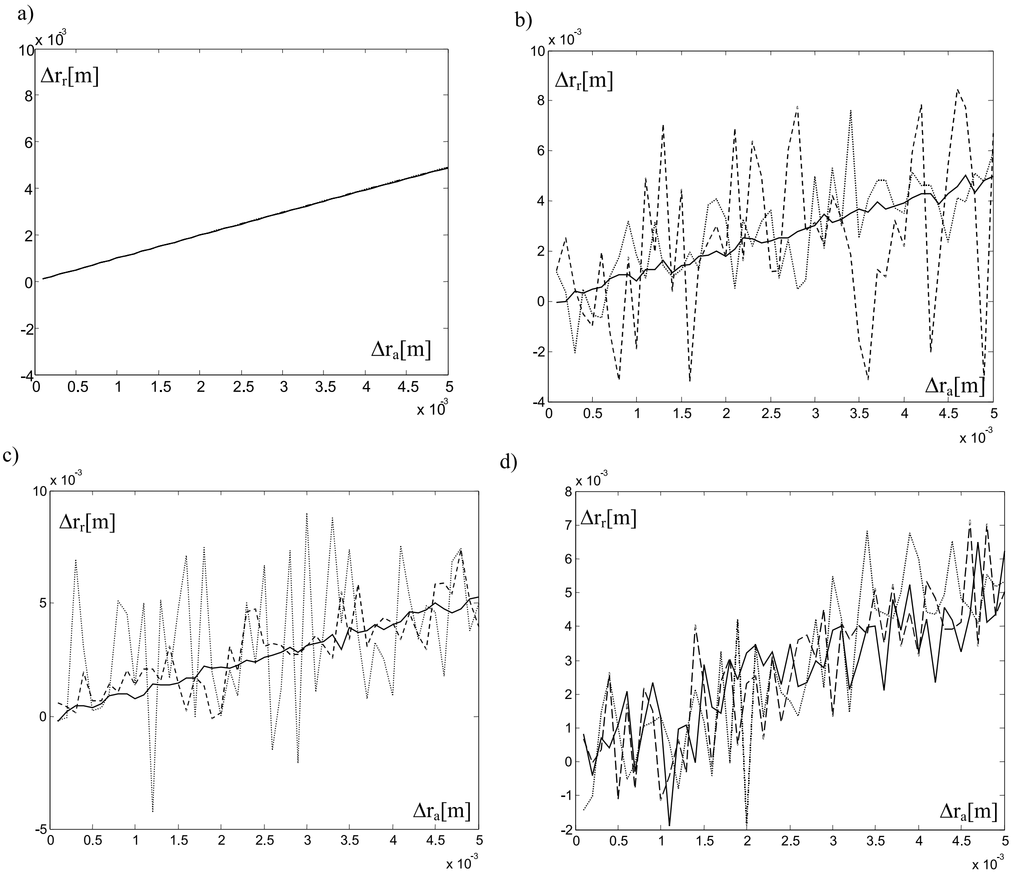


**Fig. 5.** Reconstructed $\Delta r_r$ vs. actual displacement $\Delta r_a$ of the signal source: a) simulation study for three different distances, $L = 0.3$ m, $L = 0.4$ m, and $L = 0.5$ m, between the source and the receiving coils, synthetic data for the reconstruction do not contain noise, note that all curves overlap each other; b) simulation for the fixed distance, $L = 0.5$ m, between the source and the receiving coils and for data containing noise, continuous line – SNR = 40 dB, dotted line – SNR = 26 dB, dashed line – SNR = 40 dB, reconstruction without regularization; c) simulation for the fixed distance between source and receive coils and for data containing noise, continuous line – SNR = 40 dB, dotted line – SNR = 26 dB, dashed line – SNR = 20 dB, reconstruction with regularization; d) Reconstructed displacement for data containing the same level of noise (SNR = 40 dB), however, for three different distances between signal source and array of receiving coils: $L = 0.30$ m – continuous line, $L = 0.40$ m – dotted line, and $L = 0.50$ m – dashed line

## 4. Discussion

The magnetic flux density generated by a coil of diameter $a$ decreases very fast, and non-linearly, with the distance from the coil (Fig. 4). Determination of the displacement of the signal source is consequently a strongly non-linear problem. However, the problem can be linearised in a first attempt, by assuming that the change of flux be proportional to the source deflection. This coefficient of proportionality is equal to the derivative of the magnetic flux density. In this study, a magnetic dipole approximation is assumed, which is valid as long as $z \gg a$ (Fig. 4), i.e. a large distance between the transmitting coil (the magnetic dipole) and the plane containing receiving coils. The condition $z \gg a$ also allows the relation between the change in magnetic flux density and the measured voltage to be approximated by a simple proportionality relation.

Noticeable is that the curves for $B_z$ and $dB_z/dz$ are almost horizontal for larger distances from the transmitting coil. This means, that only larger displacements may produce significant changes in flux, and thus in measured voltage. It also explains the ill-conditioned nature of the problem. The change of the measured difference may be approximated by equation (11). The derivative of this function, however, is almost zero for large distances. Taking into account that both the derivative and the displacement are small, the measured signals will also be very small, and hence be very susceptible to noise. Nevertheless, promising results are obtained for noiseless data (Fig. 5a).

The relationship between the actual displacement $\Delta r_a$ and the reconstructed $\Delta r_r$ is almost linear, and similar for all three distances between transmitting coil and array of receiving coils considered here; 0.3 m, 0.4 m and 0.5 m. However, this is a hypothetical case, as it cannot occur in real situations, such as represented in Figs. 5b, 5c and 5d. Reconstructions for different levels of noise have been carried out, with and without regularization. In the latter case the regularization coefficient is set at 0.05. All the presented results are characterised by relatively large variances. Nevertheless, it is shown that even in the presence of noise, displacement information can be gained from the measurement of variations in magnetic flux density.

The diameter of the receiving coils can be increased in further studies. This may improve the signal to noise ratio (SNR) of the measurement system. Coils may even overlay each other. In determining the distance $c$ between the receiving coils (cf. Fig. 3b), the SNR value is also a crucial factor. The smaller the distance between the receiving coils, the higher the SNR value of the measurement system should be. Notice, however, that for small values of $c$, the distances between the dipole and the receiving coils are almost equal, and according to relation (9) the measured fluxes will also be equal. This distance should be considered as a relative value, referenced to the distance between the magnetic dipole (or more generally, the signal source) and the plane containing the receiving coils.

The distance between the receiving coils was kept fixed in this study. Thus, for a fixed distance between the receiving coils, an increase of the distance between the transmitting coil and the array of the receiving coils is equivalent to the requirement of a higher SNR value.

It is important to emphasize that application of the described method is foreseen in experiments with non-magnetic soils, i.e. the magnetic permeability of the soil used in experiments should be equal to that of vacuum. Otherwise, the assumption that boundary effects can be ignored, would no longer be valid.

Another important phenomenon, which has to be taken into consideration in further studies, is the rotation of the transmitting coil. In general, transmitting coils attached to deep foundation can either be displaced or rotated. This would demand the reconstruction algorithm, currently based on relations (16) and (17), to be reformulated in more general terms, i.e. including rotational components.

## 5. Conclusions

It has been demonstrated that the displacement of a magnetic dipole can be monitored by multiple and simultaneous measurement of the magnetic flux density. The accuracy achieved is very promising, even for data contaminated by noise. It has been found that signals with an SNR as low as 40 dB are still acceptable. These values are easily achieved in practical measurement conditions, which makes the method a viable candidate to monitor the deflection of buried constructions (deep foundations) in laboratory scale studies.

## Appendix 1

In a cylindrical coordinate system, the vector potential is described by the relation (1), which for a current $I$, flowing in a coil of radius $a$, transforms to the following expression (Weber 1957)

$$A_\phi = \frac{\mu I}{4\pi}\int_0^{2\pi} \frac{\left[\mathbf{u}_r a \sin\phi + \mathbf{u}_\phi a \cos\phi\right]}{\left[(\rho - a\cos\phi)^2 + (a\sin\phi)^2 + z^2\right]^{1/2}} d\phi, \tag{A1}$$

where $\phi$ is the angle between the current position of $d\mathbf{l}$ and the projection of observation point P on plane $z = 0$ (see Fig. 2a), and $\mathbf{u}_r$ and $\mathbf{u}_\phi$ are respectively radial and tangential unit vectors, $A_\phi$ is the magnetic vector potential, while $\rho$ stands for the radial coordinate, $\phi$ for the azimuth coordinate (cylindrical coordinate system), $\mu$ for the magnetic permeability. Taking into account that the first component of (A1) is in the radial direction, and thus can be disregarded, the integral is reduced to the tangential component only. Introducing $r = \left(\rho^2 + z^2\right)^{1/2}$ (see Fig. 2a), the relationship (A1) can be rearranged as

$$A_\phi = \frac{\mu I}{4\pi}\int_0^{2\pi} \frac{\left[\mathbf{u}_r a \sin\phi + \mathbf{u}_\phi a \cos\phi\right]}{\left[a^2 + r^2 - 2\rho a \cos\phi\right]^{1/2}} d\phi. \tag{A2}$$

Assuming that the vector potential $\mathbf{A}$ is calculated for points distant from the coil, i.e. $r \gg a$, one obtains

$$A_\phi = \frac{\mu I}{4\pi}\int_0^{2\pi} \frac{\left[\mathbf{u}_r a \sin\phi + \mathbf{u}_\phi a \cos\phi\right]}{r\left[1 - \left(2\rho a/r^2\right)\cos\phi\right]^{1/2}} d\phi. \tag{A3}$$

Then, expanding the square root of the denominator binomially, and retaining the first two terms, gives

$$A_\phi \approx \int_0^{2\pi} \frac{a}{r}\cos\phi\left[1 + \frac{a\rho}{r^2}\cos\phi\right] d\phi = \frac{\mu a^2 I}{4\pi}\frac{\rho}{r^3}. \tag{A4}$$

Using spherical coordinates, with $\rho/r = \sin\theta$, and taking into account that $r \approx r'$ (see Fig. 2), the components of the magnetic fields calculated from the relation $\mathbf{B} = \nabla \times \mathbf{A}$, are given by relations (2), (3) expressed in the spherical coordination system.

## Appendix 2

The flux through the receiving coil is given by relation (9)

$$\Phi = \int_S \frac{\mu m_z}{4\pi}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right) ds. \tag{A5}$$

Taking into account that $ds = \rho d\phi d\rho$ allows relation (A5) to be transformed into the following form

$$\Phi = \frac{\mu m_z}{4\pi}\int_0^R \int_0^{2\pi} \left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right) \rho d\varphi d\rho. \tag{A6}$$

The distance $r$ between the magnetic dipole and a current point of the area of the receiving coil is given by the equation (see Fig. 2c)

$$r = \left(\rho^2 + Z^2\right)^{1/2}. \tag{A7}$$

The problem is axially symmetrical such that the double integral in (A6) reduces to

$$\Phi = \frac{\mu m_z}{2}\int_0^R \left(\frac{3Z^2}{\left(\rho^2 + Z^2\right)^{5/2}} - \frac{1}{\left(\rho^2 + Z^2\right)^{3/2}}\right) \rho d\rho. \tag{A8}$$

The solution is of the form

$$\Phi = \frac{\mu m_z}{2}\left[\left(\rho^2 + Z^2\right)^{-1/2} - Z^2\left(\rho^2 + Z^2\right)^{-3/2}\right]\Bigg|_0^R. \tag{A9}$$

Introducing the limits of integration, one obtains

$$\Phi = \frac{\mu m_z}{2}\left[\frac{1}{\left(R^2 + Z^2\right)^{1/2}} - \frac{Z^2}{\left(R^2 + Z^2\right)^{3/2}}\right], \tag{A10}$$

which after rearranging, finally yields the relation describing the flux through the receiving coil of radius $R$:

$$\Phi = \frac{\mu m_z}{2}\left[\frac{R^2}{\left(R^2 + Z^2\right)^{3/2}}\right]. \tag{A11}$$

The notational conventions are clarified in Figure 2c. Hence, the relationship between the moment of the magnetic dipole, and associated flux, can be written in the form of equation (10).

## Appendix 3
## Remarks on the Inverse Theory

Inverse theory is an organized set of mathematical and statistical techniques for retrieving useful information about a physical system from controlled observations of the system (Oristalgio, Blok 1995). It is directly concerned with the analysis of experimental data, the fitting of mathematical models to these data by estimating the unknown parameters of these models, and optimal experimental design. As a matter of fact, anyone that has fitted a line to a set of numerical data has practised inverse theory. The level of application of the inverse theory may range from the simple straight-line fitting to more sophisticated problems. In an abstract form, the inverse problem can be cast simply as the solution of a set of non-linear equations:

$$F_k[\mathbf{m}] - d_k = 0, \qquad k = 1, ..., K, \tag{A12}$$

where $m = [m_1, m_2, m_3, ..., m_N]^T$ is a model of $N$ model parameters, $F_k$ is a (non-linear) function that maps the model to the data $d_k$, with $K$ the total number of data points. Note that the displacement vector is the model parameter in this study. Equation (A12) can be treated by standard methods for the solution of non-linear equations. The most popular of these certainly are the iterative methods. Among them, the Newton iterative method is obtained by expanding $F_k[\mathbf{m}]$, with

$$F_k\left[\mathbf{m}^{(i)} + \delta\mathbf{m}^{(i)}\right] - d_k = 0, \qquad k = 1, ..., K \tag{A13}$$

in a multi-dimensional first-order Taylor series about the current estimate $\mathbf{m}^{(i)}$

$$F_k\left[\mathbf{m}^{(i)}\right] + \sum_{n=1}^{N} \frac{\partial F_k}{\partial m_n} \delta m_n^{(i)} = d_k. \tag{A14}$$

Rearranging this expression gives a set of $K \times N$ equations to be solved for the model perturbations $\delta m_n$

$$\sum_{n=1}^{N} \frac{\partial F_k}{\partial m_n} \delta m_n^{(i)} = d_k - F_k\left[\mathbf{m}^{(i)}\right]. \tag{A15}$$

The right-hand side is exactly the error (residuum) at the $i^{\text{th}}$ iteration.

$$e_k^{(i)} = d_k - F_k\left[\mathbf{m}^{(i)}\right] = \delta d_k^{(i)}. \tag{A16}$$

In matrix form this gives

$$\mathbf{S}\delta\mathbf{m} = \delta\mathbf{d}, \tag{A17}$$

where $\mathbf{S}$ is the matrix of partial derivatives. The superscripts indicating the iteration have been omitted here. A new update for the adopted model is obtained as $\mathbf{m}_{new} \leftarrow \mathbf{m}_{old} + \delta\mathbf{m}$ after solving for $\delta\mathbf{m}$. This process is repeated until convergence (or aborted if convergence does not appear). In case the number of measurements is greater than the number of model parameters the $K \times N$ system is overdetermined, and a solution is computed from the normal equations

$$\mathbf{S}^H\mathbf{S}\delta\mathbf{m} = \mathbf{S}^H\delta\mathbf{d}. \tag{A18}$$

The least squares solution is given by

$$\delta\mathbf{m} = \left(\mathbf{S}^H\mathbf{S}\right)^{-1}\mathbf{S}^H\delta\mathbf{d}, \tag{A19}$$

where the superscript $H$ denotes the Hermitian matrix. A regularized least square solution (Oristalgio, Blok 1995, Janczulewicz, Wtorek 2003, Janczulewicz, Wtorek 2003a) instead is computed according to

$$\delta\mathbf{m} = \left(\mathbf{S}^H\mathbf{S} + \lambda^2\mathbf{R}^H\mathbf{R}\right)^{-1}\mathbf{S}^H\delta\mathbf{d}, \tag{A20}$$

where $\lambda$ is the regularization coefficient, $\mathbf{R}$ is the regularization matrix. A possible choice is $\mathbf{R} = \mathbf{I}$.

**Example**

If the inverse problem can be represented by means of the explicit linear equation $\mathbf{d} = \mathbf{G}\mathbf{m}$ it is said to be linear. Here, $\mathbf{d}$, $\mathbf{G}$, and $\mathbf{m}$, stand for the data, the model, and the parameters, respectively. The problem then is to determine $\mathbf{m}$ given the data $\mathbf{d}$ and the assumed model $\mathbf{G}$. Let us consider an example of linear regression, a problem encountered in a variety of applications. For a collection of $n$ data pairs $\{(x_1, y_1), ..., (x_n, y_n)\}$ we are looking for a line $y = a + bx$. Hence, the model is described by only two parameters. For $n > m$, where $m$ is the number of parameters considered, the problem is overdetermined, i.e. number of measurements is larger than the number of unknowns. As the matrix $\mathbf{G}$ is no longer square one may compute a solution by means of the generalized inverse

$$\mathbf{m} = \left(\mathbf{G}^T\mathbf{G}\right)^{-1}\mathbf{G}^T\mathbf{d}. \tag{A21}$$

The relation between the data and the model parameters is described by the matrix equation

$$\begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{A22}$$

In the regression problem considered here, $\mathbf{G}$ and $\mathbf{G}^T$ are given by

$$\mathbf{G} = \begin{bmatrix} 1 & x_1 \\ & \cdot \\ & \cdot \\ 1 & x_n \end{bmatrix}, \quad \mathbf{G}^T = \begin{bmatrix} 1 & \cdot & \cdot & 1 \\ x_1 & \cdot & \cdot & x_n \end{bmatrix}. \tag{A23}$$

such that

$$\mathbf{G}^T\mathbf{G} = \begin{bmatrix} 1 & \cdot & \cdot & 1 \\ x_1 & \cdot & \cdot & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum\limits_{i=1}^{n} x_i \\ \sum\limits_{i=1}^{n} x_i & \sum\limits_{i=1}^{n} x_i^2 \end{bmatrix} \tag{A24}$$

while the inverse $(\mathbf{G}^T\mathbf{G})^{-1}$ can be computed analytically

$$\left(\mathbf{G}^T\mathbf{G}\right)^{-1} = \frac{1}{n\sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2} \begin{bmatrix} \sum\limits_{i=1}^{n} x_i^2 & -\sum\limits_{i=1}^{n} x_i \\ -\sum\limits_{i=1}^{n} x_i & n \end{bmatrix}. \tag{A25}$$

The vector $\mathbf{G}^T\mathbf{d}$ is determined by

$$\mathbf{G}^T\mathbf{d} = \begin{bmatrix} 1 & \cdot & \cdot & 1 \\ x_1 & \cdot & \cdot & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} \sum\limits_{i=1}^{n} y_i \\ \sum\limits_{i=1}^{n} x_i y_i \end{bmatrix}. \tag{A26}$$

Finally, the parameters of the fitted line are computed from the matrix relation

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{n\sum\limits_{i=1}^{n} x_i^2 - \left(\sum\limits_{i=1}^{n} x_i\right)^2} \begin{bmatrix} \sum\limits_{i=1}^{n} x_i^2 & -\sum\limits_{i=1}^{n} x_i \\ -\sum\limits_{i=1}^{n} x_i & n \end{bmatrix} \begin{bmatrix} \sum\limits_{i=1}^{n} y_i \\ \sum\limits_{i=1}^{n} x_i y_i \end{bmatrix}. \tag{A27}$$

Each data pair satisfies the line relation to a certain degree, i.e. $y_i = a + bx_i + \varepsilon_i$, where $\varepsilon_i$ is the vertical distance between $i^{\text{th}}$ data point and the regression line. The quantity $\varepsilon_i$ is named the residual, misfit, or prediction error.

Appropriate values for the parameters $a$ and $b$ are computed by a least squares method. In doing so, the error $\varepsilon$ will be minimised by determining the values for the parameters $a$ and $b$, such as to minimize the sum over the squared residuals

$$\varepsilon = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2. \tag{A28}$$

Differentiating $\varepsilon$ with respect to $a$ and $b$, and setting the derivatives to zero, the following equations are recovered

$$\frac{\partial \varepsilon}{\partial a} = 2\sum_{i=1}^{n} (y_i - a - bx_i)(-1) = 0; \tag{A29}$$

$$\frac{\partial \varepsilon}{\partial b} = 2\sum_{i=1}^{n} (y_i - a - bx_i)(-x_i) = 0. \tag{A30}$$

The parameters $a$, $b$ can be computed directly from (A29) and (A30)

$$a = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}; \quad b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}. \tag{A31}$$

A comparison of the results obtained by these two methods is left for the reader.

## References


Brandt S. (1988), *Data Analysis, Statistical and Computational Methods for Scientists and Engineers*, (Polish edition), PWN, Warsaw.

Dunajski Z. (1990), *Biomagnetism* (in Polish), WKiŁ, Warsaw.

Jackson J. D. (1982), *Classical Electrodynamics* (Polish edition), PWN, Warsaw.

Janczulewicz A., Wtorek J. (2003), Gauss-Newton Reconstruction – One Step vs. Iterative, *XIII Konf. Naukowa Biocybernetyka i Inżynieria Biomedyczna*, Gdańsk, 748–753.

Janczulewicz A, Wtorek J. (2003a), An EIT Reconstruction Based on Noisy Data, *I Krajowa Konf. Technologie Informacyjne*, Gdańsk, 17–26.

Januszajtis A. (1982), *Physics for Technical Universities: Fields* (in Polish), WNT, Warsaw.

Kurałowicz Z. (2001), *Lateral Load on Piles in Weak Soils* (in German), University of Kaiserslautern, Research Report.

Kurałowicz Z., Meißner H. (1994), *Laboratory Scale Examination of Lateral Load on Piles in Sandy and Clay Soil* (in German), University of Kaiserslautern, Research Report.

Kurałowicz Z., Wtorek J. (2002), *Examination of the Electromagnetic Method for Determination of Pile Deflection* (in Polish), Research project KBN 3075/T07/2001/21, Final Report, Gdańsk.

Meju M. A. (2001), Geophysical Data Analysis: Understanding Inverse Problem Theory and Practice, *Course Notes Series*, No. 6, Soc. Explor. Geophys.

Oristalgio M., Blok H. (1995), *Wavefield Imaging and Inversion in Electromagnetics and Acoustics*, Centre for Technical Geoscience, ASEE course, Delft.

Weber E. (1957), *Electromagnetic Fields: Theory and Applications*, John Wiley & Sons, Inc. NY.

Wtorek J. (2003), *Electroimpedance Techniques in Medicine* (in Polish), Gdańsk University of Technology Publishing Office, Monograph Series: 43.