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Wellposedness of optimal control problems for systems with unbounded controls and partially analytic generators

by

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Abstract: Wellposedness of differential and algebraic Riccati equations for control systems with *unbounded* control operators is considered. It is shown that the full-classical Riccati theory is recovered for a class of dynamics, whose generators are partially analytic. Partial analyticity is quantitatively expressed by the validity of the so-called "singular estimates", which is imposed on the composition operator $E^{At}B$ (A is the generator, B is unbounded control operator. This class comprises the PDE coupled systems with hyperbolic and parabolic components. Two illustrative examples are given in the paper: boundary/point control of thermal plates with hyperbolic character and point control of structural acoustic interactions. The latter are described by wave equation coupled at an interface to a plate equation.

Keywords: differential and algebraic Riccati equations, unbounded control operators, singular estimate, thermoelastic plates, structural acoustic interactions

1. Introduction

This paper deals with wellposedness theory for optimal control problems governed by PDEs with *unbounded control actions.* By unbounded, we mean control operators whose domain is either a dense, proper subspace of the state space or the range of control operator lies outside that space. The interest of considering unbounded control operators has been spurred by numerous engineering/technological applications where such operators are canonical models for actuation. Typical examples include boundary and point controls which more recently have found wide range of applicability in smart materials and, more

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generally, "smart technology" — see Hoffman and Botkin (2000), Dimitriadis, Fuller and Rogers (1991) and references therein.

In spite of large popularity of smart materials at the technological level, relatively little has been known from the point point of view of the mathematical control theory. The reason for this is simple: the unboundedness of control operators has drastic effect on the underlying PDE dynamics and many properties which hold in the context of classical, by now, "bounded control" problems are no longer valid with "unbounded controls". This fact, recognized more than 20 years ago, Balakrishnan (1975), has led to and stimulated a wide range of research in the area of "control theory of PDE with unbounded controls" — see Balakrishnan (1975), Bensoussan, Da Prato, Delfour and Mitter (1993), Lasiecka and Triggiani (2001) and references therein. In fact, the theory of optimal control and related Riccati equations, in the context of PDEs with unbounded controls, has attracted considerable attention in recent years — see books Bensoussan, Da Prato, Delfour and Mitter (1993), Lasiecka and Triggiani (2001) and references therein. The main difficulty and obstacle with respect to the classical LQR theory, Balakrishnan (1975), Lions (1968), is, of course, potential unboundedness of the so called "gain operator" which is the *formal* product of two operators: B^* — the adjoint to control operator B and P — the Riccati operator. Since the gain operator B^*P enters the nonlinear term in the Riccati equation, its wellposedness (e.g. density of the domain, boundedness, etc.) is the critical ingredient of the analysis.

In order to obtain reasonably strong results pertaining to solvability of Riccati equations, one needs to counteract the unboundedness of control operator by exploiting other properties of the dynamics. In this context the theory of LQR problems for *analytic* generators came forward in the early eighties producing an array of results which show that the gain operator B^*P, in the case of analytic dynamics, is actually *bounded* — see Bensoussan, Da Prato, Delfour and Mitter (1993), Lasiecka and Triggiani (2001) and references therein. This was the main critical step in proving almost classical solvability of Riccati equations as well as pointwise feedback synthesis for the optimal control arising in the analytic PDE dynamics.

In contrast with analytic dynamics, other PDE models, including hyperbolic, do not display the same regularizing effect. In fact, for hyperbolic dynamics one typically obtains gain operators B^*P which are intrinsincally *unbounded*, Lasiecka and Triggiani (1991), Flandoli, Lasiecka and Triggiani (1988), Da Prato, Lasiecka and Triggiani (1986). This leads, in turn, to nonstandard Riccati equations whose proper formulation requires special extensions of nonlinear terms in the equation, Barbu, Lasiecka and Triggiani (2000), Triggiani (1997). Riccati equation is no longer satisfied in a regular sense (see counterexamples in Weiss and Zwart, 1998, Triggiani, 1997).

In view of this picture one may ask the following rather natural question: what are other (than analytic) dynamics which still preserve the wellposedness with unbounded controls? It turns out that the dynamics with the so called "singular estimate" possesses this desired property. In mathematical terms this property is expressed by the so called "singular estimate" for "transfer function" of the system $y' = Ay + Bu$. This is to say one aims to obtain an estimate of the type

$$\|e^{At}B\|_{\mathcal{L}(H)} = O\left(\frac{1}{t^\gamma}\right); \quad 0 < t \leq 1$$

with a suitable value of $0 < \gamma < 1$. Here e^{At} denotes a semigroup generated by A on a Hilbert space H while B denotes an (unbounded in H) operator $U \to D(A^*)'$.

It is readily seen that the above estimate is satisfied for analytic semigroups with control operators B which are *relatively bounded* with respect to A. However, the main point we wish to make is that the class of systems with "singular estimate" is much larger than the analytic class. In fact, this class includes the so called "partially analytic" dynamics which typically arise in coupled PDE systems where only one component of the system is modeled by an analytic semigroup. For such dynamics, the main challenge is to show that the regularizing effect of analyticity is propagated from this one component onto the entire structure. Qualitatively this is described by singular estimate which amounts to saying that the "unboundedness of control operator" is offset by the dynamics once we are away from the origin.

The main goal of the present paper is twofold: (i) to present a brief exposition of the theory for partially analytic control problems and, (ii) to illustrate this theory with several concrete PDE control systems of physical interest, which fall into this class. We shall deal with both fnite and infinite horizon control problems. In the context of infinite horizon problems, questions related to stability and stabilizability will be discussed as well.

We shall begin (Section 2) by recalling abstract results on wellposedness of feedback synthesis and on unique solvability of associated Riccati Equations for *partially analytic systems.*

Sections 3 and 4 serve as an illustration of abstract theory by means of concrete PDE control systems with boundary or point control. We shall focus on two particular models: system of thermoelasticity and structural acoustic interactions. In Section 3 we consider boundary control problems associated with systems of thermoelasticity. As is known, systems of thermoelasticity combine analytic and hyperbolic properties and therefore serve as a canonical example of the model to be tested within our framework. It is shown in Section 3 that singular estimate holds for this class of problems. This allows to apply abstract results formulated in Section 2 which pertain to solvability of Riccati equations.

Section 4 deals with a more complex system, which is often referred as structural acoustic interaction Morse and Ingard (1968), Beale (1976). In fact, the mathematical model for structural acoustic interaction comprises a strongly modeled by an analytic semigroup (e.g. structurally damped plates, some thermoelastic plates) then we deal with the situation described above and the model falls, as we shall see, into the category of partially analytic. In order to establish the singular estimate for the structural acoustic model, it may be necessary (depending on the level of analyticity displayed by the plate equation) to introduce an additional boundary damping acting on the interface. This is to say that analyticity alone of one of the components may not be sufficient in order to propagate the desired regularity (singular estimate) onto the entire structure. However, addition of "strong" boundary overdamping will be shown to guarantee the validity of this estimate.

An interesting phenomenon to be observed in the context of infinite horizon problem and related stability issues is that the additional overdamping on the interface, while providing benefits locally in time, may destroy asymptotic stability properties of the overall system. In fact, strong unboundedness propagated by this damping may introduce continuous spectrum (at the point 0) to the spectrum of the generator. This leads to the question: how to remove this instability? We shall show in Section 4 that this can be done by introducing an appropriate static damping on the interface. Thus, the ultimate control model is rather complex, but it possesses all the properties which are needed for the wellposedness of Riccati theory for both finite and infinite horizon control problems associated with structural acoustic interactions.

2. Abstract theory

In this section we consider an abstract formulation of the optimal control problem governed by strongly continous semigroup with *unbounded* control operators. The semigroup in question, along with the control operator, will be eventually assumed to satisfy the "singular" estimate. Our first goal is to collect some of the basic results pertaining to solvability of this type of control problem.

2.1. Formulation of abstract control problem

Let H, U, Z be given Hilbert spaces and let the following operators be given

- A is a generator of C_0 semigroup on H with $D(A) \subset H \subset D(A^*)'$.
- The operator $B : U \to D(A^*)'$ satisfies the following condition:
$$|R(\lambda, A)Bu|_H \leq C_{Re\lambda}|u|_U; \quad \lambda \in \rho(A) \tag{1}$$
where $R(\lambda, A)$ denotes the resolvent operator for A and $\rho(A)$ denotes the resolvent set. Therefore, $B^* \in \mathcal{L}(D(A^*) \to U)$ where $(B^*v, u)_U \equiv (v, Bu)_{D(A^*),D(A^*)'}$
- The operator $R : H \to Z$ is bounded. $F \in L_1(0, T; H)$ is a given element.

With these quantities we consider the following **dynamics**

Associated with (2) is the **functional cost** where T may be finite or infinite:

$$J(u,\mathbf{y}) \equiv \int_0^T [|Ry|_Z^2 + |u|_U^2]\, dt. \tag{3}$$

Optimal Control Problem *Minimize the functional* $J(u,\mathbf{y})$ *for all* $u \in L_2(0,T;U)$ *and* $\mathbf{y} \in L_2(0,T;H)$ *which satisfy (2)*

For the *finite time horizon* problems ($T < \infty$) a standard optimization argument, Barbu (1976), Balakrishnan (1975), Lions (1968) provides existence and uniqueness of the optimal solution $u^0 \in L_2(0,T;U)$; $y^0 \in L_2((0,T);H)$ to the optimal control problem. For the *infinite time horizon* problems, the same conclusion follows provided that the so called FCC condition — *Finite Cost Condition* formulated in Hypothesis 2 (Section 2.4) is satisfied. FCC condition is implied by appropriate stabilizability properties of the dynamics generated by A, B.

Our main aim is to derive the optimal synthesis for the control problem along with a characterization of optimal control via an appropriate Riccati Equation.

2.2. Characterization of the optimal control

Without any further assumptions imposed on the problem, we are in a position to provide explicit formulas for the optimal solution. To do this we introduce the so called solution operator, often also referred as the "control-to-state" map.

$$(L_s u)(t) \equiv \int_s^t e^{A(t-z)} Bu(z)\, dz, \quad 0 \le s \le t \le T. \tag{4}$$

Condition (1) is equivalent to the statement that the control-to-state operator L_s is topologically bounded L_2 in time. This is to say, for all $T < \infty$

$$L_s \in \mathcal{L}(L_2(s,T;U) \to L_2(s,T;H)) \tag{5}$$

or equivalently

$$L_s^* \in \mathcal{L}(L_2(s,T;H) \to L_2(s,T;U)) \tag{6}$$

where

$$(L_s^* f)(t) = B^* \int_t^T e^{A^*(z-t)} f(z)\, dz, \quad 0 \le s \le t;$$
$$\text{and} \quad (L_s^* f)(t) = 0,\ s > t. \tag{7}$$

One can easily verify that the following operators are also bounded

$$[I + L^* R^* R L_s]^{-1} \in \mathcal{L}(L_2(s,T;U));$$

The effect of the deterministic noise is represented by the element

$$\mathcal{F}(t) \equiv \int_0^t e^{A(t-s)} F(s)\, ds$$

and from standard semigroup theory, Pazy (1986), with $F \in L_1(0,T;H)$; $\mathcal{F} \in C([0,T];H)$.

In order to provide an expression for the optimal synthesis it is convenient to introduce evolution operator $\Phi(t,s)$ defined by

$$\Phi(.,s)x = [I + L_s L^* R^* R]^{-1} e^{A(\cdot - s)} x \in L_2(s,T;H) \text{ for } x \in H \tag{9}$$

The evolution operator allows us to define the "Riccati operator" $P(t)$ given by the formula Balakrishnan (1975), Lasiecka and Triggiani (1991), Flandoli, Lasiecka and Triggiani (1988), Lasiecka and Triggiani (2001),

$$P(t) \equiv \int_t^T e^{A^*(s-t)} R^* R \Phi(s,t)\, ds. \tag{10}$$

From (8), (9) we infer that $P(\cdot) \in \mathcal{L}(H, C([0,T],H))$. Moreover, it is standard to show that $P(t)$ is selfadjoint and positive on H, Flandoli, Lasiecka and Triggiani (1988). We define next the "adjoint state" $p(t) \equiv \int_t^T e^{A^*(s-t)} R^* R \mathbf{y}^0(s)\, ds \in C([0,T];H)$. Finally, we define the variable

$$r(t) \equiv p(t) - P(t)\mathbf{y}^0(t) \in L_2(0,T;H). \tag{11}$$

Since by (7) $B^* p = L^* R^* R \mathbf{y}^0$, we infer by the virtue of (6) that

$$B^*[P(t)\mathbf{y}^0(t) + r(t)] \in U; \text{ a.e in } t.$$

By using the above notation one obtains, Lasiecka and Triggiani (2001), vol I, sect. 6.2.3, the explicit formulas for the optimal control. These are collected in the Lemma below:

LEMMA 1 *With reference to the control problem stated in (3), and arbitrary initial condition $\mathbf{y}_0 \in H$ and $F \in L_1(0,T;H)$, there exists a unique optimal pair denoted by $(\mathbf{u}^0, \mathbf{y}^0)$ with the following properties:*

(i) $u^0 = -[I + L^* R^* R L]^{-1} L^* R^* R[e^{A(.)}\mathbf{y}_0 + \mathcal{F}] \in L_2(0,T;U)$

(ii) $\mathbf{y}^0 = [I + L L^* R^* R]^{-1}[e^{A(.)}\mathbf{y}_0 + \mathcal{F}] \in L_2(0,T;H)$

(iii) $u^0(t) = -B^*[P(t)\mathbf{y}^0(t) + r(t)]$*; a.e in* $t \in (0,T)$*, where* $P(t), r(t)$ *are given in (10), (11).*

REMARK 1 *In the case $T = \infty$ the formulas in Lemma 1 still hold under the additional assumption of exponential stability of e^{At}.*

One of the main goals in optimal control theory is to provide an independent the most delicate point where regularity of PDE dynamics has a very strong bearing on the wellposedness of Riccati Equations.

While full characterization of optimal solution is obtainable (as we have seen above) under minimal assumption, it is the wellposedness of Riccati Equation (RE) which requires additional hypotheses. In fact, without these it is known, Triggiani (1997), Barbu, Lasiecka and Triggiani (2000), Weiss and Zwart (1998), that the classical wellposedness of RE may fail. The main technical issue is the wellposedness of the gain operator B^*P (introducing a nonlinear term in the equation) on some dense domain in H.

2.3. Differential Riccati Equations subject to a singular estimate

In this section we shall assume additional condition imposed on A and B which is referred to as "singular estimate".

HYPOTHESIS 1 **[Singular estimate]** *There exists a constant* $0 \leq \gamma < 1$ *such that*

$$|e^{At}Bu|_H \leq \frac{C}{t^\gamma}|u|_U, \quad 0 < t \leq 1$$

REMARK 2 *This estimate is trivially satisfied if* $B : U \to H$ *is bounded. Also, for analytic semigroups* e^{At} *and relatively bounded control operators* $B : U \to \mathcal{D}(A^{*\gamma})'$ *the above singular estimate follows from the analytic estimate, see Pazy (1986),* $|A^\gamma e^{At}|_{\mathcal{L}(H)} \leq \frac{C}{t^\gamma}$, $0 < t \leq 1$.

Our focus in this paper is on a class of *non-analytic* semigroups e^{At} and unbounded operators B which would still exhibit singular estimate. As we shall see later, there is a large class of dynamical systems which enjoy the above property.

We also note that Hypothesis 1 does not necessarily imply that the control operator is admissible (in the terminology of system theory — see Russell, 1978). In fact, in many situations of interest the control-to-state map denoted by L is not necessarily continuous when viewed as a map $L_2(0, T; U) \to C([0, T]; H)$. The lack of this continuity is one of a major technical difficulties in the theory.

Nevertheless, the validity of singular estimate provides, as we shall see below, full wellposedness of the Riccati theory. This includes critical statement that the gain operator $B^*P(t)$ is in fact a *bounded* operator $U \to H$.

THEOREM 1 (Lasiecka, 1998, 2001) *Consider the control problem governed by the dynamics described in (2), and the functional cost given in (3) with* $T < \infty$. *The control operator* B *is subject to Hypothesis 1. Moreover, we assume that* $F \in L_2(0, T; H)$. *Then, for any initial condition* $\mathbf{y}_0 \in H$, *there exists a unique optimal pair* $(u^0, \mathbf{y}^0) \in L_2(0, T; U \times H)$ *with the following properties:*

(ii) **[regularity of the gains and optimal synthesis]** *There exist a selfadjoint positive operator $P(t) \in \mathcal{L}(H)$ with the property*
$$B^*P(.) \in \mathcal{L}(H \to C([0,T];U));$$
$$P_t(.) \in \mathcal{L}(\mathcal{D}(A) \to C([0,T];\mathcal{D}(A^*)'))$$
and an element $r \in C([0,T];H)$ (depending on F) with the property
$$B^*r \in C([0,T];U)$$
such that: $\mathbf{u}^0(t) = -B^*P(t)\mathbf{y}^0(t) - B^*r(t);\ t \geq 0$.

(iii) **[feedback evolution]** *The operator $A_{P(t)} \equiv A - BB^*P(t) : \mathcal{D}(A_{P(t)}) \subset H \to D(A^*)'$ generates a strongly continuous evolution on H.*

(iv) **[Riccati equation]** *The operator $P(t)$ is a unique (within the class of selfadjoint positive operators subject to the regularity in part (ii)) solution of the following operator Differential Riccati Equation (**DRE**):*
$$(P_t(t)x, y)_H = (A^*P(t)x, y)_H + (P(t)Ax, y)_H + (R^*Rx, y)_H$$
$$-(B^*P(t)x, B^*P(t)y)_U;\ \text{for } x, y \in D(A). \tag{12}$$

(v) **[Equation for " r"]** *With $A_P(t)$ defined in part (iii) the element $r(t)$ satisfies the differential equation*
$$r_t(t) = -A_P^*(t)r(t) - P(t)F;\ \text{on } [D(A)]';\quad r(T) = 0. \tag{13}$$

REMARK 3 *The result of Theorem 1 was first shown in Avalos and Lasiecka (1996) in a special case of structural acoustic interaction with elastic equation on the interface modeled by plate equations with Kelvin Voight damping.*

REMARK 4 *It should be noted that the boundedness of the gain operator $B^*P(t)$ is a very special feature which is not generally expected. It is a consequence of singular estimate assumption. Indeed, in general, one does not have (unless B is bounded) the boundedness of the gains, even for the simplest scalar hyperbolic equations, Lasiecka and Triggiani (1991).*

2.4. Algebraic Riccati Equations subject to singular estimate

In order to obtain solvability of *Infinite Horizon Problem*, i.e. when $T = \infty$, one must assume that the *Finite Cost Condition* holds. *FCC* condition asserts an existence of at least one control u such that $J(u, y) < \infty$. Thus, in what follows we shall assume

HYPOTHESIS 2 **[Finite cost condition (FCC) condition]** *For any initial condition $\mathbf{y}_0 \in H$ there exists $u \in L_2(0, \infty; U)$ such that $J(u, y(u)) < \infty$, where $y(u)$ is the trajectory given by (2) with control u and originating at $\mathbf{y}_0$.*

The result stated below provides the wellposedness for Riccati Equations in the infinite horizon case.

THEOREM 2 (Lasiecka, 1998, 2001, Lasiecka and Triggiani, 2001) *Consider the control problem governed by the dynamics described in (2), and the functional*

Moreover, we assume that FCC condition in Hypothesis 2 is in place and $F \in L_2(0,\infty;H)$. Then, for any initial condition $\mathbf{y}_0 \in H$, there exists a unique optimal pair $(u^0, \mathbf{y}^0) \in L_2(0,\infty;U \times H)$ with the following properties:

(i) **[regularity of the optimal pair]** $u^0 \in C([0,\infty);U)$; $\mathbf{y}^0 \in C([0,\infty);H)$.

(ii) **[regularity of the gains and optimal synthesis]** *There exist a selfadjoint positive operator $P \in \mathcal{L}(H)$ with the property*

$$B^*P \in \mathcal{L}(H \to U);$$

and an element $r \in C([0,\infty);H)$ (depending on F) with the property

$$B^*r \in C([0,\infty);U)$$

such that: $u^0(t) = -B^*P(t)\mathbf{y}^0(t) - B^*r(t);\ t \geq 0.$

(iii) **[feedback semigroup]** *The operator $A_P \equiv A - BB^*P : \mathcal{D}(A_P) \subset H \to D(A^*)'$ generates a strongly continuous semigroup on H.*

(iv) **[Algebraic Riccati Equation]** *The operator P is a solution of the following operator Algebraic Riccati Equation (**ARE**):*

$$(A^*Px, y)_H + (PAx, y)_H + (R^*Rx, y)_H - (B^*Px, B^*Py)_U$$
$$\text{for } x, y \in D(A) \tag{14}$$

(v) **[Equation for "r"]** *With A_P defined in part (iii) the element $r(t)$ satisfies the differential equation*

$$r_t(t) = -A_P^*r(t) - PF;\ \text{on } [D(A)]';\quad \lim_{T\to\infty} r(T) = 0 \tag{15}$$

REMARK 5 *As usual, the uniqueness of ARE is guaranteed by imposing an appropriate "detectability condition". In this case the feedback semigroup $e^{A_P t}$ is exponentially stable on H. This, in turn, is guaranteed by the uniform stability of e^{At}. As we shall see later, the examples provided here fall into this category.*

REMARK 6 *As in the case of DRE, the boundedness of the gain operator B^*P is rather exceptional and results from the imposition of the singular estimate. In the general case one should not expect B^*P to be bounded, but at most densely defined, Flandoli, Lasiecka and Triggiani (1988). In addition, the quadratic term in (14) involves typically a suitable extension of B^*P rather than B^*P itself, Barbu, Lasiecka and Triggiani (2000), Triggiani (1997).*

Next two sections deal with the applicability of abstract theory to specific PDE systems with boundary and point controls.

3. Boundary control problems for thermoelastic plates

This section is devoted to an analysis of control problem governed by a thermoelastic system with boundary or point controls.

3.1. PDE model

We consider the following dynamics described by thermoelastic plates with ro-

Let $\Omega \subset R^2$ be a smooth, bounded domain with a boundary Γ. In $\Omega \times (0,T)$ we consider a vibrating plate subject to effects of thermoelasticity. The variable w denotes the vertical displacement of the plate, while θ denotes temperature. The dynamics of the plate is affected by boundary control u acting on the boundary Γ . The corresponding coupled PDE system is

$$\left.\begin{array}{r} w_{tt} - \rho\Delta w_{tt} + \Delta^2 w = \Delta\theta \\ \theta_t - \Delta\theta + \Delta w_t = 0 \end{array}\right\} \text{ in } \Omega \times (0,T)$$

with the boundary conditions on $\Gamma \times (0,T)$

$$w = \Delta w = 0; \quad \theta = u \tag{16}$$

We consider this model on the state space given by

$$H \equiv H_w \times H_\theta, \quad H_w \equiv (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega), \quad H_\theta \equiv L_2(\Omega)$$

so the initial conditions are given by $\mathbf{y}(0) \equiv (w(0), w_t(0), \theta(0)) \in H$. With equation (16) we associate the functional cost given by

$$J(u,w,\theta) \equiv \int_0^T \int_\Omega \left[\left| R_w \left(\begin{array}{c} w(t,x) \\ w_t(t,x) \end{array} \right) \right|^2 + |R_\theta \theta(t,x)|^2 \right] dx dt$$
$$+ \int_0^T \int_\Gamma |u(t,x)|^2 dx dt \tag{17}$$

where $R_w \in \mathcal{L}(H^2(\Omega) \times H_0^1(\Omega) \to L_2(\Omega))$ and $R_\theta \in \mathcal{L}(L_2(\Omega) \to L_2(\Omega))$

Optimal control problem to be considered is the following:

Boundary Control Problem: Minimize $J(u,w,\theta)$ given by (17) subject to the dynamics described by (16).

We note that when in eq. (16) the parameter $\rho = 0$, the corresponding system is analytic, Liu and Renardy (1995), Lasiecka and Triggiani (1998a), (1998b). Thus, in this case the wellposedness of standard Riccati theory follows from the "analytic LQR" theory, Bensoussan, Da Prato, Delfour and Mitter (1993), Lasiecka and Triggiani (2001). As mentioned before, our interest is in studying the non-analytic case which corresponds to $\rho > 0$. Indeed, if $\rho > 0$ the thermoelastic system is predominantly hyperbolic, i.e. it can be written as a compact perturbation of a group, Lasiecka and Triggiani (2000). Thus, the main goal here is to show that the control problem defined above is regular, by which we mean that Hypothesis 1 is satisfied. This, in turn, allows to deduce that the optimal control problem admits regular optimal synthesis including the wellposedness of Riccati equation in both finite and infinite horizon case.

3.2. Semigroup formulation

We find convenient to recast the PDE problem (16) in a semigroup framework. In order to achieve this we introduce several operators:

$$\mathcal{A} : L_2(\Omega) \to L_2(\Omega); \quad \mathcal{A} = \Delta^2,$$

$$\mathcal{D}(\mathcal{A}) = \{u \in H^4(\Omega) \cap H_0^1(\Omega);\ \Delta u = 0 \text{ on } \Gamma\}$$

$$D : L_2(\Gamma) \to L_2(\Omega);\ \Delta Dg = 0 \text{ in } \Omega, \quad Dg = g \text{ on } \Gamma$$

$$\mathcal{M} \equiv I + \gamma A_D$$

It is well known that $A_D, \mathcal{A}$ and $\mathcal{M}$ are selfadjoint, positive operators on $L_2(\Omega)$. Moreover, standard elliptic theory, Lions and Magenes (1972), gives $D \in \mathcal{L}(L_2(\Gamma) \to H^{1/2}(\Omega))$.

Notice that since $w|_\Gamma = \Delta w|_\Gamma = 0$ and $[\theta - Du]|_\Gamma = 0$,— for smooth solutions of system (16) we have that $\theta - Du \in D(A_D)$, consequently $\Delta\theta = \Delta(\theta - Du) = -A_D(\theta - Du)$ and $\Delta w = -A_D w$, $\Delta^2 w = \Delta^2 w = \mathcal{A}w$. Thus, with the above notation the semigroup representation for our PDE system (16) becomes, Lasiecka and Triggiani (2000):

$$\begin{aligned} &\mathcal{M}w_{tt} + \mathcal{A}w + A_D\theta = A_D Du \\ &\theta_t + A_D\theta - A_D w_t = A_D Du \end{aligned} \tag{18}$$

where the equalities are understood in the dual topology of $D(A_D)'$.

We introduce next $A : H \to H$, $B : L_2(\Gamma) \to \mathcal{D}(A^*)'$ given by

$$A \equiv \begin{pmatrix} 0 & I & 0 \\ -\mathcal{M}^{-1}\mathcal{A} & 0 & -\mathcal{M}^{-1}A_D \\ 0 & A_D & -A_D \end{pmatrix} \tag{19}$$

$$D(A) = \{(w, v, \theta) \in H; \mathcal{M}^{-1}\mathcal{A}w \in H_0^1(\Omega), v \in D(A_D), \theta \in D(A_D)\}$$

$$Bu \equiv \begin{pmatrix} 0 \\ \mathcal{M}^{-1}A_D Du \\ A_D Du \end{pmatrix}. \tag{20}$$

It is known, Liu and Renardy (1995), Lasiecka and Triggiani (2000), that A is a generator of a C_0 semigroup on H. Moreover, this semigroup contains a hyperbolic component, hence is non-analytic, Lasiecka and Triggiani (2000).

With the above notation our plate problem can be rewritten as a first order system:

$$\mathbf{y}_t = A\mathbf{y} + Bu \quad \text{on} \quad H;\ \mathbf{y} \equiv [w, w_t, \theta];\ u \in L_2(0, T; U);\ U = L_2(\Gamma).$$

3.3. Main result

Now we are in a position to state the main result of this section:

THEOREM 3 *Control system described by (16) satisfies assumptions required by Theorem 1. In particular, "singular estimate" of Hypothesis 1 holds with* $\gamma =$

The proof of Theorem 3 is relegated to the next subsection.

It has been shown recently, Avalos and Lasiecka (1997), (1998), that thermoelastic semigroups are exponentially stable. This is to say $|e^{At}|_{\mathcal{L}(H)} \leq Ce^{-\omega t}$, $\omega > 0,\ t \geq 0$. Since exponentially stable systems automatically satisfy the *FCC* Condition in Hypothesis 2, we obtain the following Corollary:

COROLLARY 1 *All the statements of Theorems 1 and 2 apply to Boundary Control Problem with A, B specified in (19), (20).*

REMARK 7 *Different (than hinged) boundary conditions associated with (16) can be cosidered as well. These include clamped or free homogenous boundary conditions — see Lasiecka and Triggiani (2000). The analysis presented below is not critically affected by the structure of the boundary conditions imposed for the plate. Similarly, boundary conditions associated with the heat equation can be of a Neumann type. In fact, this latter case is easier to handle.*

REMARK 8 *One could also consider the following point control problem associated with this plate.*

$$\begin{aligned} &w_{tt} - \rho\Delta w_{tt} + \Delta^2 w = \Delta\theta \ \ in\ \Omega \\ &\theta_t - \Delta\theta + \Delta w_t = \delta_{x_0} u, \quad in\ \Omega, \\ &w = \Delta w = \theta = 0\ \ on\ \Gamma \end{aligned} \tag{21}$$

where x_0 is a designated point in Ω. The corresponding functional cost can be taken as

Point control problem: *Minimize*

$$J(w,\theta,u) = \int_0^T \left[\left| R_w \begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix} \right|^2_{H_w} + |R_\theta \theta(t)|^2_{L_2(\Omega)} + |u(t)|^2 \right] dt \tag{22}$$

with $R_w \in \mathcal{L}(H_w)$, $\quad R_\theta \in \mathcal{L}(L_2(\Omega))$ and u, w satisfying (21).

All the final statements of Theorems 1, 2 and 3 apply to this model as well. In fact, the arguments are simpler that in the case of boundary control considered in Theorem 3.

The remaining part of this section is devoted to the proof of Theorem 3. In what follows we shall adopt the following notation: $|u|_{s,\Omega} \equiv |u|_{H^s(\Omega)}$, where $H^s(\Omega) = [H_0^s(\Omega)]',\ s < 0$

3.4. Proof of Theorem 3

We begin by verifying that our standing assumption $A^{-1}B \in \mathcal{L}(U \to H)$ is satisfied with $U \equiv L_2(\Gamma)$ and H defined above. Indeed, by direct computations:

$$A^{-1}B = \begin{pmatrix} 0 \\ 0 \\ \end{pmatrix} \in \mathcal{L}(L_2(\Gamma) \to H). \tag{23}$$

Thus, the operators A, B qualify for application of abstract theory. In particular, the operator $e^{At}B$ is well defined $H \to \mathcal{D}(A^*)'$. Indeed, it suffices to write $e^{At}B = Ae^{At}A^{-1}B \in \mathcal{L}(U \to \mathcal{D}(A^*)')$.

The key role in the proof is played by the singular estimate for the kernel $e^{At}B$, which establishes the meaning of this operator as acting $U \to H$.

LEMMA 2 **(Singular estimate)** *With A, B given in (19), (20) the following estimate takes place:* $\forall \epsilon > 0$

$$|e^{At}B|_{\mathcal{L}(U \to H)} \leq \frac{C}{t^{3/4+\epsilon}}; \quad 0 < t \leq 1.$$

Proof. **Step 1 — setting up integral equations:** Denote $W \equiv [w, w_t]$. Then

$$\begin{pmatrix} W(t) \\ \theta(t) \end{pmatrix} = e^{At}Bu.$$

is equivalent to:

$$\begin{cases} \mathcal{M}w_{tt} + \mathcal{A}w + A_D\theta = 0 \\ \theta_t + A_D\theta - A_D w_t = 0 \end{cases} \tag{24}$$

$$\begin{pmatrix} w(0) \\ w_t(0) \\ \theta(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{M}^{-1}A_D Du \\ A_D Du \end{pmatrix}$$

This abstract equation can be rewritten equivalently as a perturbation of damped Kirchhoff plate:

$$\begin{cases} \mathcal{M}w_{tt} + \mathcal{A}w + A_D w_t = \theta_t \\ \theta_t + A_D\theta - A_D w_t = 0 \end{cases}.$$

We introduce another operator $A_1 : H_w \to H_w$ given by

$$A_1 \equiv \begin{pmatrix} 0 & I \\ -\mathcal{M}^{-1}\mathcal{A} & -\mathcal{M}^{-1}A_D \end{pmatrix}.$$

Operator A_1 with $D(A_1) = \{(w, v) \in H_w; \mathcal{M}^{-1}\mathcal{A}w \in H_0^1(\Omega), v \in D(A_D)\}$ is a standard generator of damped Kirchhoff plate. Indeed, we have

$$W_t(t) = A_1 W(t) + \begin{pmatrix} 0 \\ \theta_t(t) \end{pmatrix}.$$

Thus, by the variation of parameters formula we obtain

$$W(t) = e^{A_1 t}W(0) + \int_0^t e^{A_1(t-s)} \begin{pmatrix} 0 \\ \mathcal{M}^{-1}\theta_t(s) \end{pmatrix} ds \tag{25}$$

$$\theta(t) = e^{-A_D t}\theta(0) + \int_0^t e^{-A_D(t-s)} A_D w_t(s) ds \tag{26}$$

These two integral equations are the main object of the analysis.

Step 2 — analysis of W equation. Integrating equation (25) by parts yields

$$W(t) = e^{A_1 t}W(0) + e^{A_1(t-s)} \begin{pmatrix} 0 \\ \mathcal{M}^{-1}\theta(s) \end{pmatrix} \Big|_0^t - \int_0^t \frac{d}{ds} e^{A_1(t-s)} \begin{pmatrix} 0 \\ \mathcal{M}^{-1}\theta(s) \end{pmatrix} ds \tag{27}$$

For $\theta(s) \in L_2(\Omega)$

$$A_1 \begin{pmatrix} 0 \\ \mathcal{M}^{-1}\theta(s) \end{pmatrix} = \begin{pmatrix} \mathcal{M}^{-1}\theta(s) \\ -\mathcal{M}^{-1}A_D\mathcal{M}^{-1}\theta(s) \end{pmatrix} \in H_w$$

we obtain

$$W(t) = e^{A_1 t}W(0) + e^{A_1(t-s)} \begin{pmatrix} 0 \\ \mathcal{M}^{-1}\theta(s) \end{pmatrix} \Big|_0^t + \int_0^t e^{A_1(t-s)} A_1 \begin{pmatrix} 0 \\ \mathcal{M}^{-1}\theta(s) \end{pmatrix} ds. \tag{28}$$

Hence

$$W(t) = e^{A_1 t}W(0) + \begin{pmatrix} 0 \\ \mathcal{M}^{-1}\theta(t) \end{pmatrix} - e^{A_1 t} \begin{pmatrix} 0 \\ \mathcal{M}^{-1}\theta(0) \end{pmatrix} + \int_0^t e^{A_1(t-s)} \begin{pmatrix} \mathcal{M}^{-1}\theta(s) \\ -\mathcal{M}^{-1}A_D\mathcal{M}^{-1}\theta(s) \end{pmatrix} ds. \tag{29}$$

The elliptic theory gives

- $\mathcal{M}^{-1} \in \mathcal{L}(L_2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)) \cap \mathcal{L}(H^{-1}(\Omega) \to H_0^1(\Omega))$
- $\mathcal{M}^{-1}A_D\mathcal{M}^{-1} \in \mathcal{L}(H^{-1}(\Omega) \to H_0^1(\Omega))$

and noticing cancellation of terms in (29) we arrive at the estimate:

$$|W(t)|_{H_w} \le C[|\theta(t)|_{H^{-1}(\Omega)} + |\theta|_{L_1(0,t;L_2(\Omega))}] \tag{30}$$

Step 3 — analysis of θ equation. Integrating by parts of the second term in (26) yields

$$\theta(t) = e^{-A_D t}\theta(0) - w_t(t) + e^{-A_D t}w_t(0) - \int_0^t e^{-A_D(t-s)} w_{tt}(s) ds. \tag{31}$$

On the other hand by recalling the original version of the plate equation in (24) we obtain:

Inserting the above relation into (31) and estimating the terms gives

$$\begin{aligned}|\theta(t)|_{0,\Omega} &\leq |e^{-A_D t} A_D D u|_{0,\Omega} + |e^{-A_D t}\mathcal{M}^{-1} A_D D u|_{0,\Omega} + |w_t(t)|_{0,\Omega} \\ &+ \Big| \int_0^t e^{-A_D(t-s)} \mathcal{M}^{-1} A_D \theta(s) ds \Big|_{0,\Omega} \\ &+ \Big| \int_0^t e^{-A_D(t-s)} \mathcal{M}^{-1} \mathcal{A} w(s) ds \Big|_{0,\Omega} \end{aligned} \tag{32}$$

Since the semigroup $e^{-A_D t}$ is analytic and the Dirichlet map satisfies, Lasiecka and Triggiani (2001),

$$D \in \mathcal{L}(L_2(\Gamma) \to \mathcal{D}(A_D^{1/4-\epsilon}))$$

we infer

$$|e^{-A_D t} A_D D|_{\mathcal{L}(L_2(\Gamma)\to L_2(\Omega))} \leq \frac{C}{t^{3/4+\epsilon}}; \quad 0 < t \leq 1. \tag{33}$$

Moreover, the analyticity of $e^{-A_D t}$ along with the fact that $\mathcal{D}(A_D^\theta) \sim H_0^{2\theta}(\Omega)$, $\frac{1}{4} < \theta \leq \frac{1}{2}$, also imply that $T_t(f) \equiv \int_0^t e^{-A_D(t-s)} f(s) ds$ satisfies, Lasiecka and Triggiani (2001), Bensoussan, Da Prato, Delfour and Mitter (1993):

$$T_t \in \mathcal{L}(L_\infty(0,t; H^{-1}(\Omega)) \to H_0^{1-\epsilon}(\Omega)); \quad \forall t > 0. \tag{34}$$

Since $\mathcal{A} = A_D^2$

$$\mathcal{M}^{-1}\mathcal{A} \in \mathcal{L}(H_0^1(\Omega) \to H^{-1}(\Omega))). \tag{35}$$

Collecting (32) , (33), (34), (35) we obtain:

$$|\theta(t)|_{0,\Omega} \leq C\Big[|w_t(t)|_{0,\Omega} + \int_0^t |\theta(s)|_{0,\Omega} ds + |w|_{L_\infty(0,t;H_0^1(\Omega))} + \frac{|u|_U}{t^{3/4+\epsilon}}\Big]; \quad 0 < t \leq 1. \tag{36}$$

Gronwall's inequality applied to the above inequality gives

$$|\theta(t)|_{0,\Omega} \leq C\Big[|w_t(t)|_{0,\Omega} + |w|_{L_\infty(0,t;H_0^1(\Omega))} + \frac{|u|_U}{t^{3/4+\epsilon}}\Big], \; 0 < t \leq 1. \tag{37}$$

Step 4 — decoupling W and θ given, respectively, by (30) and (37). From (30)

$$|W(t)|_{H_w} \leq C\Big[|\theta(t)|_{-1,\Omega} + \int_0^t |w_t(s)|_{0,\Omega} ds$$

$$+ t|w|_{L \ldots} + |u|_U \int^t \frac{C}{\ldots} ds\Big] \quad (38)$$

The H^{-1} norm of θ can be easily estimated from (26) as follows:

$$|\theta(t)|_{-1,\Omega} \leq |A_D^{-1/2} e^{-A_D t}\theta(0)|_{0,\Omega} + \left| \int_0^t e^{-A_D(t-s)} A_D^{1/2} w_t(s) ds \right|_{0,\Omega}$$
$$\leq C \left[\frac{|u|_U}{t^{1/4+\epsilon}} + t|w|_{L_\infty(0,t;H_0^1(\Omega))} \right] \tag{39}$$

where in the last estimate we have used the basic semigroup estimate only. Combining (38) and (39) allows us to decouple the W variable

$$|W(t)|_{H_w} \leq C|u|_U \left[\frac{1}{t^{1/4+\epsilon}} + |u|_U t^{1/4-\epsilon} \right] + CtW_{L_\infty(0,t,H_w)}. \tag{40}$$

Taking $t \leq \frac{1}{2C}$ we obtain

$$|W(t)|_{H_w} \leq C|u|_U \frac{1}{t^{1/4+\epsilon}} \tag{41}$$

which then combined with (37) yields

$$|\theta(t)|_{0,\Omega} \leq C \frac{|u|_U}{t^{3/4+\epsilon}}, \quad 0 < t \leq \frac{1}{2C}. \tag{42}$$

The above argument can be now bootstrapped in order to obtain the estimate for any $1/2C \leq t \leq T < \infty$. Prof of Lemma 2 is thus complete. ∎

In order to assert validity of Theorem 3 it suffices to combine the result of Lemma 2 with the fact that e^{At} is exponentially stable, Avalos and Lasiecka (1997, 1998). ∎

4. Point and boundary control problems in the acoustic-structure interactions

In this section we shall study an optimal control problem arising in an abstract model of structural acoustic interactions. In these applications the goal of control is to reduce noise entering an acoustic environment.

4.1. Description of the model

The model under consideration consists of wave equation interacting on an interface with a dynamic plate equation. This is a typical configuration arising in structural acoustic interactions, Morse and Ingard (1968), Beale (1976), Littman and Liu (1998).

Let Ω be a bounded domain in R^n, $n = 2, 3$ with a boundary Γ which consist of two parts Γ_0, Γ_1. We assume that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, Γ_0 is flat and $\Gamma_0 \cap \Gamma_1 = \emptyset$.

In practical applications Ω will describe an acoustic chamber while Γ_i, $i = 0, 1$ denote the walls of the chamber. Γ_1 is referred as "hard wall" and Γ_0 is a flexible wall where the interaction with structural medium takes place. The acoustic medium in the chamber is described by the wave equation in the variable z (where the quantity $\rho_1 z_t$ is acoustic pressure).

$$\begin{aligned}
&z_{tt} = c^2 \Delta z - d_0(x) z_t + f && \text{in } \Omega \times (0, T) \\
&\frac{\partial}{\partial \nu} z + d_1 z = 0 && \text{on } \Gamma_1 \times (0, T) \\
&\frac{\partial}{\partial \nu} z + D_0 z_t = w_t && \text{on } \Gamma_0 \times (0, T).
\end{aligned}$$

Here c^2 is the speed of sound as usual, $f \in L^2((0,T) \times \Omega)$ is an external noise, $d_0(x) \geq 0$ in Ω, $d_1 \geq 0$.

The operator $D_0 : L_2(\Gamma_0) \to L_2(\Gamma_0)$ is positive and densely defined on $L_2(\Gamma_0)$ and subject to additional assumptions specified later. The operator D_0 represents boundary damping on Γ_0 while d_0 represents internal damping. In practical applications the internal damping modeled by $d_0 z_t$ is due to viscosity of an environment, e.g. resistance of air in applications to acoustic problems. The boundary damping $D_0 z_t$ acting on a portion of the boundary Γ is typical for the so called "absorbing boundary conditions". The damping effect on the boundary may result from the effects of friction applied to an edge of a spatial domain or may be caused by structural properties of the material the boundary is build from. A common method for creating such damping involves a lamination process or the so called "constrained layer" techniques where the matrix forming the wall consists of several layers of different materials with different elastic properties. A well known form of structural damping is Kelvin Voight damping in which case the operator D corresponds to Laplace's Beltrami operator.

Our goal is to consider the controlled model describing coupled interaction between the acoustic medium and the wall structure, i.e. between wave and plate equation. To this end we introduce plate equation along with appropriate coupling. The coupling between plate and the wave represents in this example the back pressure on the wall.

Let $\mathcal{A}$ denote an elastic operator, which is positive and selfadjoint on $L_2(\Gamma_0)$. We consider the following abstract model for the plate with structural damping

$$w_{tt} + \mathcal{A} w + \rho \mathcal{A}^{\alpha} w_t + \rho_1 z_t|_{\Gamma_0} = 0 \quad \rho > 0, \ \rho_1 > 0$$

where $\rho_1 z_t$ denotes back pressure on the wall and $\rho \mathcal{A}^{\alpha} w_t$ denotes structural damping. It is known that for α between $1/2$ and 1 the above equation with $\rho_1 = 0$ generates an analytic semigroup on $\mathcal{D}(\mathcal{A}^{1/2}) \times L_2(\Gamma_0)$, Russell (1986), Chen and Triggiani (1989). A canonical realization of $\mathcal{A}$ which is of interest to

The ultimate PDE model considered in this paper (we take $\rho_1 = 0$) can be written as:

$$z_{tt} = c^2 \Delta z - d_0 z_t + f \quad \text{in} \quad \Omega \times (0,T) \tag{43a}$$

$$\frac{\partial}{\partial \nu} z + d_1 z = 0 \quad \text{on} \quad \Gamma_1 \times (0,T) \tag{43b}$$

$$\frac{\partial}{\partial \nu} z + D_0 z_t = w_t \quad \text{on} \quad \Gamma_0 \times (0,T) \tag{43c}$$

$$w_{tt} + \mathcal{A} w + \rho \mathcal{A}^\alpha w_t + z_t|_{\Gamma_0} = \mathcal{B} u \quad \text{on} \quad \Gamma_0 \times (0,T). \tag{43d}$$

Here the operator $\mathcal{B}$ is a control operator acting upon actuator $u(t)$. The control operators are typically unbounded, such as typically arise in the context of modeling smart materials.

In concrete applications to structural acoustic, the last equation in (43) models plate equation with $\mathcal{A}$ being fourth order elliptic operator. The control operator $\mathcal{B}$ represents point controls realized via smart actuators such as piezoceramic or piezoelectric patches. In such instances $\mathcal{B}$ is just a derivative of the "delta" function supported either at some interior points of Γ ($\dim \Gamma_0 = 1$) or on some closed curves in Γ_0 if $\dim \Gamma_0 = 2$, Dimitriadis, Fuller and Rogers (1991).

In this paper we shall, however, consider more general classes of operators $\mathcal{A}, \mathcal{B}$, which are defined by the following set of Hypotheses.

HYPOTHESIS 3 $\mathcal{A} : D(\mathcal{A}) \subset L_2(\Gamma_0) \to L_2(\Gamma_0)$ *is a positive, selfadjoint operator.*

HYPOTHESIS 4 *There exists a positive constant* r, $0 < r < 1/2$, *such that*

$$\mathcal{A}^{-r}\mathcal{B} \in \mathcal{L}(\mathcal{U}, L_2(\Gamma_0)); \text{ equivalently, } \mathcal{B} \text{ continuous} : \mathcal{U} \to [D(\mathcal{A}^r)]'; \tag{44}$$

where $\mathcal{U}$ *is a Hilbert space.*

The following hypothesis is made regarding the boundary operator D_0.

HYPOTHESIS 5 $D_0 : L_2(\Gamma_0) \supset \mathcal{D}(D) \to L_2(\Gamma_0)$ *is a positive, selfadjoint operator, and there exists a constant* r_0, $0 \le r_0 \le 1/4$, *and positive constants* δ_1, δ_2 *such that*

$$\delta_1 |z|_{D(\mathcal{A}^{r_0})} \le (D_0 z, z)_{L_2(\Gamma_0)} \le \delta_2 |z|_{D(\mathcal{A}^{r_0})} \; \forall z \in D(\mathcal{A}^{r_0}) \equiv D(D_0^{1/2}). \tag{45}$$

Moreover, we assume that $H^1(\Gamma_0) \subseteq D(D_0^{1/2})$.

REMARK 9 If $\mathcal{A}$ models plate equation (the case of interest to us), then with $r_0 = 1/4$ we have that $D_0(\mathcal{A}^{1/4})$ is topologically equivalent to $H^1(\Gamma_0)$ norm. In such case, the operator D corresponds to boundary structural damping modeled

The control problem considered is the following:

Control Problem: Minimize

$$J(u, z, w) = \int_0^T [|\nabla z|^2_{0,\Omega} + |z_t|^2_{0,\Omega} + |\mathcal{A}^{1/2} w|^2_{0,\Gamma_0} + |w_t|^2_{0,\Gamma_0} + |u(t)|^2_U] dt \tag{46}$$

subject to the dynamics of (43).

A cost functional of interest in applications in structural acoustic problems is a performance index which minimizes the pressure z_t in an acoustic environment and also leads to reduction of vibrations on the wall. For this problem the functional cost takes the form

$$J(u, z, w) = \int_0^T [\rho |z_t|^2_{0,\Omega} + |\Delta w|^2_{0,\Gamma_0} + |w_t|^2_{0,\Gamma_0} + |u(t)|^2_U] dt.$$

REMARK 10 *One could also consider other models for plates, including these where the analyticity is generated by thermal effects, Lasiecka (2000). Specific example can be given by considering the same system (43) with the fourth equation replaced by thermoelastic system in the variables w, θ defined on $\Gamma_0 \times (0, \infty)$.*

$$\begin{aligned} &w_{tt} + \Delta^2 w - \Delta\theta + \rho z_t = \mathcal{B} u \\ &\theta_t - \Delta\theta + \Delta w_t = 0. \end{aligned} \tag{47}$$

With the above system we associate boundary conditions either clamped $w = \frac{\partial}{\partial\nu} w = 0$, hinged $w = \Delta w = 0$ or free, Lasiecka and Triggiani (1998b). The associated control problem can be formulated as follows:

Control Problem: *Minimize*

$$J(u, z, w) = \int_0^T [|\nabla z|^2_{0,\Omega} + |z_t|^2_{0,\Omega} + |\Delta w|^2_{0,\Gamma_0} + |w_t|^2_{0,\Gamma_0} + |\theta(t)|^2_{0,\Gamma_0} + |u(t)|^2_U] dt \tag{48}$$

subject to equations (43 a–c) and (47).

Our goal is to show that structural acoustic interactions described above fit into the abstract framework of Section 2. As a consequence, the results of Theorems 1 and 2 will apply to these problems as well.

4.2. Semigroup formulation

The following operators will be used in describing the PDE model given in (43) — see Bucci, Lasiecka and Triggiani (2002).

Operators acting on Ω. (i) Let $A_N : L_2(\Omega) \supset D(A_N) \to L_2(\Omega)$ be the non-negative, self-adjoint operator defined by

$$A_N h = -c^2 \Delta h, \quad D(A_N) = \left\{ h \in H^2(\Omega) : \left(\frac{\partial}{\partial\nu} h + d h \right) \Big| = 0 \right\} \tag{49}$$

(ii) Let N be the Neumann map from $L_2(\Gamma_0)$ to $L_2(\Omega)$, defined by

$$\psi = Ng \Leftrightarrow \left\{ \Delta\psi = 0 \text{ in } \Omega; \frac{\partial}{\partial\nu}\psi\Big|_{\Gamma_0} = g, \ \left(\frac{\partial}{\partial\nu}\psi + d_1\psi\right)\Big|_{\Gamma_1} = 0 \right\}.$$

It is well known, Lions and Magenes (1972), that N continuous: $L_2(\Gamma_0) \to H^{3/2}(\Omega) \subset D(A_N^{3/4-\epsilon})$, $\epsilon > 0$. Moreover, by the Green's second theorem, the following trace results hold true:

$$N^* A_N h = \begin{cases} h|_{\Gamma_0} & \text{on } \Gamma_0, \\ 0 & \text{on } \Gamma_1 \end{cases}, \tag{50}$$

where the validity of (50) may be extended to all $h \in H^1(\Omega) \equiv D(A_N^{1/2})$.

Second order abstract model. By using the Green operators introduced above, the coupled PDE problem (43) can be rewritten as the following abstract second order system — see Bucci, Lasiecka and Triggiani (2002):

$$z_{tt} + A_N z + A_N N D_0 N^* A_N z_t + d_0 z_t - A_N N v_t = f \tag{51a}$$

$$v_{tt} + \mathcal{A} v + \rho \mathcal{A}^\alpha v_t + N^* A_N z_t = \mathcal{B} u, \tag{51b}$$

the first equation to be read on $[D(A_N)]'$, the second one on $[D(\mathcal{A})]'$.

Function spaces and operators. We define the following spaces

$$H_z \equiv \mathcal{D}(A_N^{1/2}) \times L_2(\Omega); \quad H_v \equiv \mathcal{D}(\mathcal{A}^{1/2}) \times L_2(\Gamma_0)$$

On H_z we define (unbounded) operator $A_z : H_z \to H_z$ given by

$$A_z \equiv \begin{pmatrix} O & I \\ -A_N & -A_N N D_0 N^* A_N - d_0 \end{pmatrix}. \tag{52}$$

Similarly, on H_v we define $A_v : H_v \to H_v$

$$A_v \equiv \begin{pmatrix} O & I \\ -\mathcal{A} & -\rho\mathcal{A}^\alpha \end{pmatrix}. \tag{53}$$

Coupling. Finally, we introduce the densely defined (unbounded, uncloseable) trace operator $C : H_z \supset D(C) \to H_v$ defined by

$$C \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := \begin{bmatrix} 0 \\ N^* A_N z_2 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & N^* A_N \end{pmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{54}$$

with the domain

$$D(C) = \{[z_1, z_2] \in Y_1 \, : \, N^* A_N z_2 = z_2|_{\Gamma_0} \in L_2(\Gamma_0)\}$$

so that $D(A_N^{1/2}) \times D(A_N^{1/2}) \subset D(C)$. Its adjoint $C^* : H_v \to D(A_N^{1/2}) \times [D(A_N^{1/4+\epsilon})]'$, in the sense that $(Cy_1, y_2)_{H_v} = (y_1, C^* y_2)_{H_z}$; $y_1 \in \mathcal{D}(C), y_2 \in H_v$ is given by

$$C^* \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ A_N N v_2 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A_N N \end{pmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{55}$$

where $A_N N : L_2(\Gamma_0) \to [D(A_N^{1/4+\epsilon})]'$.

First-order abstract model. Finally, from (52), (53), (54), (55) we define the operator A acting on the space $H \equiv H_z \times H_v$

$$A := \begin{pmatrix} A_z & C^* \\ -C & A_v \end{pmatrix} : \; H \supset D(A) \to H, \tag{56}$$

with the domain

$$\begin{aligned} D(A) = \{[z_1, z_2, v_1, v_2] \in H : \; z_2 \in \mathcal{D}(A_N^{1/2}),\; v_2 \in D(\mathcal{A}^{1/2}),\\ \mathcal{A}^{1-\alpha} v_1 + \rho v_2 \in D(\mathcal{A}^\alpha), z_1 + N D_0 N^* A_N z_2 - N_0 v_2 \in D(A_N)\}. \end{aligned} \tag{57}$$

By applying Lummer Phillips Theorem, Pazy (1986), it is shown, Bucci, Lasiecka and Triggiani (2002), that A generates a strongly continuous semigroup of contractions on H.

Control operator. Finally, we define the operator $B : U \to [D(A^*)]'$, dual with respect to H, as a pivot space by:

$$B \equiv [0, 0, 0, \mathcal{B}]^T. \tag{58}$$

It is readily verified that

$$A^{-1} B = [0, 0, 0, -\mathcal{A}^{-1}\mathcal{B}]^T \in \mathcal{L}(U; H),$$

so that $B \in \mathcal{L}(U; [D(A^*)]')$. Here we have used Hypothesis 4 to deduce that $\mathcal{A}^{-1}\mathcal{B} = \mathcal{A}^{r-1}\mathcal{A}^{-r}\mathcal{B}$ is bounded from $\mathcal{U}$ into $L_2(\Gamma_0)$, as $r < 1/2$.

Finally, returning to the second-order abstract model (51), we see that these equations can be rewritten as the following first-order abstract equation in the variable $\mathbf{y}(t) = [z(t), z_t(t), v(t), v_t(t)]$:

$$\mathbf{y}_t = A\mathbf{y} + B\widetilde{u} + F \quad \text{in } [D(A^*)]',\; \mathbf{y}(0) = \mathbf{y}_0 \tag{59}$$

where A, B are defined in (56), (58), respectively, and $F \equiv [0, 0, 0, f]^T$.

4.3. The finite horizon control problem

In order to apply the abstract results of Section 2, the key property to be verified is the singular estimate. As we shall see below, in the case of structural acoustic problem, this property is not always guaranteed by the analyticity of the semigroup generated by the plate equation. In fact, if the analyticity of the plate the additional regularizing effect is needed in order to offset the unboundedness of B. It turns out that a critical role in this is played by the boundary damping D. The above discussion motivates the following assumption.

HYPOTHESIS 6 *We shall assume the following relation between the parameters representing damping in the system:*
(i) either $r_0 + \frac{\alpha}{2} \geq r$
(ii) or $\alpha - 2r \geq 1/6$ *and* $H^{1/3}(\Gamma_0) = \mathcal{D}(\mathcal{A}^{\frac{1}{12}})$.

REMARK 11 *In a canonical case of a plate equation, when $\mathcal{A}$ is the fourth order differential elliptic operator, the condition $H^{1/3}(\Gamma_0) = \mathcal{D}(\mathcal{A}^{\frac{1}{12}})$ is always satisfied. This follows from the more general property $H^{4\theta}(\Gamma_0) = \mathcal{D}(\mathcal{A}^{\theta}); 0 \leq \theta < 1/8$. Moreover, in that case the first part of Hypothesis 6 is always fulfilled with $\alpha \geq 1/2$ and $r_0 = 1/4$. Thus, any structurally damped plate ($\alpha \geq 1/2$) with boundary structural damping D_0 represented by the Laplace's Beltrami operator ($r_0 = 1/4$) will always fulfill the requirements of the first part of the hypothesis.*

Regarding the second part of the hypothesis, this is always true for a strongly (Kelvin Voight) damped plate equations when $\alpha = 1$. In this latter case we do not need any additional overdamping on the interface Γ_0. This means that one can take $D_0 = 0$.

The following singular estimate has been established in Bucci, Lasiecka and Triggiani (2002).

THEOREM 4 *(Bucci, Lasiecka and Triggiani, 2002) We assume Hypotheses 3–5. In addition we assume the first part of Hypothesis 6. Then, the control system described by (59) satisfies the "singular estimate".*

$$|e^{At}Bu| \leq \frac{C}{t^{\gamma}}, \quad 0 < t \leq 1$$

with the value of γ given by

$$\gamma = \begin{cases} \frac{r}{\alpha}, & r \leq \frac{1}{2}\alpha \\ \frac{1/2 - \alpha + r}{1 - \alpha}; & r > \frac{1}{2}\alpha. \end{cases}$$

If the second part of Hypothesis is in force, then the singular estimate holds also with $D_0 \equiv 0$ and the value of γ is given by $\gamma = \frac{r}{\alpha} < 1/2$.

The proof of Theorem 4 given in Bucci, Lasiecka and Triggiani (2002) is technical and lengthy. It relies critically on two main ingredients: (i) characterization of fractional powers of elastic operators, Chen and Triggiani (1990), and (ii) sharp regularity of traces to wave equation with Neumann data, Lasiecka and Triggiani (1991). In the special case when $\alpha = 1$ (Kelvin Voight damping)

REMARK 12 *We note that in the case when the operator B is unbounded, $r > 0$, the Hypothesis 6 forces certain amount of global damping in the system. The role of the damping is to offset the unboundedness resulting from the control operator. We have two sources of damping present in the model: structural damping yielding analyticity of the "plate" component (measured by the parameter $1/2 \leq \alpha \leq 1$) and boundary structural damping due to the presence of the operator D_0, measured by the parameter $0 \leq r_0 \leq 1/4$. The following interpretation of Hypothesis 6 can be given: the more analyticity in the system (i.e. the higher value of α), the less boundary damping is needed (smaller value of r_0) in order to control singularity at the origin. In the extreme case, when the plate equation has strong analyticity properties, postulated in part (ii) of the Hypothesis 6, there is no need for boundary damping at all. In fact, the extreme case of $\alpha = 1$, treated in Avalos and Lasiecka (1996) leads to singular estimate with $D = 0$ and $\gamma = 1/2 - \epsilon$. The result presented in the second part of the theorem extends the estimate in Avalos and Lasiecka (1997) to a larger range of parameters α and also provides more precise information on the singularity.*

By applying the abstract result from Theorem 1 along with the singular estimate from Theorem 4 we infer the following final result

THEOREM 5 *Under the hypotheses of Theorem 4 and with reference to finite horizon control problem consisting of (43) with functional cost (46), all the statements of abstract Theorem 1 apply with A, B specified in Section 4.2.*

REMARK 13 *For the control problem governed by the structural acoustic interaction with thermoelasticity, see Remark 10, the validity of singular estimate with $r_0 = 1/4$ and $\gamma = 2r$ was shown in Lasiecka (2000). Thus, the same statement as in Theorem 5 is valid for this dynamics with the cost given in (48).*

4.4. The infinite horizon control problem

If the time T is infinite, the analysis of structural model is more complex. Indeed, one needs to be concerned with the validity of *Finite Cost Condition*. This is typically guaranteed by some sort of stabilization result valid for the system under considerations. Unfortunately, in the case of structural acoustic problem, the coupled system is *stable* but not *uniformly stable*, Avalos and Lasiecka (1998). Thus, in order to enforce uniform stability, the corresponding model must be more complex. It is natural to impose some viscous damping in the interior of Ω. This corresponds to taking $d_0 > 0$ in the first equation. In fact, this strategy works well, when there is no need for strong structural damping on the interface Γ_0. More precisely, the following result is known: Avalos (1996)

THEOREM 6 *Consider (43) with $f = 0$, $u = 0$, $D_0 = 0$ (or $r_0 < 1/8$), and $d_0 > 0, d_1 > 0$. Then, the corresponding system is exponentially stable on H, i.e. there exists $\omega > 0$ such that*

$$|e^{At}|_{\mathcal{L}(H)} \leq Ce^{-\omega t}; \quad t \geq 0.$$

As we already know from the results of previous section, in order to assert singular estimate for the semigroup, depending on the value of α, we may need strong structural damping on the interface (i.e. the unbounded operator D_0). One would (naively) surmise that such damping should only enhance the stability of the system. However, this is not the case as revealed in Bucci, Lasiecka and Triggiani (2002). In fact, despite strong viscous damping in Ω with $d_0 > 0$, the system (43) is *not uniformly* stable, whenever $r_0 \geq 1/8$. Thus, there is a trade-off between regularity and stability, as the result of which the overall control problem is much more subtle. A natural perception that "more damping" implies stronger decay rates is obviously false (in fact this is known among engineers as an overdamping phenomenon). In mathematical terms this is explained by noticing that the presence of strongly unbounded operator D_0 introduces an element of continuous spectrum and $0 \in \sigma_{ess}(A)$, i.e. the point 0 belongs to the essential spectrum of A. This is a new phenomenon not present in structural acoustic models without the strong damping on the interface. In view of the above, we are faced with the following dilemma. How to stabilize the system while preserving regularity guaranteed by the singular estimate?

The solution proposed below is based on the following idea: we counteract the instability of the system by introducing an additional static feedback control. The role of the static damping is to remove the *continuous* spectrum from the spectrum of the generator. This leads to the following model:

$$z_{tt} = c^2\Delta z - d_0(x)z_t + f \qquad \text{in} \quad \Omega \times (0,T) \tag{60a}$$

$$\frac{\partial}{\partial\nu} z + d_1\, z = 0 \qquad \text{on} \quad \Gamma_1 \times (0,T) \tag{60b}$$

$$\frac{\partial}{\partial\nu} z + D_0\, z_t + \beta D_0 z = w_t \qquad \text{on} \quad \Gamma_0 \times (0,T) \tag{60c}$$

$$w_{tt} + \mathcal{A}w + \mathcal{A}^{\alpha} w_t + \partial_t z|_{\Gamma_0} = \mathcal{B}u \qquad \text{on} \quad \Gamma_0 \times (0,T). \tag{60d}$$

The parameter $\beta \geq 0$ represents static damping on the interface Γ_0. If $\beta > 0$, it was shown in Bucci, Lasiecka and Triggiani (2002), Bucci and Lasiecka (2002) that the resulting system is exponentially stable also with a strong structural damping Dz_t. Precise formulation of this result is given below. Since we wish to consider cases when the damping d_0 is active only on a subportion of Ω we require the following geometric hypothesis.

HYPOTHESIS 7 *We assume that either*

- *Ω is convex and there exists $x_0 \in R^n$ such that $(x - x_0) \cdot \nu \leq 0$ on Γ_1 and $d_0(x) \geq d_0 > 0$ in $\mathcal{U}(\Gamma_0) = \{x \in \Omega, dist(\Gamma_0, \Omega) \leq \delta\}$ for some $\delta > 0$.*

THEOREM 7 *(Bucci and Lasiecka, 2002) Consider (60) with $f = 0, u = 0$ and $d_1 > 0, \beta > 0$ and d_0 subject to Hypothesis 7. Then, the corresponding system is exponentially stable on H, i.e.*

$$|e^{At}|_{\mathcal{L}(H)} \leq Ce^{-\omega t}; \quad t \geq 0$$

REMARK 14 *If the parameter $r_0 < 1/4$, one can take $\beta = 0$.*

REMARK 15 *The static damping βD_0 can be replaced by a more general operator, say D_1, which obeys the same estimates (see Hypothesis 5.) as D_0.*

The addition of static damping $\beta D_0 z$ has no effect on the validity of singular estimate. In fact, it was also shown in Bucci, Lasiecka and Triggiani (2002) that singular estimate of Theorem 4 still holds with $\beta > 0$. Thus, all the assumptions of the abstract Theorem 2 are satisfied and we conclude with our final result:

THEOREM 8 *With reference to system (60) subject to Hypothesis 3–5, 7 and functional cost given by (46), all the statements of abstract Theorem 2 remain valid with A, B introduced in Section 4.2 but with A_z replaced by*

$$A_z \equiv \begin{pmatrix} O & I \\ -A_N - \beta A_N N D_0 N^* A_N & -A_N N D_0 N^* A_N - d_0 \end{pmatrix}. \tag{61}$$

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