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Trackability and bounded output bounded input trackability of linear discrete-time systems

by

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Abstract: The notions of systems trackability and bounded output bounded input (BOBI) trackability are given. Then, necessary and sufficient conditions are formulated for trackability of linear discrete-time systems.

Keywords: controllability, trackability, linear discrete-time systems.

1. Introduction

One of the basic properties of dynamical systems are observability and controllability (Kalman, Falb and Arbib, 1969). They are the necessary and sufficient conditions for design of asymptotically stable control systems. However, in many practical problems there are additional requirements: system output should track given reference output. The tracking problem has been considered in many papers, e.g. Grizzle, Di Benedetto and Lamnabhi-Lagarrigue (1994), Hirschorn and Davies (1987), Hu and Tomizuka (1993), Man and Palaniswami (1994), Kaczorek (1981). In this paper the notions of systems trackability and bounded output bounded input (BOBI) trackability are introduced. Then, necessary and sufficient conditions are formulated for linear discrete-time systems trackability and BOBI trackability. Numerical examples are given.

2. System trackability

Given linear discrete-time system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

where $x \in R^n$ is a state vector, $u \in R^m$ is an input vector, $y \in R^p$ is an output vector, and A, B, C and D are real matrices of appropriate dimensions.

Trackability of the system is defined as follows:

DEFINITION 1. System (1) is trackable if and only if for a given reference output sequence $\{y_r(k),\ k = 0, 1, \ldots\}$ and an initial state $x(0) = x_0 \in R^n$ there are integer number $t < \infty$ and input sequence $\{u(k),\ k = 0, 1, \ldots\}$ such that $y(k) = y_r(k)$ for $k = t, t + 1, \ldots$

DEFINITION 2. The trackability index t_I of the trackable system (1) is the minimal integer number independent of the initial state and the reference output sequence such that for any reference output $\{y_r(k),\ k = 0, 1, \ldots\}$ there exists control input sequence $\{u(k),\ k = 0, 1, \ldots\}$ which gives $y(k) = y_r(k)$ for $k = t_I, t_I + 1, \ldots$.

Then, the following theorem can be proven.

THEOREM 1. *The following statements are equivalent:*

(i) system (1) is trackable,

(ii) $\operatorname{rank} \Delta(0:n) = \operatorname{rank}\Delta(0:n-1) + p$,

(iii) matrix $\Delta(n:np)$ *has full row rank*

(iv) matrix $\Delta(n:2n+1)$ *has full row rank where*

$$\Delta(0:j) = \begin{bmatrix} D & 0 & 0 & \ldots & 0 \\ CB & D & 0 & & \\ CAB & CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ CA^{j-1}B & \ldots & CAB & CB & D \end{bmatrix}$$

and

$$\Delta(i:j) = \begin{bmatrix} CA^{i-1}B & \ldots & D & & 0 \\ \vdots & & & \ddots & \\ CA^{j-1}B & \ldots & CAB & CB & D \end{bmatrix}.$$

Proof.

(ii) $\Rightarrow$ (i). From (1) we have

$$Y(0:N) = C(0:N)x(0) + \Delta(0:N)U(0:N) \tag{2}$$

where

$$Y(s:N) = \begin{bmatrix} y(s) \\ \vdots \\ y(N) \end{bmatrix}, \qquad C(s:N) = \begin{bmatrix} CA^s \\ \vdots \\ CA^N \end{bmatrix},$$

$$U(s:N) = \begin{bmatrix} u(s) \\ \vdots \\ u(N) \end{bmatrix}.$$

Thus, by calculating the input sequence one obtains the following equation

$$\Delta(0:N)U(0:N) = Y_r(0:N) - C(0:N)x(0) \tag{3}$$

where vector of reference output $Y_r(0:N)$ is analogous to $Y(0:N)$.

From Cayley–Hamilton theorem we have for $m \geq 0$

$$A^{n+m} = -\sum_{i=0}^{n-1} a_i A^{m+i}$$

where a_i are coefficients of the characteristic polynomial of matrix A

$$\det(Iz - A) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0.$$

Hence, for $N > n$ and nonsingular matrix $T_N \in R^{(N+1)m\times(N+1)m}$

$$T_N = \begin{bmatrix} I_m & & & & & 0 \\ a_{n-1}I_m & I_m & & & & \\ \vdots & & \ddots & & & \\ a_0 I_m & & & I_m & & \\ & \ddots & & & \ddots & \\ 0 & & a_0 I_m & \ldots & a_{n-1}I_m & I_m \end{bmatrix}$$

one has

$$\tilde{\Delta}(0:N) = \Delta(0:N)T_N = \begin{bmatrix} D & & & & & 0 \\ \Psi_0 & D & & & & \\ \vdots & & \ddots & & & \\ \Psi_{n-1} & & & D & & \\ & \ddots & & & \ddots & \\ 0 & & \Psi_{n-1} & \ldots & \Psi_0 & D \end{bmatrix} \tag{4}$$

where I_m denotes the $m \times m$ identity matrix and

$$\Psi_i = CA^iB - \sum_{j=0}^{i-1} a_{n-i+j}CA^jB - a_{n-i-1}D, \qquad i \in [0, n-1].$$

It is easy to see now that one can calculate appropriate control input sequence such that $Y(n:N)$ is equal to $Y_r(n:N)$ if condition (ii) is satisfied.

(i) $\Rightarrow$ (ii) Assume that condition (ii) is not satisfied. Then, there exist nonsingular matrices $W_D \in R^{p\times p}$, $V_D \in R^{m\times m}$, and

$$W_N = \operatorname{block\,diag}(W_D) \in R^{(N+1)p\times(N+1)p}$$

and

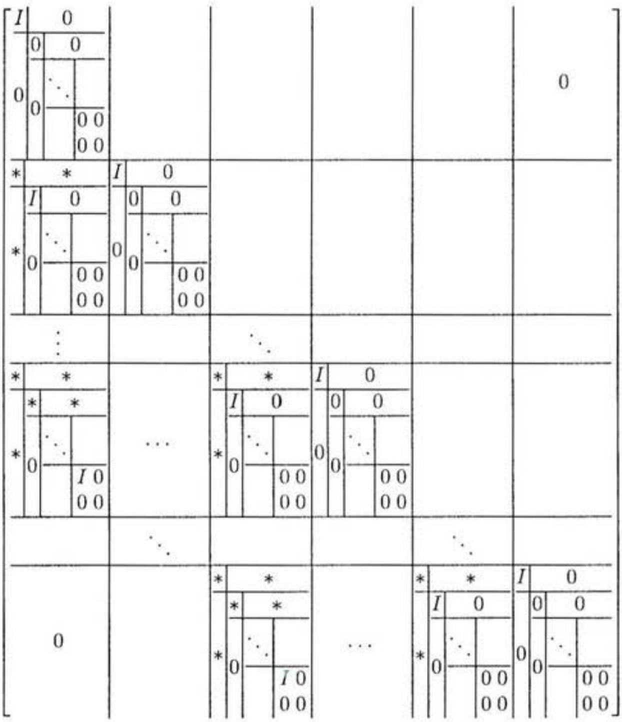
$$V_N = \operatorname{block\,diag}(V_D) \in R^{(N+1)m \times (N+1)m}$$

such that

$$W_N \tilde{\Delta}(0:N) V_N = \Lambda_N \tag{5}$$

where

$\Lambda_N =$

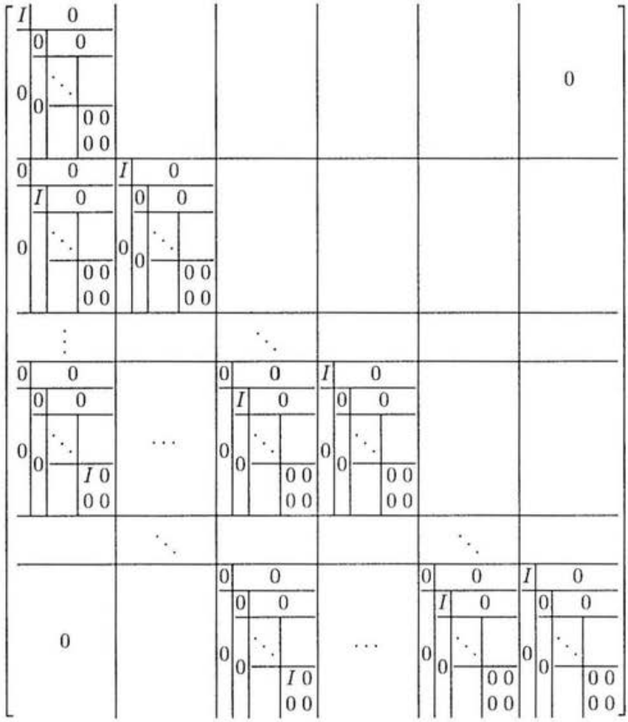


and $*$ denotes submatrices not necessarily equal to zero; each block of matrix Λ_N has dimension $p \times m$. By simple row and column operations one easily finds that the following relation occurs

$$\operatorname{rank}\ \Lambda_N = \operatorname{rank}\ \bar{\Lambda}_N$$

where

$\bar{\Lambda}_N =$



Hence

$$\operatorname{rank} \Lambda_N \leq (N+1)(p-1). \tag{6}$$

Thus, it follows from (3) that for some reference output sequence number t depends on N.

(ii) $\Rightarrow$ (iii). If condition (ii) is satisfied then for every reference output sequence one can calculate input sequence $U(0 : np)$ such that $Y(n : np) = Y_r(n : np)$. From (3) it follows that this is possible only if matrix $\Delta(n : np)$ has full row rank.

(iii) $\Rightarrow$ (ii). If condition (iii) is satisfied then rank $\Lambda_{np} \geq (np - n + 1)p$. But this contradicts (6). It can be, however, easily seen from (5) that inequality (6) is not satisfied only if condition (ii) is satisfied.

(ii) $\Rightarrow$ (iv). Proof is analogous to (ii) $\Rightarrow$ (iii) and is omitted.

(iv) $\Rightarrow$ (ii). It simply follows from $\tilde{\Delta}(0 : 2n+1)$ given by (4). ■

REMARKS. 1. It follows from condition (ii) that the necessary condition for systems trackability is $m \geq p$.

2. Clearly, system is trackable only if its output is controllable, i.e.

$$\operatorname{rank} [D \quad CB \quad CAB \quad \dots \quad CA^{n-1}B] = p$$

3. For a single output system one can easily note that output controllability is the necessary and sufficient condition for systems trackability.
4. It is easy to note that the trackability index for a trackable system is $t_I \leq n$.
5. Condition (iii) requires more matrix multiplication than condition (ii) but can be easier to check: $\det[\Delta(n:np)\Delta^T(n:np)] \neq 0$.

Then, the following corollaries can be formulated:

COROLLARY 1. *The trackability index t_I for the trackable system (1) is the smallest number such that*

$$\operatorname{rank} \Delta(0:t_I) = \operatorname{rank} \Delta(0:t_I - 1) + p$$

Proof. It follows from the proof of Theorem 1 that system (1) is trackable if

$$\operatorname{rank} \Delta(0:s) = \operatorname{rank} \Delta(0:s-1) + p$$

for $s \leq n$. The rest of the proof is obvious and is omitted. ■

REMARK. Obviously, $t_I \leq n$.

COROLLARY 2. *Trackability is a generic property for systems with $m \geq p$.*

Proof. There exist nonsingular matrices $\Delta(n:np)$, $\Delta^T(n:np)$, e.g. for $D = I$. Thus, one has, by continuity, $\det[\Delta(n:np)\Delta^T(n:np)] \neq 0$ almost always in the space of system parameters. This implies condition (iii) of Theorem 1. ■

Now consider the following example.

EXAMPLE 1. Assume system (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consideration of the system trackability leads to

$$\Delta_3 = \left[\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

It is easy to check that condition (ii), $\Delta_3 = \Delta_2 + p$, is not satisfied and system is untrackable. It can be, however, easily checked that system is controllable, observable and output controllable.

3. BOBI trackability

It is obvious that by calculating control input sequence for finite length reference bounded output sequence $\{y_r(k),\ k = 0, 1, \ldots, N\}$ one always obtains bounded control input. However, bounds for control input can be very large, the control input may increase if N increases. For this reason we introduce the following definition.

DEFINITION 3. System (1) is bounded output bounded input (BOBI) trackable if and only if it is trackable and for any bounded infinite length reference output sequence the appropriate input sequence is also bounded.

Next, in considering BOBI trackability, note that there always exist nonsingular matrices $P_0 \in R^{p\times p}$ and $T_0 \in R^{n\times n}$ such that

$$P_0 D T_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{7}$$

Hence

$$\begin{bmatrix} y_{01}(k) \\ y_{02}(k) \end{bmatrix} = \begin{bmatrix} C_{01} \\ C_{02} \end{bmatrix} x(k) + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{01}(k) \\ u_{02}(k) \end{bmatrix} \tag{8}$$

where

$$\begin{bmatrix} y_{01}(k) \\ y_{02}(k) \end{bmatrix} = P_0 y(k), \qquad \begin{bmatrix} u_{01}(k) \\ u_{02}(k) \end{bmatrix} = T_0^{-1} u(k),$$
$$\begin{bmatrix} y_{r01}(k) \\ y_{r02}(k) \end{bmatrix} = P_0 y_r(k) \quad \text{and} \quad \begin{bmatrix} C_{01} \\ C_{02} \end{bmatrix} = P_0 C.$$

Thus, output y_{01} tracks reference output y_{r01} if and only if

$$u_{01}(k) = y_{r01}(k) - C_{01}x(k).$$

The state equation can be now written as follows

$$\begin{aligned} x(k+1) &= Ax(k) + B_{01}u_{01}(k) + B_{02}u_{02}(k) \\ &= (A - B_{01}C_{01})x(k) + B_{01}y_{r01}(k) + B_{02}u_{02}(k) \\ &= A_0 x(k) + B_0 u_0(k) + r_0(k), \end{aligned}$$

where $r_0(k) = B_{01}y_{r01}(k)$, $u_0(k) = u_{02}(k)$ and, respectively

$$[B_{01} \quad B_{02}] = BT_0, \qquad A_0 = A - B_{01}C_{01}, \qquad B_0 = B_{02}. \tag{9}$$

Now, one should find an input sequence u_0 such that y_{02} will follow the reference output y_{r02}:

$$\begin{aligned} y_{02}(k) &= C_{02}x(k) = C_{02}[A_0 x(k-1) + B_0 u_0(k-1) + r_0(k-1)] \\ &= C_0 x(k-1) + D_0 u_0(k-1) + E_0 r_0(k-1) \end{aligned} \tag{10}$$

where

$$C_0 = C_{02}A_0, \qquad D_0 = C_{02}B_0, \qquad E_0 = C_{02}.$$

Then, there always exist nonsingular matrices P_1 and T_1 such that, similarly to (7), we have

$$P_1 D_0 T_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, one obtains from (10), analogously to (8),

$$\begin{bmatrix} y_{11}(k) \\ y_{12}(k) \end{bmatrix} = \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} x(k-1) + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{11}(k-1) \\ u_{12}(k-1) \end{bmatrix} + \begin{bmatrix} E_{11} \\ E_{12} \end{bmatrix} r_0(k-1),$$

where

$$\begin{bmatrix} y_{11}(k) \\ y_{12}(k) \end{bmatrix} = P_1 y_{02}(k), \qquad \begin{bmatrix} u_{11}(k) \\ u_{12}(k) \end{bmatrix} = T_1^{-1} u_0(k),$$
$$\begin{bmatrix} y_{r11}(k) \\ y_{r12}(k) \end{bmatrix} = P_1 y_{r02}(k) \quad \text{and} \quad \begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = P_1 C_0.$$

Thus, $y_{11}(k)$ will track $y_{r11}(k)$ if and only if

$$u_{11}(k-1) = y_{r11}(k) - C_{11}x(k-1) - E_{11}r_0(k-1).$$

$$\begin{aligned} x(k+1) &= A_0x(k) + B_{11}u_{11}(k) + B_{12}u_{12}(k) + r_0(k) \\ &= (A_0 - B_{11}C_{11})x(k) + B_{11}y_{r11}(k+1) - B_{11}E_{11}r_0(k) \\ &+ B_{12}u_{12}(k) + r_0(k) \\ &= A_1x(k) + B_1u_1(k) + r_1(k) \end{aligned}$$

where $u_1 = u_{12}$, $r_1(k) = (I - B_{11}E_{11})r_0(k) - B_{11}y_{r11}(k+1)$ and

$$[B_{11} \quad B_{12}] = B_0T_1, \qquad A_1 = A_0 - B_{11}C_{11}, \qquad B_1 = B_{12}.$$

Presently, one should calculate input sequence u_1 such that y_{12} will track y_{r12}:

$$y_{12}(k) = C_1x(k-2) + D_1u_1(k-2) + E_1r_1(k-2) + E_{12}r_0(k-1)$$

where

$$C_1 = C_{12}A_1, \qquad D_1 = C_{12}B_1, \qquad E_1 = C_{12} \tag{11}$$

It is easy to note that the control input $u(k)$ can be calculated in the closed loop system

$$\begin{bmatrix} u_{01}(k) \\ u_{11}(k) \end{bmatrix} = -\begin{bmatrix} C_{01} \\ C_{11} \end{bmatrix} x(k) - \begin{bmatrix} 0 \\ E_{11} \end{bmatrix} r_0(k) + \begin{bmatrix} y_{r01}(k) \\ y_{r11}(k) \end{bmatrix}.$$

It is obvious that the output of the trackable system is bounded. According to the above formula, input is also bounded if the system output $\tilde{y}(k) = \begin{bmatrix} C_{01} \\ C_{11} \end{bmatrix} x(k)$ is bounded. Thus, by repeating the above procedure the trackability index times we arrive at the following

THEOREM 2. *System (1) is BOBI trackable if and only if*

(i) it is trackable, and

(ii) the observable subsystem of the triple $(C_{1t_I}, A_{t_I}, B_{t_I})$ is stabilizable where

$$C_{1t_I} = \begin{bmatrix} C_{01} \\ \vdots \\ C_{t_I 1} \end{bmatrix} \quad \textit{and } t_I \textit{ is the trackability index.}$$

REMARKS. 1. Matrices A_i, B_i, $i = 1, 2, \ldots, t_I$, are given in (9), (10) and (11).

2. Since the trackability index $t_I \le n$ one is not obliged to calculate the trackability index but can consider the triplet (C_{1n}, A_n, B_n) instead of $(C_{1t_I}, A_{t_I}, B_{t_I})$.
3. It is easy to see that $B_{t_I} \in R^{n \times (m-p)}$.
4. Note that the stability of the trackable system (1) is neither a necessary nor a sufficient condition for its BOBI trackability.
5. Since there exists systems with $m > p$ such that the pair (A_{t_I}, B_{t_I}) is controllable, we have that BOBI trackability is generic for systems which have more inputs than outputs, $m > p$.

As a simple illustration consider the following examples.

EXAMPLE 2. Assume a single input single output system (1). From Theorem 1 one has in this case that the system is trackable if and only if

$$\begin{bmatrix} D & CB & CAB & \ldots & CA^{n-1}B \end{bmatrix} \neq 0,$$

i.e. if system's output is controllable. Then, if $D \neq 0$ the trackability index for the system is equal to 0 and, according to Theorem 2, system is BOBI trackable if and only if the matrix

$$A_1 = A - BC\frac{1}{D}$$

has all eigenvalues inside the unit circle. Otherwise, the trackability index is $t_I = h + 1$ where $CA^hB \neq 0$ and $CA^jB = 0$ for $j = 0, 1, \ldots, h-1$. The system is then BOBI trackable if and only if all eigenvalues of the matrix

$$A_{t_I} = A - BCA^{t_I}\frac{1}{CA^{t_I-1}B}, \qquad t_I \ge 1$$

are inside the unit circle.

EXAMPLE 3. Consider system (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -0.125 & 0.5 & -0.125 \\ 0 & 1 & 1.1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}, \tag{12}$$
$$C = [0 \quad 1 \quad 0.1], \qquad D = 0.$$

It can be checked that the system is controllable and observable. The system is asymptotically stable. It is easy to note that $CB = 1.05$. Thus, the system is trackable. Therefore, one can follow any reference output sequence of finite length.

It is easy to note that the trackability index for the system is equal 1: $t_I = 1$. Next, we have $C_{01} = 0$ and $B_1 = 0$ and

$$A_1 = A_0 - B_{11}C_{11} = A - BCA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.1 & -0.11 \\ 0.0625 & 0.7 & 1.1075 \end{bmatrix}.$$

Then, $K = C_{11} = C_{02}A_0 = CA = [-0.125 \quad 0.6 \quad 0.015$. It is easy to find that the pair (K, A_1) is observable, and the eigenvalues of matrix A are equal 1.0337, 0.0695 and 0.0957. Hence, the system is not BOBI trackable.

Figures 1 through 3 show the output and input to the system calculated for the reference output $y_r(k) = \begin{cases} 0.9 & \text{for } k = 0, 1, \dots, 99 \\ 0 & \text{for } k = 100, \dots, 200 \end{cases}$. Since trackability index $t_I = 1$ we have $y(k) = y_r(k)$ for $k = 1, 2, \dots, 200$, there is an error only at time instant $k = 0$ since $y(0) \neq y_r(0)$. Note that there is no transition period when output changes from 0.9 to 0 for $k = 100$. Unfortunately, control input to the system increases, Fig. 3, since system is not BOBI trackable.

Fig. 4 shows the input to the system with the same system matrices A, B, D, but $C = C' = [0 \ 1 \ 0.2]$. In this case the system is BOBI trackable, matrix A_1 has eigenvalues equal to 0.9101, 0.0975, −0.155, and the control input is bounded, significantly smaller than in the previous case, Fig. 3. Output and output error of this system are the same as for the system with matrices (12), Figs. 1 and 2.

4. Concluding remarks

Trackability and BOBI trackability have been defined for dynamical systems. Then, necessary and sufficient conditions for systems trackability have been given. It was also shown that trackability is generic for square MIMO systems and BOBI trackability for systems which have more inputs than outputs. necessary condition for system trackability is output controllability. However, in the contradiction to systems controllability, a necessary condition for systems trackability is that the number of control inputs is equal to or greater than the

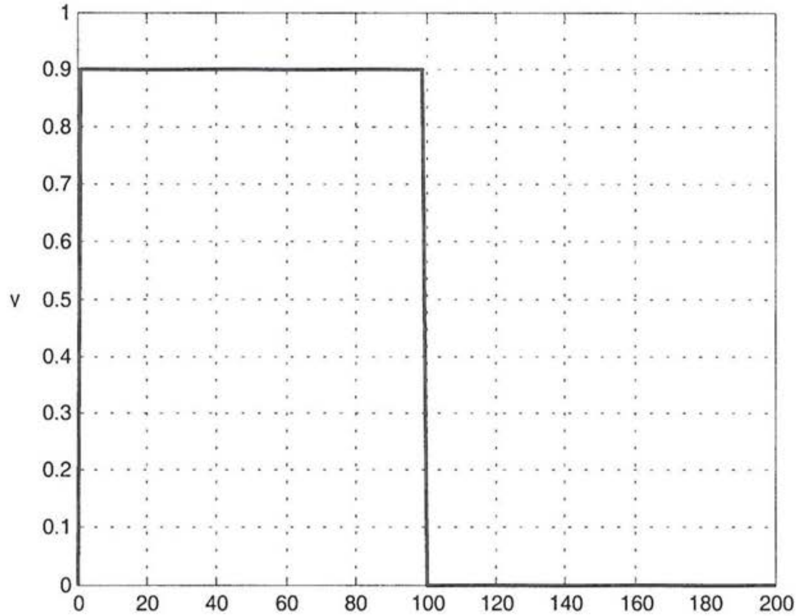


Figure 1. Output of the system

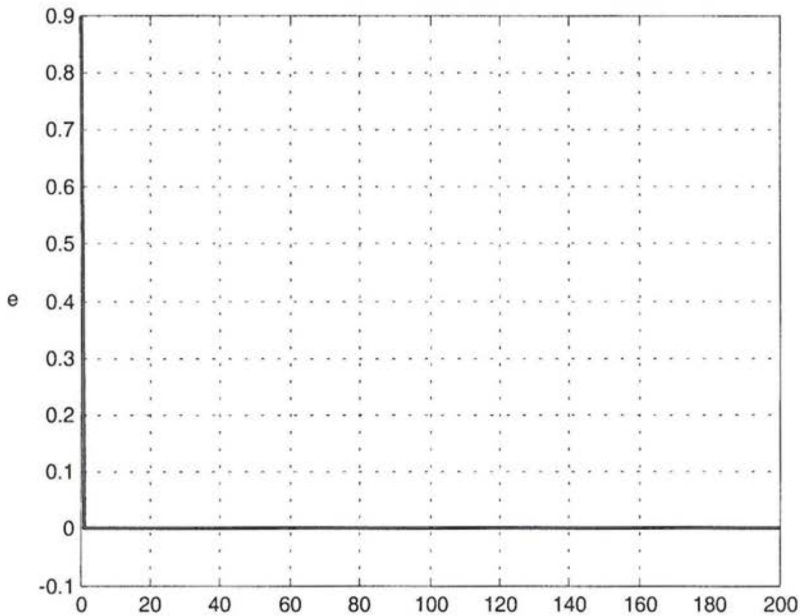


Figure 2. Output error of the system

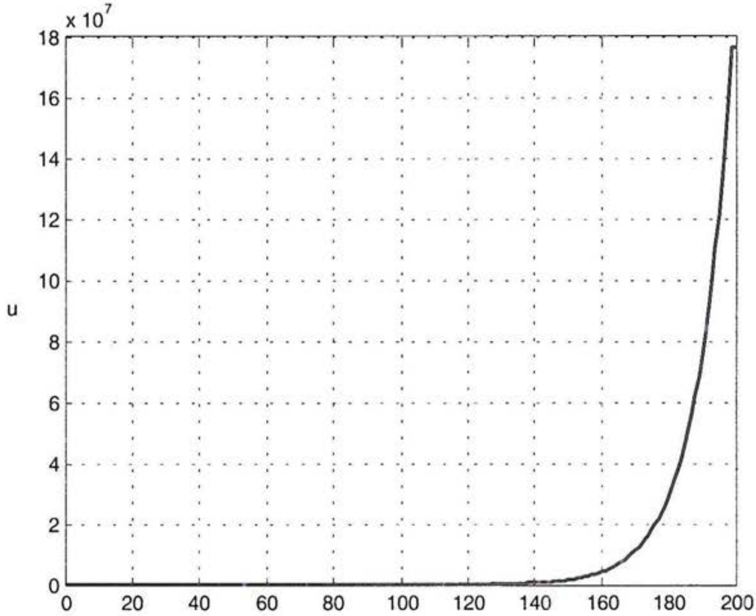


Figure 3. Control input to the system.

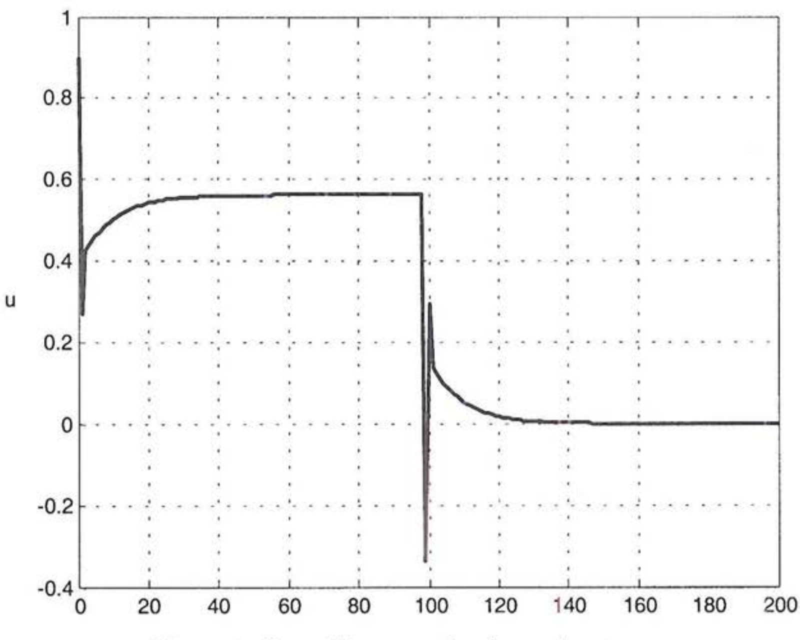


Figure 4. Control input to the changed system.

number of system outputs. Finally, let us note that in practical applications the tracking problem can be solved only if the control system is BOBI trackable. Otherwise, control input will take excessively large values.

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