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A sufficient condition for optimality in nondifferentiable invex programming

by

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Abstract: In this paper sufficient optimality condition are established for a nonlinear programming problem without differentiability assumption on the data wherein Clarke's generalized gradient is used to define invexity.

Keywords: locally Lipschitz function, Clarke's generalized gradient, invex function, semiconvex function.

1. Introduction

Many results from mathematical programming, which concern convex functions hold in fact for a considerably wider class of functions called invex functions. The first to introduce the notion of an invex functions was Hanson (1981) (it should be stressed that this definition concerned differentiable functions). Hanson's paper was an inspiration for further investigations of invexity which went in various directions, and giving the definition of nondifferentiable invex functions became the intention of many authors. In the case of quasidifferentiable functions, invexity was characterized by Craven and Glover (1985), and then, for the case of Lipschitz (not necessarily differentiable) functions, Craven (1986) gave the definition of generalized invexity.

The present paper concerns optimization problems with constraints, in which the appearing functions are (nondifferentiable) invex functions. The main result of the paper is Theorem 3 whose content is the sufficient condition for optimality in problems of this type. It is a modification of the sufficient condition from Mifflin (1977), where the respective functions were semiconvex (thus regular in the sense of Clarke, 1983).

In the final part of the paper, we give an example of an optimization problem in which one cannot apply the sufficient condition for optimality from Mifflin (1977), whereas one can make use of the sufficient condition for optimality

2. Definitions and propositions

DEFINITION 1 (Clarke, 1983) *Let $f : X \to R$ be a locally Lipschitz function in a neighbourhood of a fixed point $x \in X$. The generalized gradient (in the sense of Clarke) of f at $x \in X$ is defined by*

$$\partial f(x) := \{\xi \in R^n : \langle \xi, h \rangle \leq f^0(x;h),\ \forall h \in R^n\}, \tag{1}$$

where $f^0(x;h)$ denotes a generalized directional derivative f at a point x in the direction $h \in R^n$ defined as $f^0(x;h) := \limsup\limits_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y+th)-f(y)}{t}$.

Throughout the paper by $f'(x;h)$ we denote the directional derivative of a function $f : X \to R$ at the point $x \in X$ in the direction $h \in R^n$, that is, $f'(x;h) := \lim\limits_{t \downarrow 0} \frac{f(x+th)-f(x)}{t}$.

DEFINITION 2 (Clarke, 1983) *We say that a Lipschitz function f is regular at point x if the directional derivative $f'(x;h)$ exists in any direction h and equals the generalized directional derivative $f^0(x;h)$.*

The definition above is identical with that of a quasidifferentiable function, used in Mifflin (1977). In the present paper we shall make use of the terminology from Clarke (1983).

DEFINITION 3 (Mifflin, 1977) *Let X be any subset of R^n. A function $f : X \to R$ is called semiconvex at $x \in X$ if*

a) *f is a Lipschitz function on some ball containing the point x,*
b) *f is regular at point x,*
c) *$x + h \in X$ and $f^0(x;h) \geq 0$ imply the inequality $f(x+h) \geq f(x)$.*

We say that a function f is semiconvex (regular) on a subset $X \subset R^n$ if it is semiconvex (regular) at each point of the set X.

REMARK 1 *In Definition 3, as a matter of fact, the assumption of the regularity of the function f at the point x may be omitted. The regularity assumption was added in (Mifflin, 1977) in order to carry out the proof of the theorem containing a sufficient condition for optimality of the optimization problem considered. This proof is based, among other things, on the theorem below in which this assumption is indispensable.*

THEOREM 1 (Mifflin, 1977) *If f is semiconvex on a convex subset X of R^n, $x \in X$ and $x + h \in X$, then*

$$f(x+h) \leq f(x) \quad \Rightarrow \quad f^0(x;h) \leq 0 \tag{2}$$

DEFINITION 4 (Reiland, 1990) *Let $f : X \to R$ be a Lipschitz function on X, where X is an open subset of R^n, then f is called invex with respect to η (or shortly w.r.t. η) on X if there exists a function $\eta : X \times X \to R^n$ such that*

$$f(x) - f(y) \geq f^0(x; \eta(x, u)), \quad \forall x, u \in X. \tag{3}$$

DEFINITION 5 *If $x \in X$ is a point such that $0 \in \partial f(x)$, then x is called a stationary point of the function f.*

THEOREM 2 (Phuong, Sach, Yen, 1995) *Let $f : \Omega \to R$ be a locally Lipschitz on an open set Ω containing a nonempty subset X. The function f is invex on X if and only if each stationary point over X is a global minimum of f on X.*

COROLLARY 1 *Let X be an open subset of R^n, and $f : X \to R$ - an invex function on X. Then $u \in X$ is a global minimum point of the function f on X if and only if $0 \in \partial f(x)$.*

3. A sufficient condition for optimality

Consider an optimization problem of the form:

$$\begin{aligned} & f(x) \to \min \\ & g(x) \leq 0 \end{aligned} \tag{P}$$

where $g(x) := \max_{1 \leq i \leq m} g_i(x)$, $x \in R^n$.

DEFINITION 6 *We say that $x \in R^n$ is a feasible point of problem (P) if $g(x) \leq 0$, and a strictly feasible point when $g(x) < 0$.*

DEFINITION 7 *We say that $\overline{x} \in R^n$ is an optimal point of problem (P) if it is feasible and the inequality $f(\overline{x}) \leq f(x)$ is satisfied for all feasible points x.*

In (Mifflin, 1977, Theorem 9), which includes a sufficient condition for the point $\overline{x}$ to be the optimal solution of problem (P), it was indispensable to assume that both the objective function f and the constraint function g should be semiconvex. In reality, as was stated by the author himself (Mifflin, 1977), in the proof of this theorem in the case when $g(\overline{x}) = 0$, he needed a stronger assumption to prove optimality of the point $\overline{x}$, namely, instead of being semiconvex, the function g should be quasidifferentiable and satisfy some additional property. The above statement corresponds, in fact, to Mangasarian's optimality condition Mangasarian (1969), namely:

If $\overline{x}$ satisfies the generalized Karush–Kuhn–Tucker conditions, f is semiconvex, and is quasidifferentiable and quasiconvex, then $\overline{x}$ is the optimal point of problem (P).

In the theorem given below, which is a sufficient condition for optimality, the from Mifflin (1977), mentioned above. Also in the proof of this theorem the assumption of the quasidifferentiability of the constraint function has turned out to be dispensable and we have not made use of a certain additional property by which the constraint function in the proof of Theorem 9 in Mifflin (1977) had to be characterized.

The main role in proving a sufficient condition for optimization problem (P) is played by a multivalued mapping $M(x) : R^n \to 2^{R^n}$ defined as follows:

$$M(x) := \begin{cases} \partial f(x) & \text{if } g(x) < 0, \\ conv\{\partial f(x) \cup \partial g(x)\} & \text{if } g(x) = 0, \\ \partial g(x) & \text{if } g(x) > 0, \end{cases} \quad x \in R^n. \tag{4}$$

The above mapping was introduced by Merrill (1972) for optimization problems with differentiable and/or convex functions, i.e. problems with functions which possess gradients and/or subgradients. This mapping is also useful in the proof of our theorem including a sufficient condition for the optimality of optimization problem (P) in which the respective functions possess generalized gradients.

We prove our main result.

THEOREM 3 (a sufficient condition for the optimality of problem (P)) *If functions f and g are invex w.r.t. η on R^n and $\overline{x}$ is a point of R^n, such that $0 \in M(\overline{x})$, then the following propositions are satisfied:*

a) *If $g(\overline{x}) > 0$, then $g(x) \geq g(\overline{x}) > 0$ for all $x \in R^n$, that is, the optimization problem has no solutions.*
b) *If $g(\overline{x}) \leq 0$, then at least one of the following conditions holds:*
    i) *$\overline{x}$ is an optimal solution,*
    ii) *$g(x) \geq 0$ for all $x \in R^n$, that is, the optimization problem has no strictly feasible points.*

Proof. a) If $g(\overline{x}) > 0$, then the assumption $0 \in M(\overline{x})$ and the definition of the mapping M imply that $0 \in \partial g(\overline{x})$. Since g is an invex function w.r.t η, we have

$$g(x) - g(\overline{x}) \geq g^0(\overline{x}; \eta(x, \overline{x})) \geq 0, \quad \forall x, \overline{x} \in R^n,$$

where the second inequality follows from the fact that $0 \in g(\overline{x})$ and from the definition of the generalized gradient of g. The above inequality means that $g(x) \geq g(\overline{x}) > 0$ for all $x \in R^n$, that is, the optimization problem has no solution.

b) If $g(\overline{x}) < 0$, then the assumption $0 \in M(\overline{x})$ and the definition M imply that $0 \in \partial f(\overline{x})$, and thus, by Corollary 1, the point $\overline{x}$ is the optimal solution of problem (P).

If $g(\overline{x}) = 0$, then it follows from the definition of the mapping M that

By assumption $0 \in M(\overline{x})$ we deduce that there exist $\lambda \in [0,1]$ and $\widetilde{\xi} \in \partial f(\overline{x})$, $\widehat{\xi} \in \partial g(\overline{x})$, such that

$$\lambda\widetilde{\xi} + (1-\lambda)\widehat{\xi} = 0. \tag{5}$$

If $\lambda = 0$, then it follows from (5) that $\widehat{\xi} = 0$, and this means that $\overline{x}$ is the minimum point of the function g on R^n, and thus $g(x) \geq g(\overline{x}) = 0$ for all $x \in R^n$, that is, proposition b) ii) holds.

If $\lambda > 0$, then from (5) we get

$$\widetilde{\xi} + \frac{(1-\lambda)}{\lambda}\widehat{\xi} = 0. \tag{6}$$

By assumption, f and g are invex w.r.t. η, therefore, the following inequalities are true:

$$g(x) - g(\overline{x}) \geq g^0(\overline{x}; \eta(x,\overline{x})), \quad \forall x \in R^n,$$
$$f(x) - f(\overline{x}) \geq f^0(\overline{x}; \eta(x,\overline{x})), \quad \forall x \in R^n.$$

Since $g(\overline{x}) = 0$, we obtain from the first of the above inequalities

$$g(x) \geq g^0(\overline{x}; \eta(x,\overline{x})), \quad \forall x \in R^n. \tag{7}$$

Then, for all feasible points $x \in R^n$, that is, in conformity with the definition, such that $g(x) \leq g(\overline{x}) = 0$, from the fact that $\widehat{\xi} \in \partial g(\overline{x})$ and from (7) we get the following relations:

$$0 \geq g(x) \geq g^0(\overline{x}; \eta(x,\overline{x})) \geq \langle\widehat{\xi}; \eta(x,\overline{x})\rangle. \tag{8}$$

Consequently, by (8), we obtain

$$\langle\widehat{\xi}; \eta(x,\overline{x})\rangle \leq 0, \quad \forall x \in R^n \text{ such that } g(x) \leq 0,$$

and since $\frac{1-\lambda}{\lambda} \geq 0$, condition (6) implies

$$\langle\widetilde{\xi}; \eta(x,\overline{x})\rangle \geq 0 \quad \forall x \in R^n \text{ such that } g(x) \leq 0.$$

Since f is invex w.r.t. η and from $\widetilde{\xi} \in \partial f(\overline{x})$ it follows that

$$f(x) - f(\overline{x}) \geq f^0(\overline{x}; \eta(x,\overline{x}) \geq \langle\widetilde{\xi}; \eta(x,\overline{x})\rangle \geq 0 \text{ for } \forall x \in R^n$$
$$\text{such that } g(x) \leq 0.$$

Upon writing this inequality in a suitable form, we find that

$$f(x) \geq f(\overline{x}) \text{ for all } x \in R^n \text{ such that } g(x) \leq 0.$$

This, in turn, means that $\overline{x}$ is an optimal solution of problem (P), that is, the

In order to illustrate the results obtained, we shall give an example of an optimization problem in which the sufficient condition for the optimality of the point $\overline{x}$ will be obtained by the application of our theorem, whereas it will be impossible to apply for this purpose the theorem including the sufficient condition from Mifflin (1977).

EXAMPLE. *Consider the optimization problem (P), where the objective function is assumed to be*

$$f(x) = \begin{cases} x & \text{for } x < 0, \\ \frac{1}{2}x & \text{for } x \geq 0, \end{cases}$$

and the constraint function ($m = 2$) is of the form:

$$g(x) = \max\{g_1(x), g_2(x)\} = \{0,\, x^2\} = \begin{cases} x^2 & \text{for } x < 0, \\ 0 & \text{for } x \geq 0. \end{cases}$$

It can be shown that, in the problem under consideration, the directional derivative of the objective function at point $\overline{x}$ is not equal to the generalized directional derivative. Since $f'(0;h) \neq f^0(0;h)$, it follows from Definition 2 that f is a nonregular function (in the sense of Clarke).

Since the objective function f in the optimization problem considered is not regular, use cannot be made of the sufficient condition for optimality, included in (Mifflin, 1977, Theorem 9] in which the regularity of functions occurring in the optimization problem is an indispensable assumption.

In order to examine whether $\overline{x} = 0$ is an optimal point of the considered problem, we can be apply our sufficient condition for optimality from Theorem 3. It can be proved that the assumption $0 \in M(\overline{x})$ is satisfied and the objective function f and the constraint function g are invex with respect to the same function η, for example, of the form

$$\eta(x,u) = \begin{cases} x - u & \text{for } x \geq 0,\ u > 0 \vee x < 0,\ u < 0, \\ \frac{1}{2}x - u & \text{for } x \geq 0,\ u \leq 0, \\ 2x - u & \text{for } x < 0,\ u \geq 0. \end{cases}$$

In this way, since $g(\overline{x}) = 0$, case b) of Theorem 3 holds, that is, we can ascertain that $\overline{x} = 0$ is the optimal point of our problem and the problem possesses no strictly feasible points because $g(x) \geq 0$ for all $x \in R$.

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