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PROBLEM OF AN OPTIMAL DISCRETIZATION IN ACOUSTIC MODELLING

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In this paper two new models of an acoustic source in BEM are proposed. For simplicity a plane axisymmetric source is modelled. Up to now, the models consist of elements of the same dimension. Furthermore, the nodes are equi-spaced on each element. In contrast to these models, the first new one is composed of optimal elements on which the nodes are equi-spaced. The second new model is composed of optimal elements too but the nodes on each element are optimal (Tchebicheff nodes). Numerical calculations pointed out that the quality of the new models is better than the known ones.

List of symbols

$\mu_j, \mu_{o,j}$	break points, optimal break points; j-element $\in [\mu_{j-1}, \mu_j)$, $j = 1, 2, \ldots, n_j$, j-element in BEM is equivalent j-subinterval in mathematics,
$\nu_i, \nu_{e,i}, \nu_{T,i}$	nodes: arbitrary, equi-spaced, Tchebicheff separate numbered on each element; $i = 0, 1, \ldots, n_{ij}$,
$\mathsf{P}_{q_j}(x)$	polynomial of q_j-degree defined on j-subinterval; $q_j = n_{ij}$,
$\mathsf{P}_q(x)$	piecewise polynomial of q-degree defined on $[a, b]$ interval; $q = \max\limits_j q_j$,
$f(x), \tilde{f}(x)$	any given function, interpolating function,
$M_{\mathsf{P};n_j,n_{ij}+1}$	q-degree model with n_j-elements and $n_{ij}+1$ nodes on each j-element,
$\\|\ldots\\|_\infty$	norm of $C[\mu_{j-1}, \mu_j]$ space (Tchebicheff norm),
$\mathsf{O}(\ldots)$	Ref. [10] p. 494,
$f_{i..n}$	n-th divided difference of the function $f(x)$ at the nodes ν_i, Refs. [2], [9] p. 193,
$P_i(x)$	finite product, Refs. [2], [9] p. 193,
x_b	radius of the membrane,
$\widehat{T}_n(x)$	Thebisheff polynomials of n-th degree, Ref. [10],
$f^{(p)}(x)$	(p) derivative of $f(x)$.

1. Introduction

The studies show that the conventional BEM yields reliable results if the model of the acoustic source is composed of many elements (Ref. [8]). Using too many elements is a burden to the user and also it is not efficient computationally. This difficulty may be alleviated in one of the following three ways.

1. Application of optimal nodes

In almost all the early papers, equispaced nodes were selected on the elements. Furthermore, the elements had the same dimension, see e.g. Ref. [8]. But in Refs. [2, 3, 4, 5, 7] optimal nodes were proposed. They were found applying the Tchebicheff theorem where the optimal nodes are equal to the zeros of the Tchebicheff polynomials.

2. Application of optimal elements

Up to date, the acoustic boundary was discretized on elements of the same dimension. If the model was poor quality, a few elements or every one was subdivided into smaller elements. In this way the quality of the model increases. Thereby a good quality model consists of too many elements. To avoid the proliferation of the elements, in this paper an irregular/optimal discretization is proposed. The idea of optimal discretization is based on the minimization of the Tchebicheff metric of the functional space which is composed of the cross-section function of the source and the cross-section function of the model. One can gauge this idea by applying the theorem given in Ref. [1] p. 189. By this means a new model with optimal elements is obtained (in other words, an optimal model in the sense of discretization).

3. Application of optimal nodes on each optimal element

In this case the idea of optimal elements and the idea of optimal nodes are connected. Note that the model with optimal nodes contains elements of the same dimension. On the contrary, the model with optimal elements contains equi-spaced nodes on each element.

A next new model will be built basing on both the optimal elements and the optimal nodes; hereafter it will be called an "optimal" model. Remark that such a model is not optimal in a mathematical sense. This is because the optimal nodes were found assuming fixed boundaries of the elements and *vice versa.*

From the mathematical point of view, the optimal model should be derived minimizing the metric of the functional space by the change both of the break points and the nodes. This problem will not be solved in this paper.

In order to demonstrate the validity of new models, their quality is compared with two known models (the comparative ones). The first of them is the regular model; it contains elements of the same dimension and equi-spaced nodes on each element. The second model contains elements of the same dimension and optimal nodes (it is an optimal model in the sense of distribution of the nodes), Ref. [7].

2. Piecewise polynomial interpolation theory

Let $f(x)$ be any given function. The mathematical aim of this paper is to construct an interpolating function $\tilde{f}(x)$ which is in the form of the piecewise polynomial and which satisfies the Lagrange's interpolation condition, Ref. [9].

Let Δ_μ be any partition of the $[a, b]$ interval, i.e.,

$$\Delta_\mu: \quad a = \mu_0 < \mu_1 < \ldots < \mu_{j-1} < \mu_j < \ldots < \mu_{n_j} = b, \qquad j = 1, 2, \ldots, n_j, \tag{2.1}$$

and Δ_ν be any arbitrary partition of the j-subinterval, $x \in [\mu_{j-1}, \mu_j)$, Fig. 1,

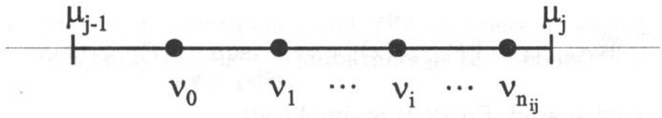

Fig. 1. General distribution of the nodes on the j-subinterval.

$$\Delta_\nu : \qquad \mu_{j-1} \le \nu_0 < \nu_1 < \ldots < \nu_{i-1} < \nu_i < \ldots < \nu_{n_{ij}} \le \mu_j , \tag{2.2}$$

where n_{ij} may be different on each j-subinterval.

Furthermore, the set of the nodal values $\{f(\nu_i)\}_0^{n_{ij}} = \{f_i\}_0^{n_{ij}}$ should be known.

2.1. Arbitrary selection of the nodes

Let q_j-degree interpolating polynomials $\mathsf{P}_{q_j}(x)$ be defined on each j-subinterval,

$$\tilde{f}_{j,n_{ij}}(x) \equiv \mathsf{P}_{q_j}(x) = \sum_{i=0}^{n_{ij}} f_{0..i} P_i(x), \qquad x \in [\mu_{j-1}, \mu_j), \quad q_j = n_{ij} . \tag{2.3}$$

Then the q-degree piecewise polynomial $\mathsf{P}_q(x)$ on the $[a,b]$ interval is defined by:

$$\tilde{f}_{n_j,n_{ij}}(x) \equiv \mathsf{P}_q(x) = \mathsf{P}_{q_j}(x), \qquad j = 1,2,\ldots,n_j , \quad q = \max_j q_j . \tag{2.4}$$

The polynomial $\mathsf{P}_{q_j}(x)$ on the j-subinterval fulfils the interpolation condition

$$\tilde{f}_{j,n_{ij}}(\nu_i) \equiv f(\nu_i) \equiv \mathsf{P}_{q_j}(\nu_i), \qquad j = 1,2,\ldots,n_j , \quad \nu_i \in [\mu_{j-1}, \mu_j). \tag{2.5}$$

The error of the piecewise polynomial interpolation can be expressed similarly as the polynomial interpolation, Ref. [6]. In this case, the error at every point of the j-subinterval can be written as follows:

$$E_{\mathsf{P};j,n_{ij}+1}(x) = f(x) - \mathsf{P}_{q_j}(x), \qquad x \in [\mu_{j-1}, \mu_j), \tag{2.6}$$

and the error of the piecewise polynomial interpolation can be expressed by the formula

$$E_{\mathsf{P};n_j,n_{ij}+1}(x) = f(x) - \mathsf{P}_q(x), \qquad x \in [a,b]. \tag{2.7}$$

Because of the reasons described in Ref. [6], only the estimation of $E_{\mathsf{P};j,n_{ij}+1}(x)$ at the point x can be calculated,

$$\|E_{\mathsf{P};j,n_{ij}+1}(x)\|_{\infty,f} \le \frac{\mathfrak{M}_{f,n_{ij}+1}}{(n_{ij}+1)!} |P_{n_{ij}+1}(x)|, \tag{2.8}$$

where

$$\mathfrak{M}_{f,n_{ij}+1} = \|f^{(n_{ij}+1)}(x)\|_\infty = \sup_{x\in[\mu_{j-1},\mu_j]} |f^{(n_{ij}+1)}(x)|. \tag{2.9}$$

In practice the estimation of $E_{\mathsf{P};j,n_{ij}+1}(x)$ plays a minor part. Then two estimations are introduced: the estimation over the j-subinterval and over the $[a,b]$ interval, i.e.

$$\|E_{\mathsf{P};j,n_{ij}+1}\|_\infty = \frac{\mathfrak{M}_{f,n_{ij}+1}}{(n_{ij}+1)!} \mathfrak{M}_{P,n_{ij}+1} , \qquad x \in [\mu_{j-1}, \mu_j), \tag{2.10}$$

$$\mathsf{E}_{\mathsf{P};n_j,n_{ij}+1} \equiv \|E_{\mathsf{P};n_j,n_{ij}+1}\|_\infty = \max_j \|E_{\mathsf{P};j,n_{ij}+1}\|_\infty , \qquad x \in [a,b], \tag{2.11}$$

where

$$\mathfrak{M}_{P,n_{ij}+1} = \|P_{n_{ij}+1}(x)\|_\infty = \sup_{x\in[\mu_{j-1},\mu_j]} |P_{n_{ij}+1}(x)|. \tag{2.12}$$

If the nodes are equi-spaced, Eq. (2.3) is simplified

$$\mathsf{P}_{q_j}(x) = \sum_{i=0}^{n_{ij}} f_{e,i}\,\mathcal{N}_i(x), \qquad x \in [\mu_{j-1},\mu_j), \tag{2.13}$$

where $f_{e,i} = f(\nu_{e,i})$ and $\mathcal{N}_i(x)$ – see Ref. [8].

Consequently the next formulae are simplified too; for further details see Ref. [11].

Assuming that $f(x)$ is continuous and $f^{(n_{ij}+1)}(x)$ is continuous or it has a finite number of singularities of different kinds, the achievable error estimation over the $[a, b]$ interval is given by the formula (Ref. [11] s. 121)

$$\mathsf{E}_{\mathsf{P};n_j,n_{ij}+1} = \mathsf{O}(1/n_j^{n_{ij}+1}). \tag{2.14}$$

Note that Eq. (2.14) does not say anything about the distribution of the break points and nodes.

2.2. Optimal nodes

The optimal nodes, $\nu_{T,i}$, are better for the interpolation than the equi-spaced ones; a detailed discussions of the mathematical aspects of the nodes $\nu_{T,i}$ can be found in Refs. [4, 11]. In this case the piecewise polynomial Eq. (2.3) can be expressed by the formula

$$\mathsf{P}_{q_i}(x) = \sum_{i=0}^{n_{ij}} f_{0..i}\widehat{T}_i(x), \qquad x \in [\mu_{j-1},\mu_j). \tag{2.15}$$

At the considerations given above, the error estimation over the $[a, b]$ interval is given by the formula (Ref. [11] s. 121)

$$\mathsf{E}_{\mathsf{P};n_j,n_{ij}+1} = \mathsf{O}(\log n_j/n_j^{n_{ij}+1}), \tag{2.16}$$

which is nearly as good as (2.14).

2.3. Optimal elements

An error estimation of the best (optimal) interpolation of the function $f(x)$ by the function $\mathsf{P}_q(x)$ is given by Eq. (2.14). It can be achieved by a special distribution of the break points. Such a distribution can be found minimising the metric of the functional space. This space consists of two functions, i.e. $f(x)$ and $\mathsf{P}_q(x)$. Next, the Tchebicheff metric $\|f(x) - \mathsf{P}_q(x)\|_\infty$ is assumed and in this way the functional space $C[a, b]$ is defined. Minimizing this metric the optimal break points $\mu_{O,j}$ are obtained; they make up the optimal elements.

In this paper the main problem is the interpolation with optimal $\mu_{O,j}$. From the mathematical point of view, the problem of the distribution of the optimal break points is considered in Refs. [1]. The distribution of $\mu_{O,j}$ is defined by the following theorem

(Ref. [1] p. 189): let $f(x)$ be continuous and $f^{(p)}(x)$ exists at each point $x \in [a,b]$, $p \in [1,\infty)$, but $f^{(p)}(x)$ may have got a finite number of discontinuous of the different kind. If

$$B = \int_a^b |f^{(p)}(x)|^{1/p}\, dx < \infty, \tag{2.17}$$

then the achievable error estimation over the $[a,b]$ interval is given by Eq. (2.14). Such an estimation assures the break points $\mu_{O,j}$ which fulfil the following condition:

$$\int_a^{\mu_{O,j}} |f^{(p)}|^{1/p}\, dx = \frac{j-1}{n_j-1} B, \qquad j = 1,2,\ldots,n_j\,, \tag{2.18}$$

where $\mu_{O,0} = a$, $\mu_{O,n_j} = b$.

3. Multi-element models of the source

3.1. Acoustic source

For simplicity, one considers a fully axisymmetric source, i.e. both the geometry and acoustic variables are independent of the angle of revolution. The membrane vibrating with an axisymmetric mode placed in an infinite baffle is chosen as the source. In this case, the function $f(x)$ may be interpreted as a cross-section of the source, hence $a = 0$, $b = x_b$; an explicite form of $f(x)$ was derived in Ref. [4]. An acoustic field of the source (an exact acoustic field), i.e. the directivity function $Q(k,\gamma)$ = function $(f(x))$ and the acoustic pressure near the source $p(k,H,x_P)$ = function $(f(x))$, was described extensively in Ref. [12] p. 594.

3.2. Models of the source

Hereafter, the function $\tilde{f}(x)$ ought to be interpreted as a cross-section of the model. Then the model M_{P} is given by Eq. (2.4): $M_{\mathsf{P}} \equiv \mathsf{P}_q(x) \equiv \tilde{f}_{n_j,n_{ij}}(x)$. It is the q-degree model with n_j elements and n_{ij} nodes on each element; the full symbol is $M_{\mathsf{P};n_j,n_{ij}+1}$. Hereafter the index "$n_j, n_{ij}+1$" will be dropped to simplify the notation. If Eq. (2.4) is substituted into the formulae for the exact acoustic field, an acoustic field of M_{P} is obtained; it is denoted by $\widetilde{Q}_{\mathsf{P}}(k,\gamma)$ = function $(\tilde{f}_{n_j,n_{ij}}(x))$ and $\widetilde{p}_{\mathsf{P}}(k,H,x_P)$ = function $(\tilde{f}_{n_j,n_{ij}}(x))$. The error of the model M_{P} constitutes the interpolation error, Eq. (2.7). The estimation of the model error on the $[a,b]$ interval, Eq. (2.11), is assumed to be a direct measure of the model quality. Furthermore, the difference between the exact acoustic field and the model acoustic field

- $\Delta Q_{\mathsf{P}}(k,\gamma) = Q(k,\gamma) - \widetilde{Q}_{\mathsf{P}}(k,\gamma)$,
- $\Delta p_{\mathsf{P}}(k,H,x_P) = p(k,H,x_P) - \widetilde{p}_{\mathsf{P}}(k,H,x_P)$,

may be interpreted as an indirect measure of the model quality; for further details see Ref. [7].

The model M_P with evenly-spaced break points $\mu_{e,j}$ and equi-spaced nodes $\nu_{e,i}$ is called the regular model $M_{\mathsf{P};R;n_j,n_{ij}+1}$ (in Fig. 2 $\mu_{e,j}$ – ∇ and $\nu_{e,i}$ – ■). The model M_P with equi-spaced break points $\mu_{e,j}$ and optimal nodes $\nu_{T,i}$ is called the model with optimal nodes $M_{\mathsf{P};O-N;n_j,n_{ij}+1}$ (in Fig. 2 $\mu_{e,j}$ – ∇ and $\nu_{T,i}$ – •). In this paper they are two comparative models.

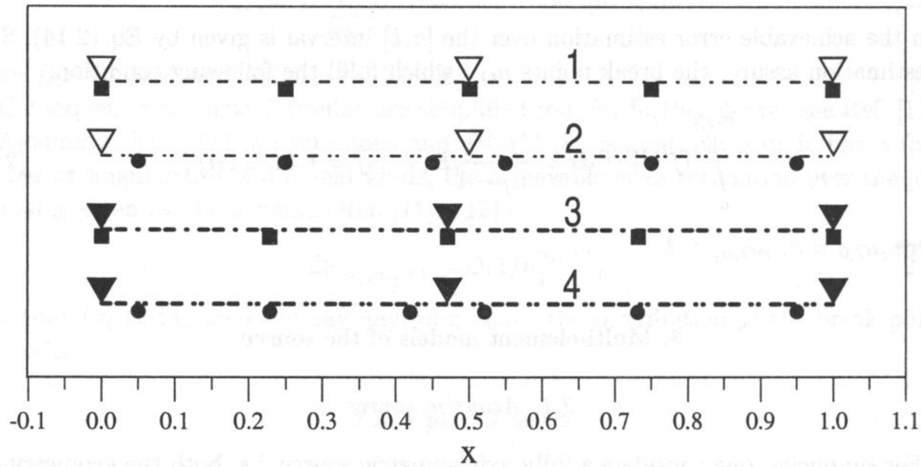

Fig. 2. Distribution of the nodes and the break points at the nodes: 1 – $M_{\mathsf{P};R;2,3}$, 2 – $M_{\mathsf{P};O-N;2,3}$, 3 – $M_{\mathsf{P};O-D;2,3}$, 4 – $M_{\mathsf{P};O;2,3}$.

3.2.1. Model with optimal elements. The model M_P with optimal break points $\mu_{O,j}$ and equi-spaced nodes $\nu_{e,i}$ is called the model with optimal elements $M_{\mathsf{P};O-D;n_j,n_{ij}+1}$ (in Fig. 2 $\mu_{O,j}$ – ▼ and $\nu_{e,i}$ – ■). The $M_{\mathsf{P};O-D}$ constitutes Eq. (2.4) but the boundaries of the elements ought to be calculated from Eq. (2.18). The acoustic field $\widetilde{Q}_{\mathsf{P};O-D}(k,\gamma)$, $\widetilde{p}_{\mathsf{P};O-D}(k,H,x_P)$ and the estimation of the model error on the $[a,b]$ interval $\mathsf{E}_{\mathsf{P};O-D}$ of $M_{\mathsf{P};O-D}$ can be derived quite similarly as those of the general model M_P.

3.2.2. "Optimal" model. The model M_P with optimal break points $\mu_{O,j}$ and optimal nodes $\nu_{T,i}$ is called the "optimal" model $M_{\mathsf{P};O;n_j,n_{ij}+1}$ (or the model with optimal discretization and optimal nodes); (in Fig. 2 $\mu_{O,j}$ – ▼ and $\nu_{T,i}$ – •). The $M_{\mathsf{P};O}$ constitutes Eq. (2.4) via Eq. (2.15) but the boundaries of the elements ought to be calculated from Eq. (2.18). As to the $\widetilde{Q}_{\mathsf{P};O}(k,\gamma)$, $\widetilde{p}_{\mathsf{P};O}(k,H,x_P)$ and $\mathsf{E}_{\mathsf{P};O}$, see the general model M_P.

Both the $M_{\mathsf{P};O}$ and $M_{\mathsf{P};O-D}$ models were not applied in BEM up to now; they are new models of the acoustic source.

4. Numerical implementation

The aim of the numerical calculations is a comparison of the quality of the new $M_{\mathsf{P};O-D}$ and $M_{\mathsf{P};O}$ models with those of the comparative models, i.e. $M_{\mathsf{P};R}$ and $M_{\mathsf{P};O-N}$.

To do this, one assumes 2-degree models ($n_{ij} = 2$) with 2 elements ($n_j = 2$) and 3 nodes on each element ($n_{ij} + 1 = 3$):

$M_{\mathsf{P};R;2,3}$ → regular model; lines 1: short + long, cf. Ref. [7],

$M_{\mathsf{P};O-N;2,3}$ → model with evenly-spaced break points and optimal nodes; lines 2: short + short + long, cf. Ref. [7],

$M_{\mathsf{P};O-D;2,3}$ → model with optimal elements and equi-spaced nodes; lines 3: short + long + long (bolted),

$M_{\mathsf{P};O;2,3}$ → model with optimal elements and optimal nodes; lines 4: short + short + long + long (bolted).

In all the figures the same kind of lines relates to the same model, cf. Ref. [7].

5. Calculations, results, conclusions

To verify the quality of $M_{\mathsf{P};O-D}$ and $M_{\mathsf{P};O}$, the direct and the indirect measures of the model quality are calculated. For this purpose the error estimations on the $[a, b]$ interval are done and plotted in Fig. 3A (the model error is not presented). As expected, the error estimations $\mathsf{E}_{\mathsf{P};O-D}$ and $\mathsf{E}_{\mathsf{P};O}$ are less than $\mathsf{E}_{\mathsf{P};R}$. Because the error estimation E_{P} is a direct measure of the model quality then,

The models $M_{\mathsf{P};O-D}$ and $M_{\mathsf{P};O}$ are of better quality than the model $M_{\mathsf{P};R}$ in the sense of the direct measure of the model quality.

This conclusion confirms the existence of the optimal discretization. As can be noticed in Fig. 3A, the model $M_{\mathsf{P};O-D}$ ought to be compared to the model $M_{\mathsf{P};R}$ and $M_{\mathsf{P};O}$ to $M_{\mathsf{P};O-N}$. This conclusion is conceivable because in $M_{\mathsf{P};O-D}/M_{\mathsf{P};R}$ and $M_{\mathsf{P};O}/M_{\mathsf{P};O-N}$, respectively, the nodes are fixed but only the discretization changes. Note that in Fig. 3A

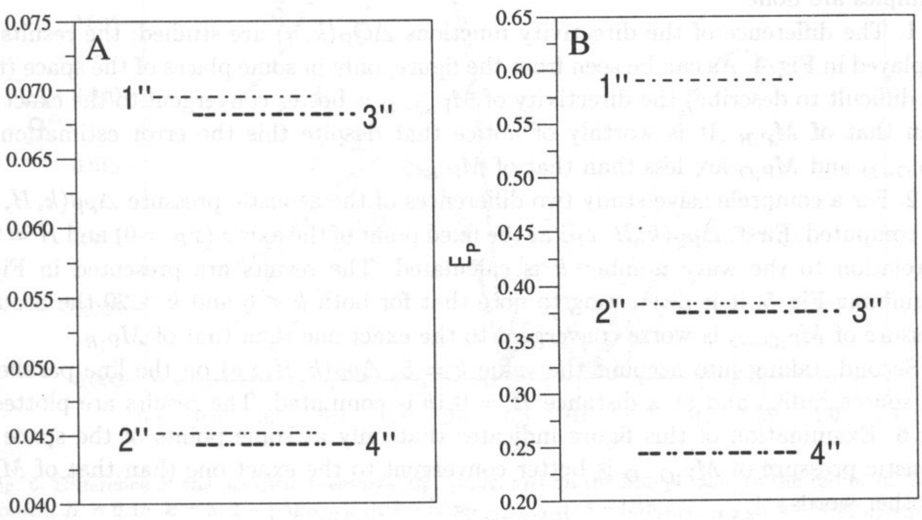

Fig. 3. Estimation of the errors: $1'' \equiv \|E_{\mathsf{P};R;2,3}\|_\infty \equiv \mathsf{E}_{\mathsf{P};R;2,3}$, $2'' \equiv \|E_{\mathsf{P};O-N;2,3}\|_\infty \equiv \mathsf{E}_{\mathsf{P};O-N;2,3}$, $3'' \equiv \|E_{\mathsf{P};O-D;2,3}\|_\infty \equiv \mathsf{E}_{\mathsf{P};O-D;2,3}$, $4'' \equiv \|E_{\mathsf{P};O;2,3}\|_\infty \equiv \mathsf{E}_{\mathsf{P};O;2,3}$.

the sensitivity of E_{P} to the discretization is little visible; better results are presented in the Appendix. Under the circumstances given above, the next conclusions will be given only for $M_{\mathsf{P};O-D}$ and $M_{\mathsf{P};R}$ because for $M_{\mathsf{P};O}$ and $M_{\mathsf{P};O-N}$ they are the same.

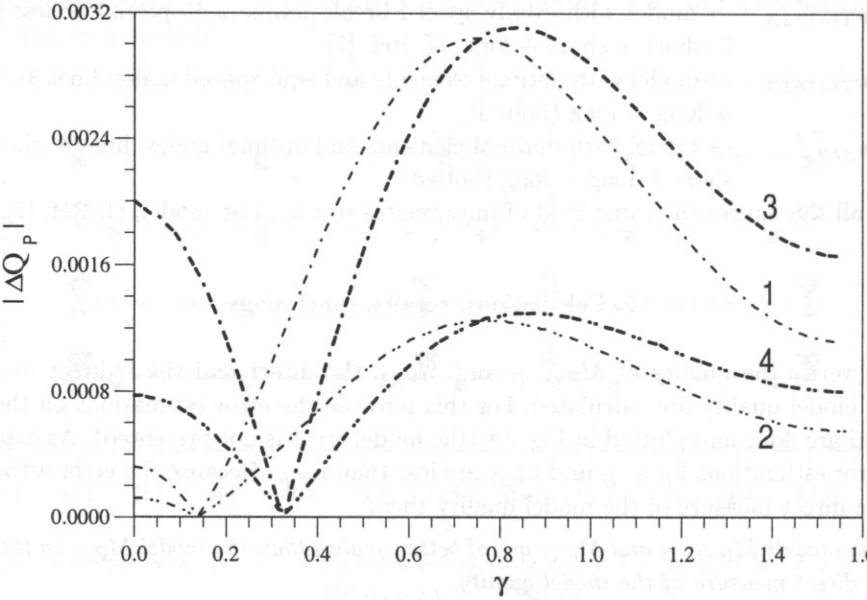

Fig. 4. Difference of the directivity functions $\Delta Q_{\mathsf{P}}(k,\gamma)$, $k=5$; 1 – $|\Delta Q_{\mathsf{P};R;2,3}|$, 2 – $|\Delta Q_{\mathsf{P};O-N;2,3}|$, 3 – $|\Delta Q_{\mathsf{P};O-D;2,3}|$, 4 – $|\Delta Q_{\mathsf{P};O;2,3}|$.

To check the quality of $M_{\mathsf{P};O-D}$ and $M_{\mathsf{P};O}$ estimated by the indirect measure, two examples are done.

1. The difference of the directivity functions $\Delta Q_{\mathsf{P}}(k,\gamma)$ are studied; the results are displayed in Fig. 4. As can be seen from the figure, only in some places of the space (they are difficult to describe) the directivity of $M_{\mathsf{P};O-D}$ is better convergent to the exact one than that of $M_{\mathsf{P};R}$. It is worthly of notice that despite this the error estimations of $M_{\mathsf{P};O-D}$ and $M_{\mathsf{P};O}$ are less than that of $M_{\mathsf{P};R}$.

2. For a comprehensive study two differences of the acoustic pressure $\Delta p_{\mathsf{P}}(k,H,x_P)$ are computed. First, $\Delta p_{\mathsf{P}}(k,H,x_P)$ at the fixed point of the axis z ($x_P=0$) and $H=0.1b$ in relation to the wave number k is calculated. The results are presented in Fig. 5. Examining Fig. 5, it is interesting to note that for both $k<6$ and $k>20$ the acoustic pressure of $M_{\mathsf{P};O-D}$ is worse convergent to the exact one than that of $M_{\mathsf{P};R}$.

Second, taking into account the value $k=5$, $\Delta p_{\mathsf{P}}(k,H,x_P)$ on the line parallel to the source radius and at a distance $H=0.1b$ is computed. The results are plotted in Fig. 6. Examination of this figure indicates that only at some points of the space the acoustic pressure of $M_{\mathsf{P};O-D}$ is better convergent to the exact one than that of $M_{\mathsf{P};R}$. In other words,

The better model derived basing on the direct measure of the quality may not be better in the sense of an indirect measure.

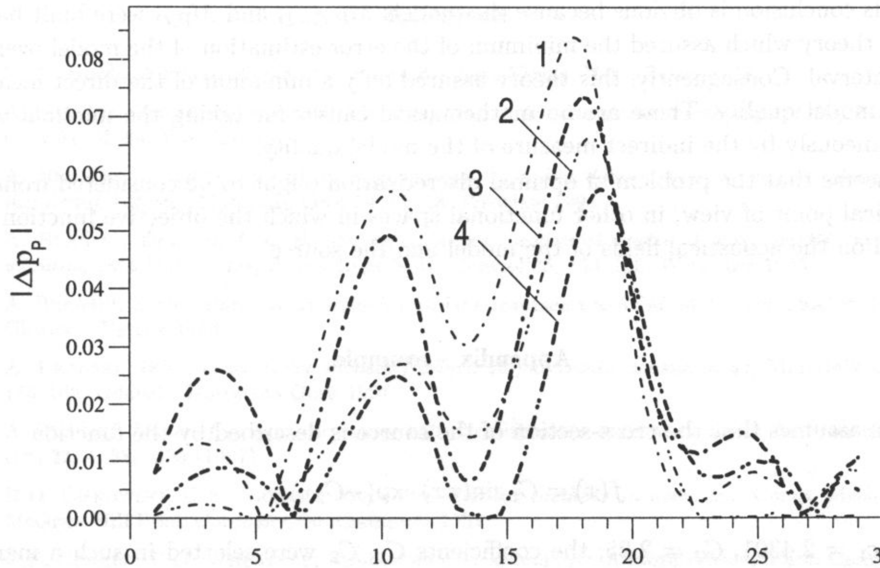

Fig. 5. Difference of the acoustic pressures $\Delta p_{\mathrm{P}}(k, H, x_P)$ at z-axis, $H = 0.1b$; 1 – $|\Delta p_{\mathrm{P};R;2,3}|$, 2 – $|\Delta p_{\mathrm{P};O-N;2,3}|$, 3 – $|\Delta p_{\mathrm{P};O-D;2,3}|$, 4 – $|\Delta p_{\mathrm{P};O;2,3}|$.

Fig. 6. Difference of the acoustic pressures $\Delta p_{\mathrm{P}}(k, H, x_P)$ on the line parallel to the radius of the source, $H = 0.1b$, $k = 5$; 1 – $|\Delta p_{\mathrm{P};R;2,3}|$, 2 – $|\Delta p_{\mathrm{P};O-N;2,3}|$, 3 – $|\Delta p_{\mathrm{P};O-D;2,3}|$, 4 – $|\Delta p_{\mathrm{P};O;2,3}|$.

This conclusion is obvious because the models $M_{\mathsf{P};O-D}$ and $M_{\mathsf{P};O}$ were built basing on the theory which assured the minimum of the error estimation of the model over the $[a, b]$ interval. Consequently, this theory assured only a minimum of the direct measure of the model quality. There are no mathematical causes for taking the minimal value simultaneously by the indirect measure of the model quality.

It seems that the problem of optimal discretization ought to be considered from the acoustical point of view, in other functional spaces in which the objective function will depend on the acoustical fields of the model and the source.

Appendix – example

One assumes that the cross-section of the source is described by the function

$$f(x) = C_1 \sin(\pi x) \exp(-C_2 x),$$

where $c_1 = 2.4301$, $C_2 = 2.25$; the coefficients C_1, C_2 were selected in such a manner that the function reaches its maximum value at the point $x = 0.3$.

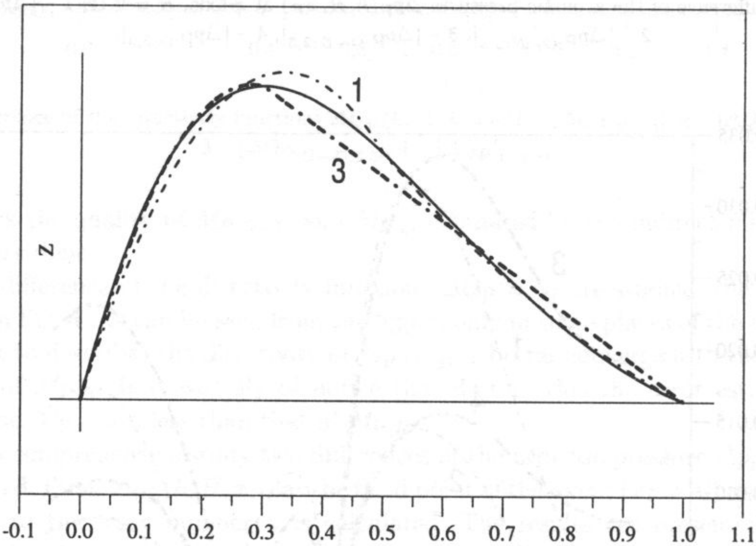

Fig. 7. Cross-sections of the source and of the models: solid line – source, 1 – $M_{\mathsf{P};R;2,3}$, 3 – $M_{\mathsf{P};O-D;2,3}$.

Two models $M_{\mathsf{P};R;2,3}$ and $M_{\mathsf{P};O-D;2,3}$ are compared. The cross-sections of the source and of the models are depicted in Fig. 7 and the error estimations are plotted in Fig. 3B. The latter figure clearly indicates that the difference between $\mathsf{E}_{\mathsf{P};O-D}$ and $\mathsf{E}_{\mathsf{P};R}$ is considerable (similarly between $\mathsf{E}_{\mathsf{P};O}$ and $\mathsf{E}_{\mathsf{P};O-N}$). But the numerical calculations (not given in this paper) confirm the conclusion given above for a membrane.

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