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Using intuitionistic fuzzy sets in group decision making

by

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Abstract: The determination of solutions in group decision making via intuitionistic fuzzy sets is considered. The point of departure is a collection of individual intuitionistic fuzzy preference relations. We also assume a (traditional) fuzzy majority equated with a fuzzy linguistic quantifier. A solution is derived either directly from the individual intuitionistic fuzzy preference relations or by constructing first a social intuitionistic fuzzy preference relation. Two solution concepts are proposed, intuitionistic fuzzy core and consensus winner.

Keywords: group decision making, fuzzy preference relations, intuitionistic fuzzy preference relations, core, consensus winner.

1. Introduction

Group decision making consists in deriving a solution (an option or a set of options) from the individual preferences over some set of options in question. The solution may be meant in various ways leading to various solution concepts. Basically, the solution contains options that "best" reflect what a majority of the involved individuals prefer. In the fuzzy context, the basic point of departure is a set of individual fuzzy preference relations. Then, a solution is derived either directly from the individual preference relations, without the derivation of a social fuzzy preference relation, or by constructing first a social fuzzy preference relation and then using it to find a solution (the so-called direct and indirect approach as defined in Kacprzyk (1986).

A straight fuzzification, just by employing fuzzy preference relations, gives rise to a multitude of solution concepts as proposed in Nurmi (1981). One can also fuzzify the very concept of majority, e.g., by employing fuzzy linguistic quantifiers exemplified by *most*, *almost all*, etc. and obtain new solution concepts proposed in Kacprzyk (1986).

In this paper we first discuss group decision making in the fuzzy context what the area by intuitionistic fuzzy sets. To be more precise, we employ intuitionistic fuzzy preference relations instead of (traditional) fuzzy preference relations. As a result we are able to take into account the situations when individuals have insufficient knowledge to describe precisely their preferences. The obtained solutions given as some intervals, make it possible to foresee the best final result (the upper bound of an obtained interval) and the worst one (the lower bound of an obtained interval).

2. Brief introduction to intuitionistic fuzzy sets

As opposed to a fuzzy set in X(Zadeh, 1965), given by

$$A^{'} = \{< x, \mu_{A'}(x) > | x \in X\} \tag{1}$$

where $\mu_{A'}(x) \in [0,1]$ is the membership function of $A^{'}$, an intuitionistic fuzzy set (Atanassov, 1999) A is given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{2}$$

where: $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \tag{3}$$

and $\mu_A(x)$, $\nu_A(x)$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively.

Obviously, each fuzzy set may be represented by the following intuitionistic fuzzy set

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > | x \in X\}. \tag{4}$$

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{5}$$

a *hesitation margin* (or an *intuitionistic fuzzy index*) of $x \in A$, expressing the lack of knowledge as to whether x belongs to A or not (see Atanassov, 1999). It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$.

On the other hand, for each fuzzy set $A^{'}$ in X, we evidently have

$$\pi_{A'}(x) = 1 - \mu_{A'}(x) - [1 - \mu_{A'}(x)] = 0 \text{ for each } x \in X. \tag{6}$$

The application of intuitionistic fuzzy sets instead of fuzzy sets means the introduction of another degree of freedom into a set description (i.e. in addition to μ_A we also have ν_A or π_A). Such a generalization of fuzzy sets gives us an additional possibility of representing imperfect knowledge what leads to describing many real problems in a more adequate way.

3. Group decision making via individual fuzzy preference relations

Suppose we have a set of n options (alternatives) $S = \{s_1, \ldots, s_n\}$ and m individuals. Each individual k, $k = 1, \ldots, m$, provides his or her own preferences over S, which are represented by individual fuzzy preference relations R_k such that:

$$\mu_{R_k} : S \times S \to [0, 1], \tag{7}$$

which may be conveniently represented by the matrix

$$R_k = [r_{ij}^k], \quad i, j = 1, \ldots, n;\ k = 1, \ldots, m, \tag{8}$$

whose elements $0 \leq r_{ij}^k \leq 1$ are such that the higher the preference of individual k of s_i over s_j the higher r_{ij}^k: from $r_{ij}^k = 0$ indicating a definite preference s_j over s_i, through $r_{ij}^k = 0.5$ indicating indifference between s_i and s_j, to $r_{ij}^k = 1$ indicating a definite preference s_i over s_j.

Moreover, it is usually assumed that the matrix R_k is reciprocal, that is

$$\begin{array}{rll} r_{ij}^k + r_{ji}^k = 1 & \text{for all} & i, j = 1, \ldots n,\ i \neq j,\ k = 1, \ldots, m \\ r_{ii}^k = 0 & \text{for} & i = 1, \ldots n \end{array} \tag{9}$$

and, since r_{ii}^k do not matter, "–" is put instead of "0".

Now we assume that all the individual fuzzy preferences relations, $R_1, \ldots, R_m$, are given and the problem is to derive some solution, that is – an option (or a set of options) which is "best" acceptable by the group.

Two main approaches are here possible (Kacprzyk, 1986):

- a *direct approach* which may be represented by the scheme : $\{R_1, \ldots, R_m\} \to$ solution;
- an *indirect approach* which may be represented by the scheme: $\{R_1, \ldots, R_m\} \to R \to$ solution.

A solution concept with much intuitive appeal is here the (traditional) core defined as

$$C = \{s_i \in S : \neg\exists\ s_j \in S \text{ such that } r_{ji}^k > 0.5 \text{ for at least } r \text{ individuals}\} \tag{10}$$

i.e. as a set of undominated options, not defeated by the required (crisp) majority $r \leq m$.

Now, if we just assume that the individual preferences are fuzzy, then the core may be extended (Nurmi, 1981) to the fuzzy α -core defined as

$$C_\alpha = \{s_i \in S : \neg\exists\, s_j \in S \text{ such that } r_{ji}^k \geq \alpha > 0.5 \text{ for at least r individuals}\}. \tag{11}$$

which is a set of options which are not sufficiently (at least to degree $1 - \alpha$) defeated by the required majority $r \leq m$.

Next, assuming also a fuzzy majority given as a fuzzy linguistic quantifier, we can define other solution concepts, such as, e.g.: the fuzzy Q-core and fuzzy α/Q-core (Kacprzyk, 1986).

For instance, Q = "most" may be given as (cf. Kacprzyk, 1986):

$$\mu_{\text{"most"}} = \begin{cases} 1 & \text{for } x \geq 0.8 \\ \text{2x-0.6} & \text{for } 0.3 < x < 0.8 \\ 0 & \text{for } x \leq 0.3 \end{cases} \tag{12}$$

and this form of "most" will be used throughout this paper.

Informally, the fuzzy Q-core is defined as a fuzzy set of options, such that Q individuals are not against them (not defeated by Q individuals) (Kacprzyk, 1986).

To derive a formal definition, let us first denote:

$$h_{ij}^k = \begin{cases} 1 & \text{if } r_{ij}^k < 0.5, \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

i.e. $h_{ij}^k = 1$ means that individual k prefers s_j over s_i. (If not otherwise specified, $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$ throughout the paper). Then

$$h_j^k = \frac{1}{n-1} \sum_{i=1,\, i \neq j}^{n} h_{ij}^k \tag{14}$$

is to what extent individual k is not against s_j : from 0 for certainly against, to 1 for certainly not against, through all intermediate values. Next

$$h_j = \frac{1}{m} \sum_{k=1}^{m} h_j^k \tag{15}$$

is to what extent all the individuals are not against s_j, and

$$v_Q^j = \mu_Q(h_j) \tag{16}$$

is to what extent Q individuals are not against s_j. Then, the fuzzy Q-core is defined as

$$C_Q = v_Q^1/s_1 + \ldots + v_Q^n/s_n. \tag{17}$$

Example 1 Let us have four individuals, $k = 1, 2, 3, 4$, whose individual fuzzy preference relations are, respectively,

$$R_1 = \begin{bmatrix} - & 0.3 & 0.7 & 0.4 \\ 0.7 & - & 0.6 & 0.9 \\ 0.3 & 0.4 & - & 0.5 \\ 0.6 & 0.1 & 0.5 & - \end{bmatrix} \quad R_2 = \begin{bmatrix} - & 0.4 & 0.6 & 0.5 \\ 0.6 & - & 0.7 & 0.7 \\ 0.4 & 0.3 & - & 0.4 \\ 0.5 & 0.3 & 0.6 & - \end{bmatrix}$$

$$R_3 = \begin{bmatrix} - & 0.5 & 0.7 & 0.3 \\ 0.5 & - & 0.8 & 0.7 \\ 0.3 & 0.2 & - & 0.5 \\ 0.7 & 0.3 & 0.5 & - \end{bmatrix} \quad R_4 = \begin{bmatrix} - & 0.4 & 0.7 & 0.6 \\ 0.6 & - & 0.4 & 0.6 \\ 0.3 & 0.6 & - & 0.4 \\ 0.4 & 0.4 & 0.6 & - \end{bmatrix}.$$

To determine $C_{\text{“most”}}$, we start with (13) and obtain:

$$[h_{ij}^1] = \begin{bmatrix} - & 1 & 0 & 1 \\ 0 & - & 0 & 0 \\ 1 & 1 & - & 0 \\ 0 & 1 & 0 & - \end{bmatrix} \quad [h_{ij}^2] = \begin{bmatrix} - & 1 & 0 & 0 \\ 0 & - & 0 & 0 \\ 1 & 1 & - & 1 \\ 0 & 1 & 0 & - \end{bmatrix}$$

$$[h_{ij}^3] = \begin{bmatrix} - & 0 & 0 & 1 \\ 0 & - & 0 & 0 \\ 1 & 1 & - & 0 \\ 0 & 1 & 0 & - \end{bmatrix} \quad [h_{ij}^4] = \begin{bmatrix} - & 1 & 0 & 0 \\ 0 & - & 1 & 0 \\ 1 & 0 & - & 1 \\ 1 & 1 & 0 & - \end{bmatrix}.$$

From (14) and (15) we have:

$$[h_j] = [h_1, h_2, h_3, h_4] = \left[\frac{5}{12}, \frac{10}{12}, \frac{1}{12}, \frac{4}{12}\right],$$

thus, from (16):

$$[v^j_{\text{“most”}}] = \left[\frac{7}{30}, 1, 0, \frac{1}{15}\right]$$

and (17) gives:

$$C_{\text{“most”}} = \frac{7}{30}/s_1 + 1/s_2 + \frac{1}{15}/s_4,$$

which means that s_2 is certainly an element of the fuzzy “most”-core, while s_1 belongs to this core to the extent 7/30, s_4 to the extent 1/15, i.e. not too high; on the other hand s_3 is certainly not in this core. ■

The strength of a defeat of some prespecified level may also be accounted for in the definition of a core. Namely, a fuzzy α/Q-core, $C_{\alpha/Q}$, has been defined by Kacprzyk (1986) as a fuzzy set of options such that Q individuals are not sufficiently (at least to degree $1-\alpha$) against them.

One can also explicitly account for how strongly individual k prefers s_j over s_i, and use Kacprzyk's (1986) definition of a fuzzy s/Q-core, $C_{s/Q}$, which is a fuzzy set of options such that Q individuals are not strongly (to a specific degree from $[0,1]$) against them.

Formally, similarly as for (13)–(17), we have

$$h_{ij}^k(\alpha) = \begin{cases} 1 & \text{if } r_{ij}^k < \alpha < 0.5, \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

$$h_j^k(\alpha) = \frac{1}{n-1} \sum_{i=1,\, i\neq j}^{n} h_{ij}^k(\alpha) \tag{19}$$

$$h_j(\alpha) = \frac{1}{m} \sum_{k=1}^{m} h_j^k(\alpha) \tag{20}$$

$$v_Q^j(\alpha) = \mu_Q(h_j(\alpha)) \tag{21}$$

and the fuzzy α/Q-core, $C_{\alpha/Q}$ is defined as

$$C_{\alpha/Q} = v_Q^1(\alpha)/s_1 + \ldots + v_Q^n(\alpha)/s_n. \tag{22}$$

Finally, one can also explicity account for how strongly individual k prefers s_j over s_i, and use Kacprzyk's definition (Nurmi and Kacprzyk, 1991) of a fuzzy s/Q-core, $C_{s/Q}$, which is a fuzzy set of options such that Q individuals are not strongly against them.

Formally, one can calculate

$$\underline{h}_{ij}^k = \begin{cases} 2(0.5 - r_{ij}^k) & \text{if } r_{ij}^k < 0.5, \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

which expresses how strongly individual k prefers s_j over s_i.

Then, following (13)–(17), we have

$$\underline{h}_j^k = \frac{1}{n-1} \sum_{i=1,\, i\neq j}^{n} \underline{h}_{ij}^k \tag{24}$$

$$\underline{h}_j = \frac{1}{m} \sum_{k=1}^{m} \underline{h}_j^k \tag{25}$$

$$\underline{v}_{s/Q}^j = \mu_Q(\underline{h}_j) \tag{26}$$

and the fuzzy s/Q-core, $C_{s/Q}$, is defined as

$$C_{s/Q} = \underline{v}_{s/Q}^1/s_1 + \ldots + \underline{v}_{s/Q}^n/s_n. \tag{27}$$

Now we will extend the above concepts of fuzzy cores to the case of intuitionistic fuzzy preference relations.

4. Group decision making under individual intuitionistic fuzzy preference relations

Each individual $k = 1, 2, 3, 4$ gives his or her own preferences not only as a preference matrix R_k (as in Section 3.) but also as a matrix Π_k of intuitionistic fuzzy indices. The intuitionistic fuzzy indices $0 \leq \pi_{ij}^k \leq 1$ are such that the higher π_{ij}^k the higher the hesitation margin of individual k as to the preference between s_i and s_j, whose intensity is given by r_{ij}^k. Intuitionistic indices let us calculate the best final result (and the worst one) we can expect in a process leading to a final group decision. During that process an individual k can change his preferences in the following way. He can (maximally) increase his preference by adding the value of the intuitionistic index (i.e. $r_{ij(\max)}^k = r_{ij}^k + \pi_{ij}^k$). So, in fact, his preference lies in the interval $[r_{ij}^k,\ r_{ij}^k + \pi_{ij}^k]$.

Example 2 Let the individual intuitionistic fuzzy preference relations be, respectively:

$$[R_1] = \begin{bmatrix} - & 0.3 & 0.7 & 0.4 \\ 0.7 & - & 0.6 & 0.9 \\ 0.3 & 0.4 & - & 0.5 \\ 0.4 & 0 & 0.3 & - \end{bmatrix} \quad [R_2] = \begin{bmatrix} - & 0.4 & 0.6 & 0.5 \\ 0.6 & - & 0.7 & 0.7 \\ 0.4 & 0.3 & - & 0.4 \\ 0.3 & 0.1 & 0.3 & - \end{bmatrix}$$

$$[\Pi_1] = \begin{bmatrix} - & 0 & 0 & 0.2 \\ 0 & - & 0 & 0.1 \\ 0 & 0 & - & 0.2 \\ 0.2 & 0.1 & 0.2 & - \end{bmatrix} \quad [\Pi_2] = \begin{bmatrix} - & 0 & 0 & 0.2 \\ 0 & - & 0 & 0.2 \\ 0 & 0 & - & 0.3 \\ 0.2 & 0.2 & 0.3 & - \end{bmatrix}$$

$$[R_3] = \begin{bmatrix} - & 0.5 & 0.7 & 0.3 \\ 0.5 & - & 0.8 & 0.7 \\ 0.3 & 0.2 & - & 0.5 \\ 0.4 & 0.1 & 0.2 & - \end{bmatrix} \quad [R_4] = \begin{bmatrix} - & 0.4 & 0.7 & 0.6 \\ 0.6 & - & 0.4 & 0.6 \\ 0.3 & 0.6 & - & 0.4 \\ 0.1 & 0.1 & 0.1 & - \end{bmatrix}$$

$$[\Pi_3] = \begin{bmatrix} - & 0 & 0 & 0.3 \\ 0 & - & 0 & 0.2 \\ 0 & 0 & - & 0.3 \\ 0.3 & 0.2 & 0.3 & - \end{bmatrix} \quad [\Pi_4] = \begin{bmatrix} - & 0 & 0 & 0.3 \\ 0 & - & 0 & 0.3 \\ 0 & 0 & - & 0.5 \\ 0.3 & 0.3 & 0.5 & - \end{bmatrix}.$$

Let us notice that for the above matrices, R_k, the following condition: $r_{ij}^k + r_{ji}^k = 1$, which was fulfilled in Example 1, is not valid any more. Now, the following condition must be fulfilled

$$r_{ij}^k + r_{ji}^k + \pi_{ij}^k = 1 \qquad (28)$$

or:

$$r_{ij}^k + r_{ji}^k + \pi_{ji}^k = 1 \qquad (29)$$

because the matrix Π_k is symmetric. It is obvious that the hesitation margin on decision s_i over s_j is the same as on decision s_j over s_i.

The values $\pi_{ij}^k = 0$ mean that the individual k is certain as to his or her preference of decision s_i over s_j (even when $r_{ij} = 0.5$, which means that there is no preference between s_i and s_j and, moreover, an individual will not change his or her opinion).

From (13) we obtain

$$[h_{ij}^1] = \begin{bmatrix} - & 1 & 0 & [1] \\ 0 & - & 0 & 0 \\ 1 & 1 & - & 0 \\ [1] & 1 & 1 & - \end{bmatrix} \quad [h_{ij}^2] = \begin{bmatrix} - & 1 & 0 & 0 \\ 0 & - & 0 & 0 \\ 1 & 1 & - & [1] \\ 1 & 1 & [1] & - \end{bmatrix}$$

$$[h_{ij}^3] = \begin{bmatrix} - & 0 & 0 & [1] \\ 0 & - & 0 & 0 \\ 1 & 1 & - & 0 \\ [1] & 1 & 1 & - \end{bmatrix} \quad [h_{ij}^4] = \begin{bmatrix} - & 1 & 0 & 0 \\ 0 & - & 1 & 0 \\ 1 & 0 & - & [1] \\ 1 & 1 & [1] & - \end{bmatrix}. \tag{30}$$

It is easy to notice that in the above matrices not always $h_{ij}^k + h_{ji}^k = 1$, which is the effect of (28)–(29), and such cases are indicated in those matrices as "[1]".

For instance in Example 2, shown above, we have: $h_{14}^1 = h_{41}^1 = 1$, $h_{34}^2 = h_{43}^2 = 1$, $h_{14}^3 = h_{41}^3 = 1$, and $h_{34}^4 = h_{43}^4 = 1$. The case: $h_{ij}^k = h_{ji}^k = 1$ means that neither s_i nor s_j is preferred (the final result will be given by the value π_{ij}^k). In this case it is necessary to assume:

$$h_{ij}^k = h_{ji}^k = 0 \tag{31}$$

i.e. to assume that the available knowledge and the individual's opinion make it only possible to say that no option is in fact preferred, which is denoted by "[0]". Finally, due to (31), we obtain the following h_{ij}^k's due to (30):

$$[h_{ij}^1] = \begin{bmatrix} - & 1 & 0 & [0] \\ 0 & - & 0 & 0 \\ 1 & 1 & - & 0 \\ [0] & 1 & 1 & - \end{bmatrix} \quad [h_{ij}^2] = \begin{bmatrix} - & 1 & 0 & 0 \\ 0 & - & 0 & 0 \\ 1 & 1 & - & [0] \\ 1 & 1 & [0] & - \end{bmatrix}$$

$$[h_{ij}^3] = \begin{bmatrix} - & 0 & 0 & [0] \\ 0 & - & 0 & 0 \\ 1 & 1 & - & 0 \\ [0] & 1 & 1 & - \end{bmatrix} \quad [h_{ij}^4] = \begin{bmatrix} - & 1 & 0 & 0 \\ 0 & - & 1 & 0 \\ 1 & 0 & - & [0] \\ 1 & 1 & [0] & - \end{bmatrix}.$$

From (14) and (15) we have

$$[h_j] = \left[\frac{6}{12}, \frac{10}{12}, \frac{2}{12}, 0\right] \tag{32}$$

and (16) gives $[v^j_{\text{"most"}}] = [\frac{4}{10}, 1, 0, 0]$.

But now it is also necessary to take into account the hesitation margin of the individual k, given by Π_k. In the algorithm presented (13)–(17), only the formulas (14) and (15) will be applied now. So, all calculations are performed on the values of the matrix Π_k, i.e. we obtain the aggregated values

$$\pi_j^{'k} = \frac{1}{n-1} \sum_{i=1, i\neq j}^{n} \pi_{ij}^k \tag{33}$$

$$\pi_j' = \frac{1}{m} \sum_{k=1}^{m} \pi_j^{'k} \tag{34}$$

and (33) and (34) yield

$$[\pi_j'] = [\pi_1', \pi_2', \pi_3', \pi_4'] = \left[\frac{1}{12}, \frac{8}{120}, \frac{13}{120}, \frac{31}{120}\right] \tag{35}$$

Hence, the hesitation margin $[\pi_j']$ defined by (35) must be taken into account in the values $[h_j]$ previously calculated by (32). (The value π_j' added to h_j gives the upper bound of the interval). In the discussed example (32) and (35), the values h_j can lay in the following ranges:

$$h_j = \left[\left(\frac{6}{12}, \frac{7}{12}\right), \left(\frac{100}{120}, \frac{108}{120}\right), \left(\frac{20}{120}, \frac{33}{120}\right), \left(0, \frac{31}{120}\right)\right]$$

and thus from (16)

$$v^j_{\text{``most''}} = \left[\left(\frac{4}{10}, \frac{17}{30}\right), (1), (0), (0)\right],$$

and (17) gives

$$C_{\text{``most''}} = (4/10,\, 17/30)/s_1 + 1/s_2,$$

which means that s_2 (as before) is certainly an element of the intuitionistic fuzzy "most"-core, s_3 and s_4 - certainly are not, whereas s_1 belongs to this core to the extent given by a value from the interval $(\frac{4}{10}, \frac{17}{30})$ (in the best situation s_1 can belong to the core to the extent $\frac{17}{30}$, in the worst one - to the extent $\frac{4}{10}$). ■

Taking into account all the previous considerations on the hesitation margin, we can use formulas (18)–(22) and apply them to the intuitionistic fuzzy α/Q-core, $C_{\alpha/Q}$.

Let us analyse again data from Example 2 and find $C_{0.3/\text{``most''}}$

Example 3 First, due to (18)

$$[h_{ij}^1(0.3)] = \begin{bmatrix} - & 1 & 0 & 0 \\ 0 & - & 0 & 0 \\ 1 & 0 & - & 0 \\ 0 & 1 & 1 & - \end{bmatrix} \quad [h_{ij}^2(0.3)] = \begin{bmatrix} - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 1 & - & 0 \\ 1 & 1 & 1 & - \end{bmatrix}$$

$$[h^3_{ij}(0.3)] = \begin{bmatrix} - & 0 & 0 & 1 \\ 0 & - & 0 & 0 \\ 1 & 1 & - & 0 \\ 0 & 1 & 1 & - \end{bmatrix} \quad [h^4_{ij}(0.3)] = \begin{bmatrix} - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 1 & 0 & - & 0 \\ 1 & 1 & 1 & - \end{bmatrix}.$$

From (19)–(20)

$$[h_j(0.3)] = \left[\frac{5}{12}, \frac{7}{12}, \frac{4}{12}, \frac{1}{12}\right],$$

and after taking into account the hesitation margin – the same as in Example 2, i.e.:

$$[\Pi'_j] = \left[\frac{1}{12}, \frac{8}{120}, \frac{13}{120}, \frac{31}{120}\right],$$

we have

$$[h_j(0.3)] = \left[\left(\frac{5}{12}, \frac{6}{12}), \left(\frac{7}{12}, \frac{78}{120}\right), \left(\frac{4}{12}, \frac{53}{120}\right), \left(\frac{1}{12}, \frac{41}{120}\right)\right].$$

Thus, (21) gives

$$[v^j_{\text{“most”}}(0.3)] = \left[\left(\frac{7}{30}, \frac{4}{10}\right), \left(\frac{17}{30}, \frac{21}{30}\right), \left(\frac{1}{15}, \frac{17}{60}\right)\left(0, \frac{1}{12}\right)\right].$$

And, finally, (22) yields

$$C_{0.3/\text{“most”}} = (7/30,\, 4/10)/s_1 + (17/30,\, 21/30)/s_2 + (1/15,\, 17/60)/s_3 + (0,\, 1/12)/s_4.$$

This means that s_1 belongs to this core to the extent given by a value from the interval $(7/30, 4/10)$, s_2, s_3, s_4 - to the extent given by the values from intervals: $(17/30, 21/30)$, $(1/15, 17/30)$, $(0, 1/12)$ respectively. ■

Example 4 Let the data be the same as in Example 2. We will determine $C_{s/\text{“most”}}$.

First, due to (23)

$$[h^1_{ij}] = \begin{bmatrix} - & 0.4 & 0 & 0.2 \\ 0 & - & 0 & 0 \\ 0.4 & 0.2 & - & 0 \\ 0.2 & 1 & 0.4 & - \end{bmatrix} \quad [h^2_{ij}] = \begin{bmatrix} - & 0.2 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0.2 & 0.4 & - & 0.2 \\ 0.4 & 0.8 & 0.4 & - \end{bmatrix}$$

$$[h^3_{ij}] = \begin{bmatrix} - & 0 & 0 & 0.4 \\ 0 & - & 0 & 0 \\ 0.4 & 0.6 & - & 0 \end{bmatrix} \quad [h^4_{ij}] = \begin{bmatrix} - & 0.2 & 0 & 0 \\ 0 & - & 0.2 & 0 \\ 0.4 & 0 & - & 0.2 \end{bmatrix},$$

and by (25)

$$[\underline{h}_j] = \left[\frac{3}{12}, \frac{54}{120}, \frac{24}{120}, \frac{1}{12}\right].$$

Then, taking into account the hesitation margin (calculated in Example 1) we have

$$[\underline{h}_j] = \left[\left(\frac{3}{12}, \frac{4}{12}\right), \left(\frac{54}{120}, \frac{62}{120}\right), \left(\frac{24}{120}, \frac{37}{120}\right), \left(\frac{1}{12}, \frac{41}{120}\right)\right]$$

which gives

$$[v^j_{s/\text{“most”}}] = \left[\left(0, \frac{1}{15}\right), \left(\frac{36}{120}, \frac{52}{120}\right), \left(0, \frac{2}{120}\right)\left(0, \frac{1}{12}\right)\right]$$

and finally

$$C_{s/\text{“most”}} =$$
$$(0,\ 1/15)/s_1 + (36/120,\ 52/120)/s_2 + (0,\ 2/120)/s_3 + (0,\ 1/120)/s_4,$$

which means that s_1, s_2, s_3, and s_4 belong to this core to the extent given by values from the intervals: $(0, 1/15)$, $(36/120, 52/120)$, $(0, 2/120)$, $(0, 1/120)$ respectively. ■

5. Group decision making via a social fuzzy preference relation

Now, in the derivation of a solution, we follow the scheme: $\{R_1, \ldots, R_m\} \to$ $\to R \to$ solution, where R is a social fuzzy preference relation. Therefore, first, an aggregation of the individual fuzzy preference relations into a social fuzzy preference relation should be performed. It is assumed that the social fuzzy preference relation $R = [r_{ij}]$ is determinded as follows (Kacprzyk, 1986):

$$r_{ij} = \begin{cases} \frac{1}{m} \sum_{k=1}^{m} a^k_{ij} & \text{for } i \neq j, \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

$i, j = 1, \ldots, n$, where

$$a^k_{ij} = \begin{cases} 1 & \text{if } r^k_{ij} > 0.5, \\ 0 & \text{otherwise,} \end{cases} \tag{37}$$

$i, j = 1, \ldots, n;\ k = 1, \ldots, m$. R need not to be reciprocal, though $r_{ij} \leq 1 - r_{ji}$.

Example 5 For the four individual intuitionistic fuzzy preference relations from Example 2, we have from (36) and (37)

$$R = [r_{ij}] = \begin{bmatrix} - & 0 & 1 & \frac{1}{2} \\ \frac{3}{4} & - & \frac{3}{4} & 1 \\ 0 & \frac{1}{4} & - & 0 \\ 0 & 0 & 0 & - \end{bmatrix}. \quad ■ \tag{38}$$

We will now consider some counterparts of the two popular solutions concepts: the consensus winner and the α- consensus winner.

The consensus winner is defined as

$$s_i \in W \Leftrightarrow \forall s_j \neq s_i : r_{ij} > 0.5, \tag{39}$$

i.e. an option s_i belongs to the set of consensus winners W if and only if there is no other option preferred over s_i (see Nurmi, 1981).

By introducing a fuzzy majority given as a fuzzy linguistic quantifier Q, the concept of a fuzzy Q-consensus winner may be introduced that is defined as a fuzzy set of options that are preferred over Q other options (Kacprzyk, 1986).

Formally, first we use

$$g_{ij} = \begin{cases} 1 & \text{if } r_{ij} > 0.5, \\ 0 & \text{otherwise,} \end{cases} \tag{40}$$

and

$$g_i = \frac{1}{n-1} \sum_{j=1,\, j \neq i}^{n} g_{ij} \tag{41}$$

is the extent to which s_i is preferred. Then

$$z_Q^i = \mu_Q(g_i) \tag{42}$$

is the extent to which a_i is preferred over Q other options.

The fuzzy Q-consensus winner is then defined as

$$W_Q = z_Q^1/s_1 + \ldots + z_Q^n/s_n. \tag{43}$$

Example 6 Let us determine $W_{\text{“most”}}$ for the social fuzzy preference relation from Example 5. First, from (40), we have

$$[g_{ij}] = \begin{bmatrix} - & 0 & 1 & 0 \\ 1 & - & 1 & 1 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{bmatrix} \tag{44}$$

and from (41),

$$[g_i] = [g_1, g_2, g_3, g_4] = \left[\frac{1}{3}, 1, 0, 0\right]$$

Next, from (42),

$$z^i_{\text{“most”}} = \left[\frac{1}{15}, 1, 0, 0\right]$$

and finally (43) yields:

$$W_{\text{“most”}} = \frac{1}{15}/s_1 + 1/s_2. \qquad \blacksquare$$

The α-consensus winner is in turn defined as (Nurmi and Kacprzyk, 1991)

$$s_i \in W_\alpha \Leftrightarrow \forall s_j \neq s_i : r_{ij} > \alpha \geq 0.5, \tag{45}$$

i.e. an option s_i belongs to the set of α-consensus winners W if and only if there is no other option sufficiently (at least to the degree α) preferred over s_i.

And analogously, a fuzzy α/Q-consensus winner may be defined (Nurmi and Kacprzyk, 1991) as a fuzzy set of options that are sufficiently (at least to degree α) preferred over Q other options.

Formally, first

$$g_{ij}(\alpha) = \begin{cases} 1 & \text{if } r_{ij} > \alpha \geq 0.5, \\ 0 & \text{otherwise,} \end{cases} \tag{46}$$

and

$$g_i(\alpha) = \frac{1}{n-1} \sum_{j=1,\, j\neq i}^{n} g_{ij}(\alpha) \tag{47}$$

is the extent to which s_i is sufficiently (at least to degree α) preferred.

Then

$$z_Q^i(\alpha) = \mu_Q(g_i(\alpha)) \tag{48}$$

is the extent to which s_i is sufficiently preferred over Q other options.

The fuzzy α/Q-consensus winner is then defined as

$$W_{\alpha/Q} = z_Q^1(\alpha)/s_1 + \ldots + z_Q^n(\alpha)/s_n \tag{49}$$

Example 7 For the social fuzzy preference relation from Example 5 we seek $W_{0.8/\text{“most”}}$. First, due to (46)

$$[g_{ij}(0.8)] = \begin{bmatrix} - & 0 & 1 & 0 \\ 0 & - & 0 & 1 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{bmatrix}; \tag{50}$$

next, (47) gives

$$[g_i(0.8)] = [g_1, g_2, g_3, g_4] = \left[\frac{1}{3}, \frac{1}{3}, 0, 0\right]$$

and from (48),

$$z^i_{\text{“most”}}(0.8) = \left[\frac{1}{15}, \frac{1}{15}, 0, 0\right].$$

Hence, (49) yields:

$$W_{0.8/\text{“most”}} = \frac{1}{15}/s_1 + \frac{1}{15}/s_2.$$ ■

We can also introduce the strenght of preference into (40), and define the s/Q-consensus winner (Nurmi and Kacprzyk, 1991), $W_{s/Q}$, that is a fuzzy set of options which are strongly preferred over Q other options.

Formally, first we use

$$\underline{g}_{ij} = \begin{cases} 2(r_{ij} - 0.5) & \text{if } r_{ij} > 0.5, \\ 0 & \text{otherwise} \end{cases} \tag{51}$$

and analogously to (41)–(43),

$$\underline{g}_i = \frac{1}{n-1} \sum_{j=1,\, j \neq i}^{n} \underline{g}_{ij} \tag{52}$$

$$\underline{z}_Q^i = \mu_Q(\underline{g}_i) \tag{53}$$

and the fuzzy s/Q consensus winner is defined as

$$W_{s/Q} = \underline{z}_Q^1/s_1 + \ldots + \underline{z}_Q^n/s_n. \tag{54}$$

Example 8 Let us determine $W_{s/\text{``most''}}$ for the social fuzzy preference relation from Example 5. First, (51) gives

$$[g_{ij}] = \begin{bmatrix} - & 0 & 1 & 0 \\ 0.5 & - & 0.5 & 1 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{bmatrix} \tag{55}$$

and from (52),

$$[g_i] = [g_1, g_2, g_3, g_4] = \left[\frac{1}{3}, \frac{2}{3}, 0, 0\right]$$

which implies by (53),

$$z^i_{\text{``most''}} = \left[\frac{1}{15}, \frac{11}{15}, 0, 0\right],$$

Hence, from (54),

$$W_{s/\text{``most''}} = \frac{1}{15}/s_1 + \frac{11}{15}/s_2.$$

■

6. Group decision making via a social intuitionistic fuzzy preference relation

We will now extend the above concept of social fuzzy preference relations to the case of social intuitionistic fuzzy preference relations.

As we consider intuitionistic fuzzy sets, social intuitionistic fuzzy preference relations must take into account not only the formulas (36)–(37), but also the hesitation margin. As before, all calculations are performed on the values of the Π_k matrix, i.e.:

$$\pi_{ij} = \begin{cases} \frac{1}{m} \sum_{k=1}^{m} \pi_{ij}^k & \text{for } i \neq j, \\ 0 & \text{for } i = j, \end{cases} \tag{56}$$

$i, j = 1, \ldots, n; k = 1, \ldots, m.$

Example 9 For four individuals with the intuitionistic fuzzy preference relations from Example 2, we have from (37) and (36) the matrix R as in Example 5, and from (56) we have:

$$[\pi_{ij}] = \begin{bmatrix} - & 0 & 0 & \frac{1}{4} \\ 0 & - & 0 & \frac{1}{5} \\ 0 & 0 & - & \frac{13}{40} \\ \frac{1}{4} & \frac{1}{5} & \frac{13}{40} & - \end{bmatrix}$$ ■

A similar procedure should be applied in case of the consensus winner and α-consensus winner. In addition to the formulas (40)–(43) for R, one should derive similar formulas for the hesitation margins. So, to (41) we should add

$$\pi_\iota = \frac{1}{n-1} \sum_{j=1,\, j \neq i}^{n} \pi_{ij}. \tag{57}$$

which yields for the data from Example 2:

$$\Pi = \left[\frac{1}{12}, \frac{1}{15}, \frac{13}{120}, \frac{31}{120} \right]. \tag{58}$$

Taking (58) into account (as before, the value π_ι added to z_i gives the upper bound of the interval) we have for the data from Example 6:

$$z^i_{\text{"most"}} = \left[\left(\frac{1}{15}, \frac{3}{20} \right), 1, \left(0, \frac{13}{120} \right), \left(0, \frac{31}{120} \right) \right] \tag{59}$$

and

$$W_{\text{"most"}} = \left(\frac{1}{15}, \frac{3}{20} \right) / s_1 + 1/s_2 + \left(0, \frac{13}{120} \right) / s_3 + \left(0, \frac{31}{120} \right) / s_4. \tag{60}$$

In an analogous way the procedure can be augmented in case of the α-consensus winner, i.e. for (46) - (49). Now the fuzzy solution obtained in Example 7 (α =0.8) must be modified by taking into account (58) what leads to the intuitionistic fuzzy solution:

$$z^i_{\text{"most"}}(0.8) = \left[\left(\frac{1}{15}, \frac{3}{20} \right), \left(\frac{1}{15}, \frac{2}{15} \right) \left(0, \frac{13}{120} \right), \left(0, \frac{31}{120} \right) \right] \tag{61}$$

$$W_{0.8/\text{``most''}} = \left(\frac{1}{15}, \frac{3}{20}\right)/s_1 + \left(\frac{1}{15}, \frac{2}{15}\right)/s_2 + \left(0, \frac{13}{120}\right)/s_3 + \left(0, \frac{31}{120}\right)/s_4. \tag{62}$$

We proceed analogously in the case of the s/Q-consensus winner (51)-(54). Taking into account (58) (the value π_i added to z_i gives the upper bound of the interval), we have

$$z^i_{\text{``most''}} = \left[\left(\frac{1}{15}, \frac{3}{20}\right), \left(\frac{11}{15}, \frac{12}{15}\right)\left(0, \frac{13}{120}\right), \left(0, \frac{31}{120}\right)\right]. \tag{63}$$

Hence from (54)

$$W_{s/\text{``most''}} = \left(\frac{1}{15}, \frac{3}{20}\right)/s_1 + \left(\frac{11}{15}, \frac{12}{15}\right)/s_2 + \left(0, \frac{13}{120}\right)/s_3 + \left(0, \frac{31}{120}\right)/s_4. \tag{64}$$

7. Concluding remarks

In this paper we have proposed some solution concepts in group decision making under "intuitionistic fuzziness", i.e. with intuitionistic (individual and social) fuzzy preference relations. The proposed solution concepts seem to well reflect real perception and intuition as to how group decision are to be made, and the use of intuitionistic fuzzy sets seems to provide a tool for accommodating some more general concept aspect of fuzziness.

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