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Some remarks on the concept of stream tubes for numerical simulations of complex fluid flows — applications

by

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Abstract: This paper presents new theoretical elements for numerical simulation of two- and three-dimensional flows, based on the concept of streamlines and domain decomposition. The so-called "stream-tube method", considered previously particularly for flows involving open streamlines, is extended to general streamline configurations. It is shown how local transformation functions may be defined in order to simulate flows of complex fluids, notably those requiring evaluation of particle time history. The specific features (for example: mass conservation, simplicity in handling time-dependent constitutive equations) of the stream-tube methods previously investigated numerically are still preserved in the new formulation. An example of calculations is given in the case of the two-dimensional flow of a Newtonian fluid between two eccentric cylinders where results are found to be consistent with literature data.

Keywords: incompressible fluid, stream tubes, streamlines, constitutive equations, viscoelasticity, transformations, domain decomposition.

1. Introduction

The main purpose of this paper is to present theoretical elements permitting computation of general complex flows of various fluids, notably those obeying more complex constitutive equations than the Newtonian model. Significant ten years, particularly for fluids modelled by non-linear constitutive equations, which generally lead to highly complicated governing equations to be solved. Different numerical methods have been presented, particularly for rheological equations involving viscoelasticity. If differential models present difficulties that are generally overcome, memory-integral constitutive equations are still more delicate to handle, mainly because of the so-called "particle tracking" problem which is to be considered for general time-dependent equations of the type

$$\mathbf{T}(t) = \Phi^{t}_{-\infty}[\mathbf{K}_i(t,\tau)] \tag{1}$$

In equation (1), $\mathbf{T}$ denotes the stress tensor expressed at time t in terms of a functional tensor Φ of a finite number of kinematic tensors $\mathbf{K}_i$ evaluated at times τ, where $-\infty < \tau \leq t$. The kinematic (rate or deformation) tensors involved in such models have to be computed at every time for every spatial point corresponding to the position of the material point on the streamline or pathline (in the case of a steady flow situation). When evaluating numerically flow characteristics by means of a set of governing equations, the consideration of a fluid obeying such constitutive equations leads to significant problems. The material points do not necessarily pass through the mesh points. Accuracy problems in calculating the kinematic and stress tensors may lead to failure of the numerical procedure.

The numerous papers devoted to the numerical simulation of complex flows of non-Newtonian fluids have generally considered finite element methods, using the velocity components and the pressure as primary variables. In two-dimensional flow situations, there are now classical finite elements verifying basic conditions that can be successfully used in flow calculations, for various non-Newtonian constitutive models, particularly differential models (see, for example, Marchal and Crochet, 1987, Caswell, 1996). Specific approaches for memory-integral equations have been also developed, notably in two-dimensional flow situations, Luo and Tanner (1988), Luo and Mitsoulis (1990), Goublomme et al. (1992). In some cases, the discretizing mesh built on the streamlines was updated at every step of the iterative procedure, Luo and Tanner (1986). Other authors developed interpolation functions in order to approximate the kinematic quantities related to a given element. The so-called Protean coordinates, introduced by Duda and Vrentas (1967), were also used (e.g. Papanastasiou et al., 1987) in order to evaluate the kinematics and related quantities. In the Protean system, one coordinate is the stream function Ψ. In three-dimensional flow situations, streamlines or pathlines are warping curves and require significant effort for approximating the kinematic quantities (rate- of-deformation and deformation tensors), when using time-dependent constitutive equations. Successful three-dimensional approaches have been proposed for differential viscoelastic models, Tran-Cong and Phan-Thien (1988), Shiojima and Shimazaki (1990), but at the present time there are very few numerical studies involving memory-integral constitutive equations. In a recent paper, Broszeit (1997) considered the circulating steady flow in a single-flow constitutive equation by developing particle-tracking methods for open and closed pathlines, and a mixed Galerkin formulation for computation of the unknowns.

Besides the significant work reported in literature on grid generation methods widely investigated in the field of computational solid and fluid mechanics (e.g. Smith, 1982, Lau et al., 1997), some authors developed specific approaches to simulate two- and three-dimensional flows. For example, Greywall proposed a method related to the evaluation of the streamwise velocity, Greywall (1985, 1988), and free surface equations, in order to compute 2D and 3D potential flows.

The so-called "stream-tube method", based on geometrical considerations, was introduced by Clermont (1983). This analysis has proved to be an appropriate answer to computation of different complex flows of fluids obeying memory-integral constitutive equations since the particle-tracking problem is avoided. The boundary of the material may be free or confined. The calculations are performed in a transformed domain Ω^* of the physical flow domain Ω, Clermont (1983), André and Clermont (1990), Clermont et al. (1991), where the mapped streamlines are straight lines parallel to a mean flow direction (open stream lines) or concentric circles (closed streamlines), Clermont et al. (1991), in the case of pure recirculating flows (Fig. 1). The primary unknowns of the

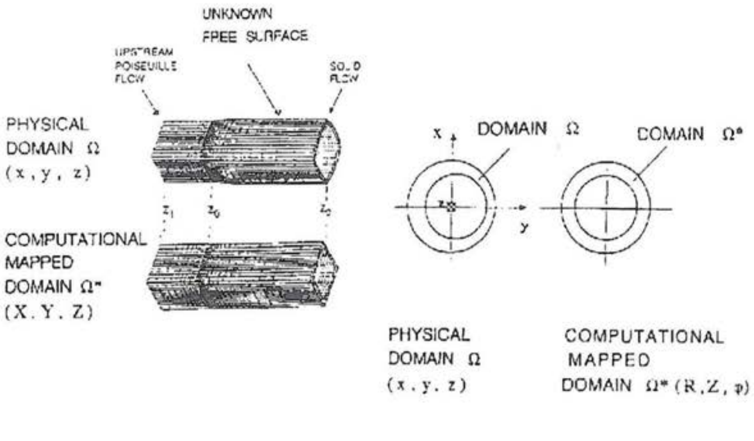

Figure 1. Mapping of the physical flow domain in specific stream-tube situations: (a) Flow with open streamlines; (b) Flow with closed streamlines

problem are, together with the pressure, the transformation between domains Ω^* and Ω. Calculations are possible under the condition of non-singularity of the Jacobian of the transformation $\mathcal{T} : \Omega^* \rightarrow \Omega$, that involved one or two coordinate mapping functions for two- and three-dimensional

3D main flow conditions, the basic equations relating coordinates (x, y, z) of Ω to (X, Y, Z) in the mapped domain Ω^* where the transformed streamlines are rectilinear (Fig. 1a) are given by

$$x = f(X, Y, Z);\ y = g(X, Y, Z);\ z = Z. \tag{2}$$

For 2D pure circulating flows, an unknown mapping function λ was defined using coordinates (x, y, z) in the physical domain Ω and (R, Z, φ) in the mapped domain Ω^* of concentric circular streamlines by means of the following equations (Fig. 1b)

$$x = a + R\lambda(R, \varphi)\sin\varphi,\ y = b + R\lambda(R, \varphi)\cos\varphi;\ z = Z. \tag{3}$$

The non-singularity of the Jacobian in both cases points out limitations of the method to computation of:

- simply-connected (with regard to the streamlines) flow regions, for the transformation expressed by equations (2) which means that the recirculations are not taken into account explicitly,
- pure vortex flows (doubly-connected domains) with closed streamlines, in relation to mapping equations (3).

The elements given in the present paper aim at providing possibilities of calculation of main flow zones as well as secondary flow regions. Some features previously depicted in studies related to the "classical" stream-tube method are still used, particularly those concerning the simplicity of handling memory-integral constitutive equations. Additional elements based on geometrical considerations are given, allowing computation of a general flow field by still considering the concept of streamlines and stream tubes. It should also be pointed out that practical issues of this possibility are based upon previous computational results on flows involving open streamlines (Clermont and de la Lande, 1993, Normandin and Clermont, 1996, Guillet et al., 1996) and from recent numerical studies on flow involving pure recirculating regions (Clermont and Radu, 1999a, 1999b). The approach is presented here for steady flows of incompressible fluids. Numerical results are given for the steady flow of a Newtonian fluid in the annulus of a cylinder.

2. General transformations — Basic computational results with the stream-tube method

2.1. Basic equations for general transformations

Let x^i ($x^1 = x$, $x^2 = y$, $x^3 = z$) be the cartesian coordinates related to an Euclidean basis $\boldsymbol{\varepsilon}_i$ $(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3)$ for a material point M occupying the position $\mathbf{X}$ (x, y, z) in $\mathcal{D}$ $(\Re^3 \supset \mathcal{D})$. When considering another coordinate system ξ^j

$x^i(\xi^j)$ from $\mathcal{D}^*$ to $\mathcal{D}$ may be defined by the following relations:

$$\begin{aligned} x &= \alpha(X, Y, s) \\ y &= \beta(X, Y, s) \\ z &= \gamma(X, Y, s). \end{aligned} \tag{4}$$

The Jacobian $\Delta = |\partial(x^i)/\partial(\xi^j)|$, assumed to be non-zero, can be expressed by the following equation

$$\Delta = \alpha'_X(\beta'_Y\gamma'_S - \beta'_S\gamma'_Y) - \beta'_X(\alpha'_Y\gamma'_S - \gamma'_Y\alpha'_S) + \gamma'_X(\alpha'_Y\beta'_S - \beta'_Y\alpha'_S) \tag{5}$$

where $\alpha'_{\dots}$, $\beta'_{\dots}$ etc. stand for partial derivatives. In the following, different partial derivative operators lead us to define quantities denoted by A_1, A_2, A_3, $B_1, B_2, B_3, C_1, C_2, C_3$ and given by the relations

$$A_1 = \alpha'_Y\beta'_S - \alpha'_S\beta'_Y;\ A_2 = \beta'_Y\gamma'_S - \beta'_S\gamma'_Y;\ A_3 = \gamma'_Y\alpha'_S - \gamma'_S\alpha'_Y \tag{6}$$
$$B_1 = \alpha'_X\beta'_S - \alpha'_S\beta'_X;\ B_2 = \beta'_X\gamma'_S - \beta'_S\gamma'_X;\ B_3 = \gamma'_X\alpha'_S - \gamma'_S\alpha'_X \tag{7}$$
$$C_1 = \alpha'_X\beta'_Y - \alpha'_Y\beta'_X;\ C_2 = \beta'_X\gamma'_Y - \beta'_Y\gamma'_X;\ C_3 = \gamma'_X\alpha'_Y - \gamma'_Y\alpha'_X \tag{8}$$

These equations lead to writing the Jacobian Δ as

$$\Delta = \alpha'_X A_2 + \beta'_X A_3 + \gamma'_X A_1. \tag{9}$$

Then, the natural basis $\mathbf{e}_i$ $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ related to the coordinates (X, Y, s) is expressed by

$$\begin{aligned} \mathbf{e}_1 &= \alpha'_X\varepsilon_1 + \beta'_X\varepsilon_2 + \gamma'_X\varepsilon_3 \\ \mathbf{e}_2 &= \alpha'_Y\varepsilon_1 + \beta'_Y\varepsilon_2 + \gamma'_Y\varepsilon_3 \\ \mathbf{e}_3 &= \alpha'_S\varepsilon_1 + \beta'_S\varepsilon_2 + \gamma'_S\varepsilon_3 \end{aligned} \tag{10}$$

Conversely, from equations (10), the cartesian basis vectors may be given in terms of the natural basis by

$$\varepsilon_1 = (1/\Delta)[A_2\mathbf{e}_1 + B_2\mathbf{e}_2 + C_2\mathbf{e}_3] \tag{11}$$
$$\varepsilon_2 = (1/\Delta)[A_3\mathbf{e}_1 + B_3\mathbf{e}_2 + C_3\mathbf{e}_3] \tag{12}$$
$$\varepsilon_3 = (1/\Delta)[A_1\mathbf{e}_1 + B_1\mathbf{e}_2 + C_1\mathbf{e}_3]. \tag{13}$$

It can also be shown that the derivative operators $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ are expressed in terms of $\partial/\partial\xi^j$ by the following equations

$$\partial/\partial x = (1/\Delta)[A_2\partial/\partial X - B_2\partial/\partial Y + C_2\partial/\partial_s] \tag{14}$$
$$\partial/\partial y = (1/\Delta)[A_3\partial/\partial X - B_3\partial/\partial Y + C_3\partial/\partial_s] \tag{15}$$

2.2. Transformation of subdomains

As pointed out previously, applications of stream-tube analysis have concerned calculations of main flows for various viscoelastic fluids obeying differential (Clermont and Radu, 1999b) and integral equations (e.g. Guillet et al., 1996) and also pure recirculating flows related to closed streamlines in a two-dimensional journal bearing geometry (Clermont and Radu, 1999a, 1999b). In both cases, the formulation allowed simple mapped computational domains Ω^* to be defined in order to solve the governing equations.

For purposes of performing simultaneous computations of main and recirculation flow regions in general situations for a bounded physical domain Ω, we now consider sub-regions Ω_i that may involve open or closed elementary streamtubes or both. These non overlapping subdomains are defined such that:

$$\Omega = \bigcup_{m=1}^{m=m_0} \Omega_m \tag{17}$$

Starting from an original section z_1, we limit each sub-domain Ω_m by two cross-section planes z_m and z_{m+1} $(m = 1, 2, \ldots, m_0)$. These sub-domains may involve open and closed streamtubes, denoted, respectively, by $\mathcal{B}_o$ and $\mathcal{B}_c$. They are a priori unknown and correspond to main flow and vortex regions of the total flow domain Ω.

Let us consider a sub-domain Ω_m of Ω limited by two cross-section planes at z_m and z_{m+1} (Fig. 2) in an axisymmetric flow situation. We select in Ω_m a cross-section S_m at $z = \zeta_m$ $(z_m \leq \zeta_m \leq z_{m+1})$, used as reference section. The mapped domain Ω^*_m is a straight cylinder (Fig. 2), of basis $\mathcal{S}^*_m$ identical, in shape, to the reference section $\mathcal{S}_m$. The cylinder consists of mapped straight lines of streamlines of the physical sub-domain Ω_m, parallel to the direction of the generants. These transformed streamlines are related to a local variable s in Ω^*_m, to be used as computational sub-domain. The mapped domains $\mathcal{B}^*_o$ and $\mathcal{B}^*_c$ of the respective open and closed stream tubes $\mathcal{B}_o$ and $\mathcal{B}_c$ are elementary straight cylinders of Ω^*_m. The basis of the cylinder is related to local variables (X, Y).

Let Ω^*_m be elementary subdomains of rectilinear parallel streamlines related to a reference cross-section S^*_m and involving a finite number of mapped stream tubes $\mathcal{B}^*_i$ $(\Omega^*_m = \bigcup \mathcal{B}^*_{im})$. We then investigate elementary transformations $\mathcal{T}_m$ defined from a mapped subdomain Ω^*_m towards a subregion Ω_m of number m, defined between two cross-sections z_m and z_{m+1} in the physical domain Ω, such that

$$\mathcal{T}_m : \Omega^*_m \to \Omega_m$$

with

$$M^*(X, Y, s) \to M(x, y, z) \tag{18}$$

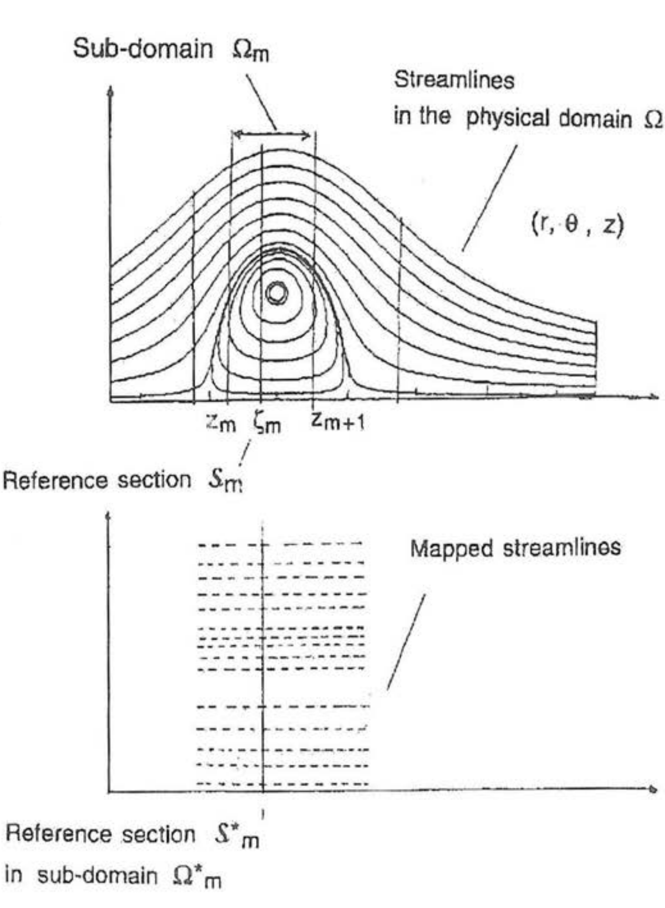

Figure 2. Subdomain Ω_m and its mapped domain involving rectilinear streamlines for a two-dimensional (axisymmetric) flow

streamline (in a steady flow configuration) defined by

$$\chi = \chi_0 + \int_{t_0}^{t} |\mathbf{V}(\tau)|\, d\tau \tag{19}$$

where χ_0 denotes the abscissa of the reference section $\mathcal{S}_m^*$ of the mapped subdomain Ω_m^*, $|\mathbf{V}(\tau)|$ the modulus of the velocity vector $\mathbf{V}$ on the streamlines points, expressed by

$$\mathbf{V} = u(x,y,z)\boldsymbol{\varepsilon}_1 + v(x,y,z)\boldsymbol{\varepsilon}_2 + w(x,y,z)\boldsymbol{\varepsilon}_3. \tag{20}$$

In equation (19), the times t_0 and t are associated to positions s_0 and s of a

Defining by s_M ($s_M < +\infty$) the given total length of the streamline in the transformed domain and by χ_M ($\chi_M < +\infty$) the maximum curvilinear abscissa on every streamline $\mathcal{L}$ originated at a point M_o of the stream tube, we may define the variable s by the following relationship

$$s = (\chi/\chi_M)s_M. \tag{21}$$

In order to define the variable s for a streamline $\mathcal{L}$ of zero velocity (presence of a wall where the fluid adheres), the coordinate χ related to a point M is assumed to be the curvilinear distance $\mathbf{M_0M}$. Such definitions contrast to the previous specific formulations of stream-tube method which do not involve variables directly related to the kinematics.

Fig. 3 illustrates physical and mapped stream tubes of length s_M corresponding to open and closed streamlines in a three-dimensional flow situation. The case of open stream tubes B_o is shown in Fig. 3a. Fig. 3b depicts a flow

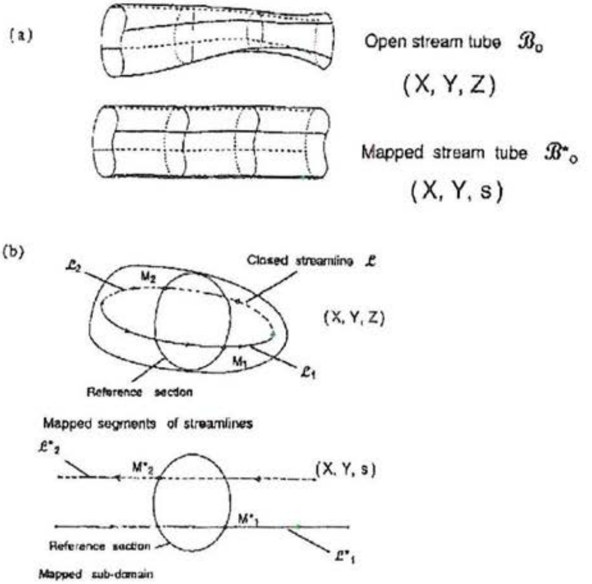

Figure 3. Elementary subdomains: (a) Physical open stream tube $\mathcal{B}_o$ and its transformed domain $\mathcal{B}_o^*$. (b) Physical closed stream tube $\mathcal{B}_c^*$ and its mapped domain $\mathcal{B}_c^*$

subdomain of Ω_m involving closed stream tubes, related to a reference section, for $s = s_0$. The sub-region of closed streamtubes, doubly-connected, is ob-

Fig. 3b, it may be observed that the closed stream tube involves two separated regions where the main velocity component does not change its sign. A closed streamline $\mathcal{L}$ can be divided into two curves $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ according to the positions of streamline points of zero velocity. The curves $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect the reference section at points M_1 and M_2, respectively. The mapped rectilinear lines of $\mathcal{L}_1$ and $\mathcal{L}_2$ are represented in the mapped domain as distinct segments passing through M_1^* and M_2^*, respective transformed points of M_1 and M_2. The mapped closed stream tube defines a straight cylinder, of basis included in the reference section of the mapped subdomain Ω_m^* of Ω_m.

2.3. Kinematics — Basic equations and unknowns

Referring to the general definitions for flows of an incompressible fluid, we may write the velocity vector in terms of a pair of stream functions ψ_1 and ψ_2 as

$$\mathbf{V} = \nabla\psi_1(x, y, z) \times \nabla\psi_2(x, y, z) \tag{22}$$

The use of this equation at a reference section $\mathcal{S}(X, Y, s_0)$ where $x = X$, $y = Y$ and $z = s_0$ leads to writing the w-velocity component as

$$\begin{aligned} w(X, Y, s_0) &= \partial\psi_1(x, y, z)/\partial X . \partial\psi_2(x, y, z)/\partial Y_{|(X,Y,s_0)} \\ &- \partial\psi_1(x, y, z)/\partial Y . \partial\psi_2(x, y, z)/\partial X_{|(X,Y,s_0)}. \end{aligned} \tag{23}$$

We define the reference kinematic function $\phi(X, Y)$ by the following relation

$$\phi(X, Y) = w(X, Y, s_0). \tag{24}$$

From equations (14–16), we may express the components (u, v, w) of the velocity vector $\mathbf{V}$ as

$$u = \alpha'_s\phi(X, Y)/\Delta;\ v = \beta'_s\phi(X, Y)/\Delta;\ w = \gamma'_s\phi(X, Y)/\Delta, \tag{25}$$

where Δ denotes the Jacobian, assumed to be non-singular, given by a relation of the type of equation (4).

We can assume the following relations at the reference section $\mathcal{S}$ for (X, Y, s_0)

$$\begin{aligned} &\alpha'_X(X, Y, s_0) = 1,\ \alpha'_Y(X, Y, s_0) = 0 \\ &\beta'_X(X, Y, s_0) = 0,\ \beta'_Y(X, Y, s_0) = 1 \\ &\gamma'_X(X, Y, s_0) = 0,\ \gamma'_Y(X, Y, s_0) = 0 \end{aligned} \tag{26}$$

For a subdomain Ω_m, the primary unknowns to be considered are the three mapping functions α, β, γ and the pressure p. In accordance with the definition of the variable s, the set of the governing equations should also involve the following relation (with $w(X, Y, s) \neq 0$), for points that are not located at the boundary where generally the fluid adheres:

$$z = z_0 + \int^s w(X, Y, \zeta)\, d\zeta \tag{27}$$

or, equivalently

$$\gamma(X,Y,s) = \gamma(X,Y,s_0) + \phi(X,Y) \int_{s_0}^{s} [\gamma_s'(X,Y,\xi)/\Delta(X,Y,\xi)]\, d\xi \tag{28}$$

In isothermal conditions, the momentum conservation law provides three dynamic equations. Though the writing of those equations is possible by classic tools of tensorial analysis, we find it of more practical use to write them by means of cartesian coordinates, following the approach already defined in specific stream-tube analysis (e.g. Clermont and de la Lande, 1993). Accordingly, the superscripts i and j,k, respectively, associated to components b^i and A^{jk} of a vector $\mathbf{b}$ and a tensor $\mathbf{A}$ are still related to $x = 1$, $y = 2$ and $z = 3$, but the derivatives $\partial/\partial X^i$ are expressed by means of derivative operators $\partial/\partial \xi^j$ defined by equations (14–16) that involve the variables (X,Y,s) of the mapped computational domains.

2.4. Examples

Viscometric shearing flow situations provide simple examples of transformation functions involving global mapping functions in the total flow domain. For the plain Poiseuille case, the functions may be written as

$$x = \alpha(X) = X;\; y = \beta(Y) = Y;\; z = \gamma(Y) = s. \tag{29}$$

The pure rotating Couette flow in a plane (x,y) between two concentric cylinders of radii R_0 and R_1 $(R_0 < R_1)$ involves global mapping functions expressed by

$$\begin{aligned} x &= \alpha(Y,s) = Y\cos(s/Y) \\ y &= \beta(Y,s) = Y\sin(s/Y) \\ z &= \gamma(X) = X. \end{aligned} \tag{30}$$

The flow streamlines of Fig. 2, previously considered, illustrate a two-dimensional analytical case requiring local transformation functions (Normandin and Clermont, 1994). In this situation, the upstream section z_1 allows a reference section $\mathcal{S}_1$ to be defined such that the corresponding reference function ϕ_1 (see equation (23)) at section $\mathcal{S}_1$ is known. Various subdomains Ω_m^*, of reference sections $\mathcal{S}_m$, may involve stream tubes of different types.

The elements shown in Fig. 4 provide an illustrative example of a three-dimensional flow situation, with different subdomains Ω_m^* of reference sections $\mathcal{S}_m$. The stream-tube cross-sections of Fig. 3b and the cut section of Fig. 3c in the mean flow direction point out the presence of a main flow region

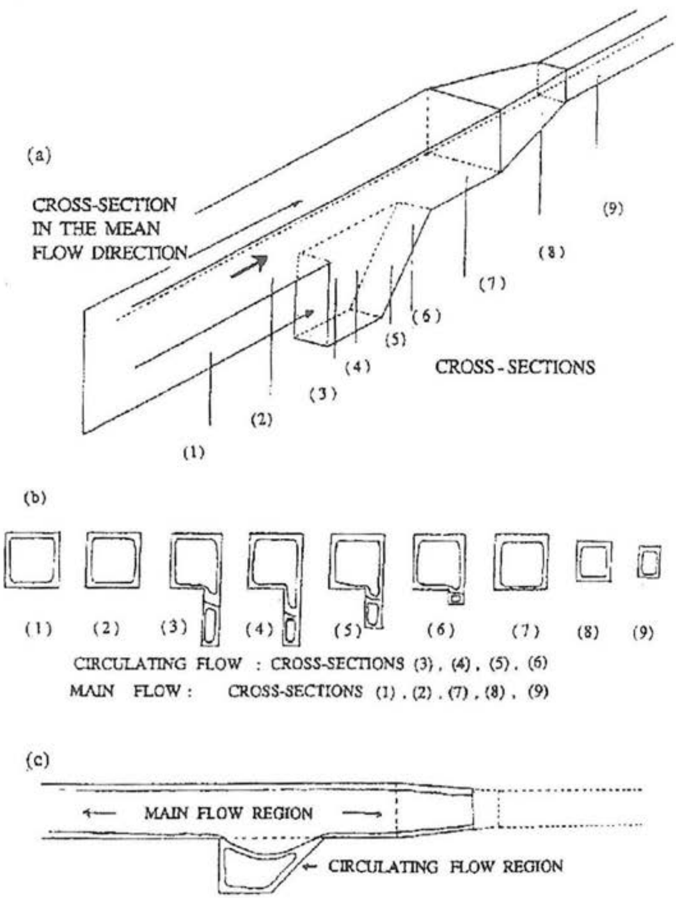

Figure 4. Example of a three-dimensional flow. (a) Duct geometry, involving the positions of cut and cross-sections. (b) Cross-sections of the stream tubes at $z =$ Constant. (c) Cut section of stream tubes in the mean flow direction

3. Specific properties — Computational considerations

3.1. Specific features of the analysis

1) The formulation proposed here by means of local transformations $\mathcal{T}_m$ can be proved to verify the incompressibility condition (mass conservation) $\nabla.\mathbf{V} = 0$, by using equations (14–16) and (25), similarly to the previous formulations related to open (Clermont and de la Lande, 1993) and closed (Clermont et al., 1991, Clermont and Radu, 1999a, 1999b) streamlines. Consequently, given

$\nabla.\boldsymbol{\sigma} = \mathbf{F}$ ($\boldsymbol{\sigma}$ denotes the stress tensor, $\mathbf{F}$ the external forces), formally written as

$$\mathcal{E}_{mj}(\alpha_i, \beta_i, \gamma_i, p_i) = 0 \ (j = 1, 2, 3), \text{ for } m = 1, 2, \ldots, m_0 \quad (31)$$

are to be considered under isothermal conditions.

2) Using equations (11–13) and (25), it can be shown that the velocity vector $\mathbf{V}$ in a stream tube is given in the natural basis $\mathbf{e}_i$ ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) by the simple relation

$$\mathbf{V} = [\phi(X, Y)/\Delta]\mathbf{e}_3. \quad (32)$$

3) The analysis enables the computations of the unknowns to be performed in simple mapped subdomains where the streamlines are rectilinear and parallel to the generants of the straight cylinders defined for the transformation (Fig. 2).

4) As for the particular cases of flows involving specifically open or closed streamlines, the present formulation avoids particle tracking problems for fluids obeying time-dependent constitutive equations. The time evolution of particles on their pathlines may be easily evaluated using the w-velocity component given by equation (25), such that

$$t - t_o = \int_{s_o}^{s} d\xi / w(X, Y, \xi) = [1/\phi(X, Y)] \int_{s_o}^{s} [\Delta/\gamma_s']_{(X,Y,\xi)} \, d\xi \quad (33)$$

where t_o denotes the reference time corresponding to the position of the material point at the reference section (X, Y, s_o) in the sub-domain considered.

5) It should be pointed out that the mapping of coordinate equations (2–3) cannot be considered as particular cases of the relevant transformation equations (18), essentially because the coordinate s is directly related to the curvilinear abscissa for a material point on its pathline. However, the definition of this coordinate allows to consider mapped stream tubes of given length, as it was the case in the previous stream-tube formulations.

3.2. Computational considerations

Recalling the case of specific "open stream-tube" problems where the flow does not involve recirculating regions, a single function ψ^* introduced similarly to the reference kinematic function ϕ of equations (22–23) is known and remains unchanged, as also the corresponding reference section, during the iterative numerical process (see, for example, Clermont, 1983, André and Clermont, 1990, Clermont et al., 1991). However, for specific "closed stream-tube" situations (pure recirculating flows), Clermont and Radu (1999a, 1999b), Clermont and Normandin (1993), that require a reference section z_o to be selected arbitrarily to define the streamline transformations, the function ψ^* corresponding to the reference function ϕ was unknown and had to be determined iteratively in the

Concerning general features of the numerical computations, it should be pointed out that, according to the geometry of the streamlines in the computational domain, simple meshes can be defined for approximating the equations and unknowns involved in the present formulation. The procedure to be achieved here for computing the flow characteristics is essentially related to domain decomposition methods, requiring the writing of compatibility equations at the interfaces of the sub-domains. The solution of the problem in the total domain Ω should be obtained by considering local sub-problems stated on the subdomains Ω_m. This analysis generalizes the previous stream-tube formulations already depicted. Field calculations using subdomains for different flow conditions have been around since quite a long time (see for example Dinh et al., 1984).

The basic ideas for solving the problem can be summarized as follows:

The geometrical domain Ω of boundary Γ is divided into M sub-domains Ω_m ($m = 1, 2, \ldots, M$). We denote by Γ_{m-1} ($m = 2, 3, \ldots, M$) the interface of Ω_{m-1} and Ω_m (see the example of Fig. 5).

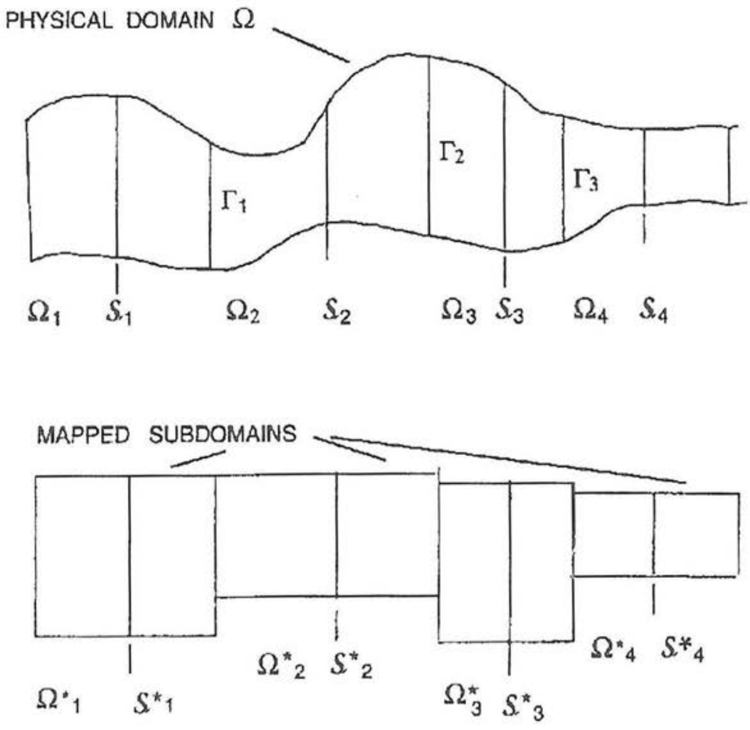

Figure 5. Physical domain Ω divided into subregions Ω_m and computational

If p_i corresponds to the restriction of the pressure p to Ω_i, we may consider local transformations $\alpha_i(X,Y,s)$, $\beta_i(X,Y,s)$ and $\gamma_i(X,Y,s)$ verifying:

— the kinematic equation (27);

— the dynamic equations (31);

— the following boundary condition equations on $\Gamma \cap \Omega_i$:

$$\begin{aligned} &a_j(\alpha_i,\beta_i,\gamma_i,p_i) = 0 \ (j=1,2,3) \\ &b_j(\alpha_i,\beta_i,\gamma_i,p_i) = 0 \ (j=1,2,3) \\ &p_i = \pi_i; \end{aligned} \tag{34}$$

— the compatibility conditions at the common boundary Γ_i of the subdomains Ω_i and Ω_{i+1}, such that:

$$p(M_i^*) = p(M_{i+1}^*) \text{ on } \Gamma_i \tag{35}$$

$$\mathbf{V}(M_i^*) = \mathbf{V}(M_{i+1}^*) \text{ on } \Gamma_i \tag{36}$$

where $M_i^* \in \Omega_i$, $M_{i+1}^* \in \Omega_{i+1}$, with $x(M_i^*) = x(M_{i+1}^*)$, $y(M_i^*) = y(M_{i+1}^*)$, $z(M_i^*) = z(M_{i+1}^*)$ on Γ_i.

For every subdomain Ω_i, we consider a reference section S_i to which a kinematic function ϕ_i corresponds. This function is generally unknown for the subdomains, as pointed out previously.

To solve the problem, the main features of the algorithm can be written according to the following process, given a numerical procedure for solving the equations, upon convergence criteria:

(i) Initialization:

Definition of the subdomains Ω_i $(i = 1,2,\ldots,M)$ and choice of the reference sections $\mathcal{S}_i$.

A geometrical shape of the streamlines is assumed, as also the kinematic function $\phi_{i[0]}$ in every mapped subdomain Ω_i^*, related to the reference section $\mathcal{S}_i$. The initial guess of the streamlines correspond to an estimate of the local functions α_i, β_i and γ_i.

(ii) Solve the following set of equations:

— the kinematic equation (27),

— the dynamic equations (31) and boundary conditions (34–36) in the total flow domain Ω^*.

The Levenberg–Marquardt optimization algorithm (see, for example, Gourdin and Boumahrat, 1989) is assumed to be used to obtain new iterates and achieving the process. This method combines the Newton and gradient algorithms and should allow the procedure to converge for an initial estimate rather far from the solution. This algorithm, previously used in applications of the stream tube method (Clermont and de la Lande, 1993, Clermont and Radu, 1999b, Clermont and Normandin, 1993) that also required direct computations of streamlines, has proved to be robust and efficient. It should be underlined that in such problems, slight modifications in position of the streamline points

4. Application to the two-dimensional flow between eccentric cylinders

We now consider the steady flow of a Newtonian fluid, where the stress tensor is given by the following equation

$$\mathbf{T} = 2\mu\mathbf{D} \tag{37}$$

in the annulus of a cylinder, as shown in Fig. 1b, when involving recirculations. In equation (37), $\mathbf{D}$ denotes the rate-of-deformation tensor; μ the constant viscosity of the fluid. Such flows have been extensively studied in the literature (e.g. Ballal and Rivlin, 1977, Ramesh and Lean, 1991) and are notably related to lubrication problems in journal bearings, for small gaps.

The half-plane physical domain Ω between the two eccentric cylinders (Fig. 6a), is referred to polar coordinates ($x^1 = r$, $x^2 = \theta$) limited by the azimuth angles $\theta = 0$ and $\theta = \pi$. The inner cylinder, of radius r_0, rotates with an angular velocity ω. The outer cylinder is at rest. The parameter e denotes the distance between the axes of the cylinders. According to results obtained for such geometry with a Newtonian fluid and ignoring inertial effects, Kelmanson (1984), we may consider two elementary sub-domains Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. The angle θ_1 corresponds to the limiting section between subdomains Ω_1 and Ω_2 (Fig. 6) and is a priori unknown. Two reference kinematic functions ϕ_1 and ϕ_2 are defined at sections $\theta = 0$ and $\theta = \pi$, respectively. On these

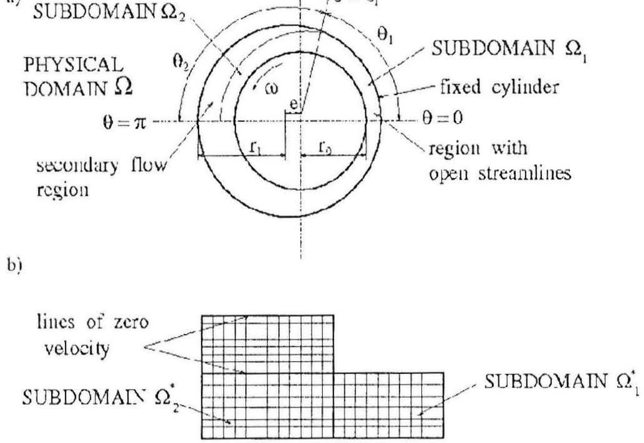

Figure 6. (a) Physical domain Ω divided into subregions Ω_1 and Ω_2 related to the two-dimensional flow between two cylinders. (b) Mapped computational subdo-

subdomains, we define two local transformations $\mathcal{T}_m : \Omega_m^* \to \Omega_m$ $(m = 1, 2)$, where $M^*(s, R) \to M(r, \theta)$. The variable s is defined according to equation (21) and the variable R corresponds to the radius r at the reference sections of the subdomains. Fig. 6b shows the mapped subdomains Ω_1^* and Ω_2^*, where the streamline lengths have been renormalized for computational purposes. The first subdomain Ω_1 involves only open streamlines. For subdomain Ω_2, open and closed streamlines are to be taken into account. The dynamic equations are written in terms of variables (s, R) for each mapped subdomain. The Levenberg–Marquardt optimization algorithm is used to compute the unknowns (mapping functions and pressure). A typical grid used for discretizing the equations and unknowns is shown in Fig. 6b.

The code was implemented on a workstation (Pentium 333 MHz processor). The number of equations and unknowns was approximately 1000. Fig. 7 presents our numerical results (solid lines) for the streamlines, in the case of a large gap between the cylinders, compared to those previously provided by Kelmanson (1984). A good agreement may be observed. Another example of computed

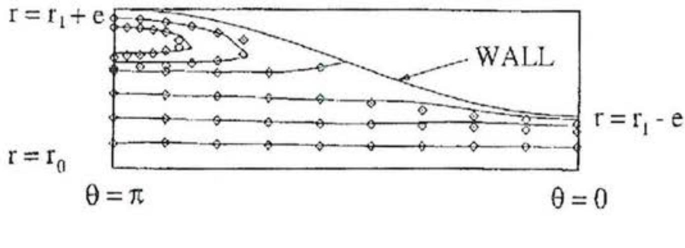

Figure 7. Computational results for the streamlines (Newtonian fluid): — : present work; $\diamondsuit$: Kelmanson data. The geometrical parameters are: $r_0 = 15$ mm, $r_1 = 30$ mm, $e = 7.5$ mm. The inner cylinder rotates at an angular velocity $\omega = 250$ rad/s

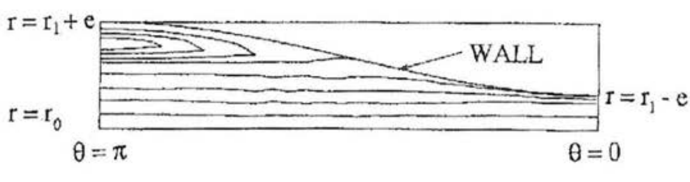

Figure 8. Computed streamlines in the conditions of a journal bearing flow (Newtonian fluid). The geometrical parameters are: $r_0 = 29.9$ mm, $r_1 =$

streamlines, corresponding to the case of a journal bearing geometry (small gap and eccentricity), is given in Fig. 8.

5. Concluding remarks

The formulation presented in this paper provides a unified approach for the simulation of complex flows involving open streamlines, closed streamlines or both situations, by using geometrical considerations allowing to rewrite the equations in terms of streamline coordinates. Distinguishing features of the analysis are related to the definition of local transformations leading to application of domain decomposition methods. The automatic verification of mass conservation from the basic equations and the reduction of difficulties in particle-tracking problems related to constitutive equations with history-dependence are significant possibilities of the approach to be underlined. Though the analysis involves great similarities with the previous studies using the concept of streamlines and stream tubes, the corresponding basic equations of those specific stream tube methods cannot be considered as particular cases. The numerical application related to the flow in the annulus of two cylinders has provided consistent results with those in the literature.

References

André, P. and Clermont, J.R. (1990) Experimental and numerical study of the swelling of a viscoelastic liquid using the stream-tube method and a kinematic singularity approximation. *J. Non-Newtonian Fluid Mech.*, **38**, 1-29.

Ballal, B.Y. and Rivlin, R.S. (1977) Flow of Newtonian fluid between eccentric rotating cylinders: inertia effects. *Arch. Rat. Mech. Anal.*, **62**, 237-294.

Broszeit, J. (1997) Finite-element simulation of circulating steady flow for fluids of the integral type: flow in a single-screw extruder. *J. Non-Newtonian Fluid Mech.*, **70**, 35-58.

Caswell, B. (1996) Report on the IXth International Workshop on Numerical Methods in Non-Newtonian Flows. *J. Non-Newt. Fluid Mech.*, **62**, 99-110.

Clermont, J.R. (1983) Sur la modélisation numérique d'écoulements plans et méridiens de fluides incompressibles. *C. R. Acad. Sci. Paris Sér.* 2.1, **297**.

Clermont, J.R. and de la Lande, M.E. (1993) Numerical simulation of three-dimensional duct flows of incompressible fluids by using the stream-tube method. Part I: Newtonian equation. *Theor. Comput. Fluid Dynamics*, **4**, 129-149.

Clermont, J.R., de la Lande, M.E., Pham Dinh, T. and Yassine, M.
(1991) Analysis of plane and axisymmetric flows of incompressible fluids

with the stream-tube method: Numerical simulation by Trust-Region Optimization algorithm. *Int. J. Num. Meth..Fluids*, **13**, 371-399.

CLERMONT, J.R. and NORMANDIN, M. (1993) Numerical simulation of extrudate swell for Oldroyd-B fluids using the stream-tube analysis and a streamline approximation. *J. Non-Newtonian Fluid Mech.*, **50**, 193-215.

CLERMONT, J.R. and RADU, D. (1999A) Formulations pour le calcul d'écoulements dans des paliers cylindriques. *Revue Roumaine des Sciences et Techniques* (Roumanian Academy of Sciences), in press.

CLERMONT, J.R. and RADU, D. (1999B) Calcul d'écoulements de fluides viscoélastiques a formulation intégrale dans des paliers cylindriques. submitted to the *Revue Roumaine des Sciences et Techniques* (Roumanian Academy of Sciences), in press.

DINH, Q.V. GLOVINSKI, R. and PERIAUX, J. (1994) Solving elliptic problems by domain decomposition methods with applications. In: *Elliptic Problem Solvers*, C. Birkoff and A. Schoensdat, eds., Academic Press, New York.

DUDA, J.L. and VRENTAS, J.S. (1967) Fluid mechanics of laminar liquid jets. *Chem. Eng. Sci.*, **22**, 855-869.

GOUBLOMME, A. and CROCHET, M.J. (1992) Numerical prediction of extrudate swell of a high-density polyethylene. *J. Non-Newt. Fluid Mech.*, **44**, 171-195.

GOURDIN, A. and BOUMAHRAT, M. (1989) *Méthodes numériques appliquées, Technique et Documentation.* Editions Lavoisier, Paris.

GREYWALL, M.S. (1985) Streamwise computation of Two-dimensional incompressible potential flows. *J. of Comp. Physics*, **59**, 224-231.

GREYWALL, M.S. (1988) Streamwise computation of Three-dimensional incompressible potential flows. *J. of Comp. Physics*, **59**, 178-193.

GUILLET, J., REVENU, P., BÉREAUX, Y. and CLERMONT, J.R. (1996) Experimental and numerical study of entry flow of low-density polyethylene melts. *Rheol. Acta*, **35**, 494-507.

KELMANSON, M.A. (1984) A boundary integral equation method for the study of slow flow in bearings with arbitrary geometries. *Trans. of the ASME*, **108**, 260-264.

LAU, T.S., LO, S.H. and LEE, C.K. (1997) Generation of quadrilateral mesh over analytical curved surfaces. *Finite Elements Anal. Des.*, **27**, 251-272.

LUO, X.L. and MITSOULIS, E. (1990) An efficient algorithm for strain history tracking in finite element computations of non-Newtonian fluids with integral constitutive equations. *Int. J. Num. Meth. Fluids*, **11**, 1015-1031.

LUO, X.L. and TANNER, R.I. (1986) A streamline element scheme for solving viscoelastic flow problems. Part II. *J. Non-Newt. Fluid Mech.*, **22**, 61-69.

LUO, X.L. and TANNER, R.I. (1988) Finite element simulation of long and short die extrusion experiments using integral models. *Int. J. Num. Methods Eng.*, **25**, 9-22.

MARCHAL, J.M. and CROCHET, M.J. (1987) A new finite element for calculating viscoelastic flows. *J. Non-Newt. Fluid Mech.*, **26**, 77-114.

NORMANDIN, M. and CLERMONT, J.R. (1994) Theoretical analysis of extensional flows in relation to processing rheology and predictions of some constitutive equations. *Rheol. Acta*, **33**, 145-157.

NORMANDIN, M. and CLERMONT, J.R. (1996) Three-dimensional extrudate swell: Formulation with the stream-tube method and numerical results for a Newtonian fluid. *Int. Journal for Num. Meth. Fluids*, **23**, 937-952.

PAPANASTASIOU, A.C., SCRIVEN, L.E. and MACOSKO, C.W. (1987) A finite-element method for liquid with memory. *J. Non-Newt. Fluid Mech.*, **22**, 271-288.

RADU, D. (1997) Méthode des tubes de courant pour des écoulements dans des paliers cylindriques 2D — Étude théorique et numérique. Internal Report. Laboratoire de Rhéologie, Grenoble, July 1997.

RAMESH, P.S. and LEAN, M.H. (1991) A boundary-integral equation for Navier–Sokes equations — Application to flow in annulus of eccentric cylinders. *Int. J. Num. Meth. Fluids*, **13**, 3, 355-369.

SHIOJIMA, T. and SHIMAZAKI, Y. (1990) Three-dimensional finite element method for extrudate swells of a Maxwell fluid. *J. Non-Newtonian Fluid Mech.*, **34**, 269-288.

SMITH, R.E. (1982) Algebraic grid generation. *Appl. Math. Comp.*, **10/11**, Nashville, 137-170.

TRAN-CONG T. and PHAN-THIEN, N. (1988) Three-dimensional study of extrusion processes by boundary element method 2. Extrusion of a viscoelastic fluid. *Rheol. Acta*, **27**, 639-648.