

## Two factors utility approach

by

Roman Kulikowski

Systems Research Institute, Polish Academy of Sciences,  
Newelska 6, 01-447 Warszawa, Poland

**Abstract:** This paper deals with optimization of portfolios composed of securities (equities). The drawbacks of existing methodologies, based on a single factor utility function, are indicated. The two-factor utility function introduced takes into account the expected excess return and expected worst case return (both in monetary units). Assuming that utility is "risk averse" and "constant returns to scale", a theorem on existence of optimum strategy of investments is proven. The optimum strategy is derived in an explicit form. A numerical example is also given.

**Keywords:** portfolio optimization, utility function, investment allocation, risk aversion, expected return, worse case return, optimum investment strategies, portfolio variance

### 1. Introduction

The basic portfolio optimization methodology rests formally on the conditional optimization problems, see e.g. Elton, Gruber (1994), Markowitz (1952). An objective function, called utility, such as e.g. the expected portfolio return, is maximized subject to the constraints including a risk measure, such as variance. A concrete example of such an approach, called mean-variance, is provided by the well known paper by H. Markowitz.

It should be observed that in order to describe properly investor's behaviour, such as risk aversion, one has to deal with a nonlinear (increasing, concave) utility function. Unfortunately the exact analytical form of that function is unknown. Assuming a concrete utility function (from the class of possible risk averse functions) one gets a solution, which generally depends on the analytic form of utility function adopted.

Another class of simplified alternatives to the expected utility approach, stems from the belief that investors prefer to apply criteria that concentrate on worse outcomes (returns). The first criterion developed by Roy, Elton, Gruber (1994) states that the best portfolio has the smallest probability of return below

a specified level. The Kataoki and Telsar criteria, Elton, Gruber (1994), also belong to the class of worst case approaches. An obvious drawback of worse case approaches is the absence of the risk aversion mechanism (i.e. the decreasing marginal return of utility), which characterizes most of the investors.

In the present paper an attempt has been made to incorporate worse case criterion into the utility function as an additional factor. In other words the two-factor utility function is proposed, with expected return and worse case return, as the main factors describing decision makers behaviour.

It is also assumed that factors are expressed in monetary units and utility is a homogeneous, constant return to scale function. Since utility cannot be changed by a change of monetary units such an assumption is obvious.

Then one can show that an optimal strategy, determining the structure of optimal portfolio of assets exists and can be derived in an explicit manner.

The solution does not depend on the exact analytic form of utility function (unless it does not belong to the class of strictly concave, scale preserving functions). Being "universal" within that class of functions the two-factor utility function is able to represent and satisfy different individual decision makers.

The two-factor approach is also convenient for system analysts who construct portfolio decision support systems. They do not need to worry about the identification of an investor's utility function.

It can be used for optimization of derivative securities with asymmetrical probability distribution functions. It should also be noted that the two-factor utility approach has already been used for the optimum allocation of labour resources, Kulikowski (1993, 1994). The present paper can therefore be regarded as an extension of the two-factors approach to the capital allocation problems.

## 2. Single factor utility functions

There exists an impressive literature on the single factor utility theory (see e.g. Elton, Gruber, 1994; Zenios, 1993). In the present paper the utility function  $\bar{U}(z_1, z_2, \dots, z_n)$  of the portfolio consisting of  $n$  assets, generating the monetary returns  $z_1, \dots, z_n$ , will be used. The monetary, one period, return for equities is defined as follows:

$$z_i(t) = P_i(t+1) - P_i(t) + D_i(t+1)$$

where

$P_i(t)$  is the price of the  $i$ -th security in period  $t$ ,

$D_i(t)$  is the dividend received in period  $t$ .

The notion of return (non monetary):

$$R_i(t) = z_i(t) : P_i(t)$$

will be also used.

The utility function  $\bar{U}(z_1, z_2, \dots, z_n)$  is usually employed

## 2.1. Additivity

For analytic convenience it is usually assumed for the utility of the portfolio that

$$\bar{U}(z_1, \dots, z_n) = \frac{1}{n} \sum_{i=1}^n U(z_i) \quad (1)$$

The formal justification for that assumption can be based on the well known Von Neumann and Morgenstern theorem that (under the specified set of five axioms) the utility of a gamble equals the expected utility of its outcomes, which is known as the "expectation principle". Obviously, the expectation principle has an appealing psychological interpretation.

## 2.2. Risk aversion

The analytical form of the  $U(z)$  function is generally unknown. However, the psychological considerations suggest that  $U(z)$  should be continuously increasing and concave. Such a utility is called risk averse (R.A.). Additional property is connected with marginal effects with respect to the wealth level of the investor. Generally, the richer the investor, the more he is inclined to invest. In formal terms the coefficient

$$a(z) = -z \frac{U''(z)}{U'(z)}, \quad (2)$$

called the relative risk aversion, should be negative, Elton, Gruber (1994).

When  $a(z) < 0$  the percentage of funds invested in assets increases as wealth increases.

## 3. Portfolio optimization

The single factor portfolio optimization problem can be formulated as follows.

Introduce the variables  $x_i = X_i/X$ , where  $X$  ( $X_i$ ) is the total (asset  $i$ ) funds the investor is willing to invest in risky assets (labelled by the index  $i = 1, \dots, n$ ). Investors want to find such a vector  $x = \hat{x}$ , that

$$\max_{x \in \Omega} \sum_{i=1}^n U_i(x_i) = \sum_{i=1}^n U_i(\hat{x}_i), \quad (3)$$

where  $\Omega$  is the admissible set.

For example:

$$\Omega = \left\{ x : \sum_{i=1}^n x_i = 1, \quad x_i \geq 0, \quad V(x) \leq V_0 \right\},$$

$V(\tau)$  – the risk measure,  $V_0$  – given number

When  $U(x_i)$  is the expected return, i.e.  $U(x_i) = \bar{R}_i x_i$  and  $V(x)$  – portfolio variance:

$$V(x) = \sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j \neq i}^n x_i x_j \sigma_i \sigma_j \rho_{ij}, \quad (4)$$

$\sigma_i$  = standard deviations,  $\rho_{ij}$  correlation coefficients between assets labelled  $i, j$ ; one gets the classical “mean-variance” portfolio optimization problem.

It can be observed that the optimization strategy  $\hat{x}$  (if it exists) depends on the analytic form of the utility function  $U(x)$ . That form is generally unknown though some properties such as, risk aversion,  $a(x) < 0$ , etc. can be postulated. Assuming a concrete form of  $U(x)$ , e.g.

$$U(x) = x - bx^2, \quad b = \text{positive constant},$$

one can show that the optimum solution obtained by a particular methodology (e.g. mean-variance) is compatible with one property (risk aversion) but at the same time is not compatible with the other, e.g.  $a(x) < 0$  (for discussion of such situations see Elton, Gruber, 1994).

There exists, of course, the possibility of identification of the utility function by experiments, conducted with real investors. It seems, however, that investors do not like identification experiments. Besides, the identified functions are not stable in time. They depend on age, financial status and emotions of the decision makers.

#### 4. Worse case return

An alternative (to the maximization of utility) approach to the decision problems stems from the belief that decision makers concentrate on the bad outcomes mostly. For example, the approach developed by Roy (1952) states that the best asset (having the return  $R_i$ ) is the one that has the smallest probability of producing returns below a specified ( $R_F$ ) level, i.e.

$$\hat{R}_i = \min_{j \in J} \text{Prob}(R_j < R_F), \quad J = \text{the set of all assets}.$$

If returns are normally distributed then the optimum asset would be the one which corresponds to the maximum number of standard deviations ( $\sigma_i$ ) away from the mean ( $\bar{R}_i$ ).

That criterion is, obviously, equivalent to

$$\hat{R}_i = \max_j \frac{R_j - R_F}{\sigma_j} \quad (5)$$

Another possible approach is the one which assumes that the “worse case” probability is fixed. For example, one can assume that the probability distribution of the returns of the asset is normal and is illustrated by Fig. 1. Every



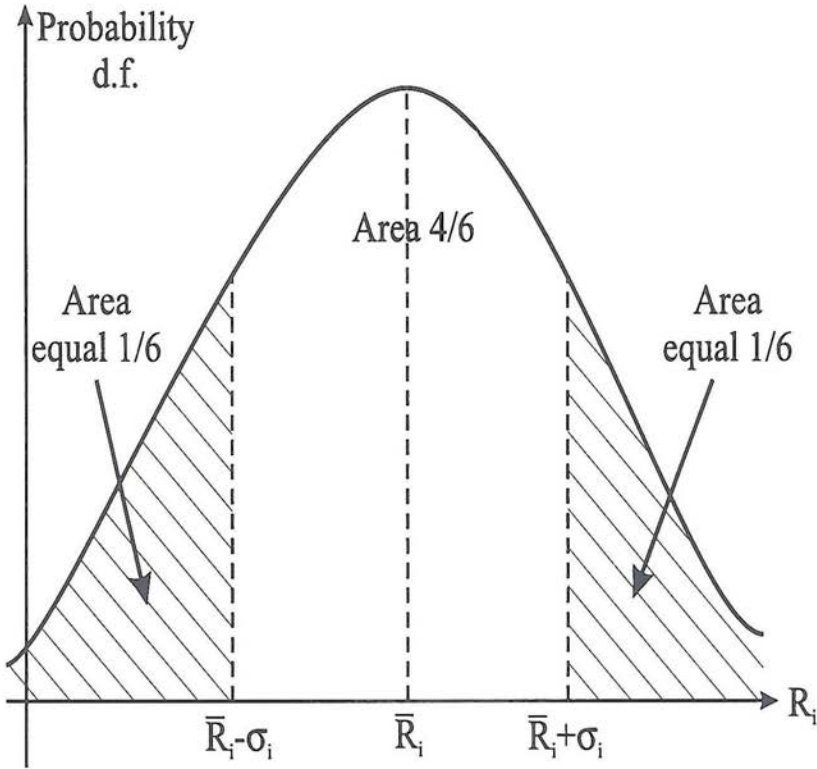


Figure 1.

out of six actual outcomes should, on average, lie within one standard deviation  $\sigma_i$  of the expected outcome  $\bar{R}_i$ . However, two times, out of six, the outcome can be expected to lie outside one standard deviation and one out of six will lie below  $\bar{R}_i - \sigma_i$ .

The best asset corresponds here to  $\hat{R}_i = \max_i(\bar{R}_i - \sigma_i)$ .

The present approach can be generalized to the situation where the probability distribution is not normal. For that purpose the Tchebyshev inequality, see Korn (1968):

$$\text{Prob}(|R_i - \bar{R}_i| \geq a_i) \leq \left(\frac{\sigma_i}{a_i}\right)^2, \quad \forall a_i > 0. \quad (6)$$

can be used.

Assuming  $\left(\frac{\sigma_i}{a_i}\right)^2 = p_0$ , where  $p_0$  is a given number ( $p_0 \in [0, 1]$ ), e.g.  $p_0 = \frac{1}{3}$ , one finds:  $a_i = \frac{\sigma_i}{\sqrt{p_0}} = 1.732 \sigma_i$ . Then, one out of three actual outcomes should

on the average lie outside the interval

$$|R_i - 1.732 \sigma_i|$$

If  $R_i$  has a continuous unimodal distribution, an estimate of probabilities stronger than (6) can be used, see Korn (1968):

$$\text{Prob}(|R_i - \bar{R}_i| \geq a_i) \leq \frac{4}{9} \cdot \frac{1 + s_i^2}{\left(\frac{a_i}{\sigma_i} - |s_i|\right)^2}, \quad (7)$$

where  $s_i$  is the Pearson measure of asymmetry ( $s_i = \frac{\bar{R}_i - R_M}{\sigma_i}$ ,  $R_M$  = return corresponding to the max of distribution function, when  $s_i = 0$ , one gets the distribution which is symmetrical with respect to  $R_M$ ).

Assuming

$$\frac{4}{9} \cdot \frac{1 + s_i^2}{\left(\frac{a_i}{\sigma_i} - |s_i|\right)^2} = p_0, \quad (0 < p_0 < 1),$$

one gets

$$a_i = \varphi(P_0), \text{ where } \varphi(P_0) = \left[ \frac{2}{3} \sqrt{\frac{1 + s_i^2}{p_0}} + |s_i| \right].$$

For  $s_i = 0$ , and  $p_0 = 1/3$  one obtains  $a_i = 1.155 \sigma_i$ . Then for the symmetrical p.d.f. one out of six actual outcomes should lie below the  $\bar{R}_i - a_i = \bar{R}_i - 1.155 \sigma_i$  level. Observe also that when  $p_0$  increases then all levels  $a_i$  are decreasing.

It should be observed that the worse case approach does not take into account the concavity (risk aversion) of the utility of investors. For that reason it cannot be recommended as a general criterion for portfolio optimization methodology.

## 5. Two-factors utility approach

The two-factor approach stems from the belief that in order to properly describe the investor's behaviour one should take into account two factors: monetary expected return and "the worse case" monetary return. It should be noticed, that from the formal point of view, the risk measure, in the single-factor-utility optimization problems, enters into the constraints. In the two-factor approach it is incorporated in the structure of the utility function.

In other words the utility of the  $i$ -th asset can be written

$$U(Z_i x_i, Y_i),$$

where

$Z_i = P_i \bar{R}_i$  monetary expected return on one unit of  $i$ -th asset, having price  $P_i$

$x_i$  the number of  $i$ -th assets in the portfolio (a decision variable),  
 $Y_i = P_i(\bar{R}_i - \kappa\sigma_i)$  worse case monetary return, where  $\sigma_i$  is the standard deviation.

The threshold levels  $\kappa\sigma_i$  depend on the asset's p.d. function. For a normal distribution, as already mentioned it can be assumed that, e.g.,  $\kappa = 1$ . For a unimodal symmetrical distribution  $\kappa = 1.155$ .

Since worse case frequency (i.e.  $p_0$ ) is assumed to be given (e.g.  $1/6$ ) one can say that  $Y_i$  represents the worse expected monetary return level. One can expect returns to be not more than  $\bar{R}_i - a_i$  once out of six periods.

On the other hand  $Z_i$  represents the expected monetary return. Generally the parameter  $\bar{R}_i$  can be regarded as the individual investor's expectation, which may differ from the mean value, based on historical observations, or – from the market expectations.

It is also assumed that the investor is driven by the desire to get maximum utility from the portfolio (which consists of  $n$  assets, each purchased in quantity  $x_i$ ) and he, or she wants to get a given value ( $Z$ ) of total return:

$$\sum_{i=1}^n Z_i x_i = Z.$$

In order to solve the portfolio optimization problem explicitly the following important assumption should be introduced.

Constant return to scale (CRS) and risk averseness (RA)

The function  $U(Z_i x_i, Y_i)$  is CRS, (i.e. homogeneous degree one) so it can be written in the following form:

$$U(Z_i x_i, Y_i) = Y_i F\left(\frac{Z_i}{Y_i} x_i\right) = Z_i A_i F\left(\frac{x_i}{A_i}\right),$$

where  $F$  is strictly concave;  $F(\cdot) > 0$ ,  $F'(\cdot) > 0$ ,  $F''(\cdot) < 0$ , and

$$A_i = \frac{Y_i}{Z_i} = 1 - \kappa_i \frac{\sigma_i}{\bar{R}_i} > 0, \quad \bar{R}_i > 0, \quad \forall i. \quad (8)$$

The number  $A_i$ , which reflects the investor's confidence in  $i$ -th asset, can be called the coefficient of assurance.

One can observe that utility introduced concerns the expected return  $Z_i$ , the risk measure  $A_i$  and the number of securities  $x_i$ . For risk free asset  $A_i = 1$  and the utility reduces to the classical single factor utility function  $U_i = P_i \bar{R}_i F(x_i)$ , which is used commonly in economic sciences.

It is also assumed that the risk averse (RA) investor is interested in assets with positive worse case return only, i.e. one assumes  $P_i, \bar{R}_i, A_i$ , to be positive  $\forall i$ . Then the utilities

$$U_i = P_i \bar{R}_i A_i F\left(\frac{x_i}{A_i}\right), \quad \forall i \text{ are strictly concave.}$$

REMARK 5.1 Observe that when  $U_i$  is not CRS one can generate additional utility by changing monetary units (e.g. US \$ for cents) of the factors  $Z_i, Y_i$ .

The following are two examples of CRS functions:

$$1. U(Z_i x_i, Y_i) = (Z_i x_i)^\alpha (Y_i)^{1-\alpha}, \quad F(\cdot) = \left(\frac{Z_i}{Y_i} x_i\right)^\alpha$$

$$2. U(Z_i x_i, Y_i) = [\alpha (Z_i x_i)^\nu + (1-\alpha) Y_i^\nu]^{1/\nu}, \quad F(\cdot) = \left[\alpha \left(\frac{Z_i}{Y_i} x_i\right)^\nu + 1 - \alpha\right]^{1/\nu}$$

where  $\alpha, \nu$  are positive numbers ( $0 < \alpha < 1$ ), ( $0 < \nu < 1$ ).

THEOREM 5.1 A unique investment strategy  $x \triangleq \hat{x}$ , for an investor with strictly concave utility exists, such that:

$$\max_{x \in \Omega} \sum_{i=1}^n Y_i F\left(\frac{x_i}{A_i}\right) = Y F\left(\frac{Z}{Y}\right), \quad (9)$$

where

$$\Omega = \{x : \sum_{i=1}^n Z_i x_i = Z, \quad x_i \geq 0, \quad \forall i\},$$

$$Y = \sum_{i=1}^n Y_i = \text{total worse case return},$$

$$A = \sum_{i=1}^n A_i, \quad Z = \text{required level of return}.$$

The optimum strategy becomes

$$\hat{x}_i = A_i \frac{Z}{Y}, \quad \forall i \quad (10)$$

and

$$U(Z, Y) = Y F\left(\frac{Z}{Y}\right).$$

**Proof.** The problem reduces to  $\max_x \sum_i Y_i F(x_i/A_i)$  subject to the constraint:

$$\sum_i Z_i x_i = Z.$$

The Lagrangean of the problem becomes

$$\phi(x, \lambda) = \sum_{i=1}^n Y_i F\left(\frac{x_i}{A_i}\right) + \lambda \left[ Z - \sum_{i=1}^n Z_i x_i \right],$$

where  $\lambda$  = Lagrange multiplier.

The necessary conditions of optimality require that

$$\phi'_{x_i} = Y_i/A_i F'\left(\frac{\hat{x}_i}{A_i}\right) - \lambda Z_i = 0, \quad \forall i, \quad (11)$$

$$\phi_\lambda = Z - \sum_{i=1}^n Z_i \hat{x}_i = 0. \quad (12)$$



Since  $Y_i/A_i = Z_i$ ,  $\lambda = F(Z/Y) = \text{const}$ , the strategy (10) satisfies the first order condition (11). Setting that strategy in (12) one finds also that

$$\sum_{i=1}^n Z_i \frac{A_i}{Y} Z = \frac{Z}{Y} \sum_{i=1}^n A_i Z_i = Z,$$

and

$$\sum_{i=1}^n Y_i F\left(\frac{\hat{x}_i}{A_i}\right) = Y F\left(\frac{Z}{Y}\right) = U(Z, Y).$$

In order to complete the proof of optimality of (10) it is necessary to check the Kuhn-Tucker conditions at  $x_i = 0$ ,  $\forall i$ .

In our simple case these conditions require that for  $x_i = 0$ ,  $\forall i$  the derivatives  $U(x_i)_{x_i=0}$  be positive, i.e.

$$U'_{x_i} = \frac{Y_i}{A_i} F'\left(\frac{x_i}{A_i}\right) \Big|_{x_i=0} = P_i \bar{R}_i F'(0) > 0, \quad \forall i$$

Since  $F'(0) > 0$ ,  $\bar{R}_i > 0$ ,  $\forall i$ , by assumptions, the Kuhn-Tucker conditions hold.

Since the objective function is strictly concave in  $\Omega$  the strategy (10) is unique and the sufficiency condition holds. ■

It can be observed that for CRS, RA utility, the strategy (10) represents also the solution of the problem:

$$\max_{x \in \Omega'} U\left(\sum_{i=1}^n Z_i x_i, Y\right) = U(Z, Y),$$

where

$$\Omega' = \left\{ x : \sum_{i=1}^n Z_i x_i \leq Z, \quad x_i \geq 0, \quad \forall i \right\}$$

$$Y = \sum_{i=1}^n Y_i = \sum_{i=1}^n P_i \bar{R}_i A_i.$$

Indeed, since within  $\Omega'$  there is no stationary point, i.e.

$$U'_{x_i} = Y F'\left(\frac{\sum_{i=1}^n Z_i x_i}{Y}\right) \frac{Z_i}{Y} > 0, \quad \text{for } x_i \in \Omega, \quad \forall i$$

the optimum solution (according to the Weierstras theorem or Kuhn-Tucker condition) is located on the border line i.e.  $\sum_{i=1}^n Z_i \hat{x}_i = Z$

REMARK 5.2 The total return level  $Z$  in the formulation of the Theorem can be replaced by the given total initial investment value,  $\sum_i P_i \hat{x}_i$  denoted by  $X$ . Indeed,

$$X = \sum_i P_i \hat{x}_i = \frac{Z}{Y} \sum_i P_i A_i, \quad (13)$$

so

$$\frac{Z}{Y} = \frac{X}{P}, \text{ where } P = \sum_i P_i A_i, \text{ and}$$

$$\hat{x}_i = A_i \frac{X}{P}, \quad \forall i \quad (14)$$

Then one obtains

$$U = YF(X/P).$$

REMARK 5.3 The optimum strategies (10), (14) do not depend on the individual investor's utility function (unless it is not RA & CRS function). Though optimal strategies are "universal", each individual can enjoy his own level of utility, which is specified by his individual  $F$ -function (expressed in monetary units of  $Y$  with  $F(Z/Y)$  as a dimensionless multiplier). Suggesting these strategies to an investor one should not worry about identification of the investor's utility function. However, in such a case it is important to check: does the investor accept the two factor utility as a function which, in the best sense, reflects his or her tastes and preferences? That is especially important for system analysts who construct the portfolio decision support systems.

REMARK 5.4 The optimum strategies do not allow for the so called "short selling", i.e.  $\hat{x}_i < 0$ . In the case when, for an asset,  $A_i \leq 0$ , the asset should be dropped from the portfolio. In such a case it is also possible to decrease the general level of all  $a_i$  by increasing the probability  $p_0$ , as already mentioned.

REMARK 5.5 The two-level approach enables one to optimize portfolios with the asymmetric probability distribution function, such as derivative securities (e.g. equities + options).

## 6. An application

The two-factor utility theory can be used to derive the optimum portfolio consisting of equities. In particular, it can be used for  $n$  fully diversified portfolios of equities. Since diversification removes the unsystematic risk component it is convenient to deal with portfolios which are diversified at the preliminary

$i$	$R_i$	$\beta_i \sigma_m$	$A_i$	$P_i/X$	$\hat{x}_i$	$\hat{x}_i P_i/X$
1	0.19	0.10	0.47	0.033	10.77	0.356
2	0.23	0.15	0.35	0.025	8.01	0.200
3	0.14	0.05	0.64	0.025	14.66	0.366
4	0.25	0.20	0.20	0.017	4.58	0.078

Table 1.

The systematic risk component can be derived by employing the so called beta coefficient, Elton, Gruber (1994):

$$\beta_i = \sigma_{im} / \sigma_m^2, \quad i = 1, \dots, n,$$

$\sigma_{im}$  covariance between market and  $i$ -th portfolio,  
 $\sigma_m^2$  market variance.

The parameters  $\sigma_{im}$ ,  $\sigma_m^2$  and  $\beta_i$  can be derived using historical data. The systematic risk component  $\sigma_i$  can be derived using relation, see Elton, Gruber (1994):

$$\sigma_i = \beta_i \sigma_m, \quad \forall i.$$

Since  $\beta_i$  is a measure of correlation between the  $i$ -th asset and the market the risk (expressed by  $\sigma_i$ ) decreases along with  $\beta_i$ . At the same time the coefficient of assurance increases and so does the number of shares chosen by formulae (14).

When the numerical values of parameters  $\bar{R}_i$ ,  $\beta_i$ ,  $\sigma_m$  and  $P_i/X$  are given one can derive  $\hat{x}_i$  explicitly. As an example consider four diversified portfolios with  $\sigma_m = 0.10$  and the rest of parameters given in Table 1.

Since  $P/X = \frac{1}{X} \sum_{i=1}^n A_i P_i = \frac{0.0437}{X}$ , one gets  $X/P = 22.90$ . Then the optimum strategy  $\hat{x}_i$  and the investment shares  $\hat{x}_i P_i/X$ ,  $i = 1, 2, 3, 4$ , can be derived, as shown in Table 1.

## References

- ELTON, E.J., GRUBER, M.J. (1994) *Modern Portfolio Theory and Investment Analysis*. V-th Edition, New York, John Wiley & Sons.
- KORN, G.A., KORN, T.M. (1968) *Mathematical Handbook*. New York, McGraw-Hill.
- KULIKOWSKI, R. (1994) A theory of motivation and satisfaction with application to decision support. *Annals of Operation Research*, **51**, 265-281.
- KULIKOWSKI, R. (1993) Decision support systems in allocation of capital and labour resources. *Systems Analysis and Decision Support in Economics and Technology: Proc. of Symposium*. Eds. R. Kulikowski, K. Szkatuła, J. Kacprzyk. Warsaw, Omnitech Press.

- MARKOWITZ, H. (1952) Portfolio selection. *Journal of Finance*, **7**, 77-91.
- ROY, A.D. (1952) Safety first and holding of assets. *Econometrica*, **20**, 431-449.
- ZENIOS, S.E. Ed. (1993) *Financial Optimization*. Cambridge, Cambridge University Press.