Control and Cybernetics

vol. **31** (2002) No. 3

Solution tubes to differential inclusions within a collection of sets

by

Marc Quincampoix[1] **and Vladimir M. Veliov**[2]

[1] Département de Mathématiques, Faculté des Sciences et Techniques,
Université de Bretagne Occidentale,
6 Avenue Victor Le Gorgeu, BP 809, 29285 Brest, France

[2] Institute for Econometrics, Operations Research and Systems Theory,
Vienna University of Technology,
Argentinierstrasse 8, A-1040 Vienna, Austria;
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria

Abstract: This paper develops the theory of solution tubes to differential inclusions (uncertain systems) within a prescribed collection of sets. The notion is defined as a minimal invariant tube with values in the collection. Under certain requirements for the collection we prove existence and Lipschitz-like stability of the solution tubes. The theory is relevant to problems of systems estimation in the context of control or differential games.

Keywords: differential inclusions, control systems, uncertain systems, reachable sets, deterministic estimation.

1. Introduction

The reachable set of a differential inclusion (the latter interpreted as an uncertain system) is the minimal guaranteed estimation of the current state. Therefore, to calculate reachable sets is a cornerstone of the deterministic estimation and control of uncertain systems (see e.g. Kurzhanski and Filippova, 1993). A lot of work has been done for developing numerical approximation methods, see the surveys of Dontchev and Lempio (1992), and Lempio and Veliov (1998). Since the geometry of the reachable sets can be quite complex, specific subclasses of sets are usually used as approximation tools: boxes, polyhedral

This work was started during a visit of the second author in the Département de Mathématiques of the Université de Bretagne Occidentale. The work was partially supported

sets, ellipsoids (see Chernousko, 1988, Kurzhanski and Valyi, 1997, Chernousko and Rokityanskii, 2000, Kurzhanski and Varaiya, 2000), box or polyhedral complexes (Saint-Pierre, 1994, Häckl, 1992–93, Ushakov and Khripunov, 1994, Cardaliaguet, Quincampoix and Saint-Pierre, 1999), etc. In some cases convergence results are obtained, but usually to achieve a good approximation one has to use rather complex approximating sets. On the other hand, in problems of control of uncertain systems and differential games, where the systems estimation is just an auxiliary tool, one has to employ only fairly simple sets[1]. In such cases the issue of *approximation* is not that relevant. A different problem arises: to obtain inclusions of the reachable sets in sets from a prescribed collection $\mathcal{E}$. That is, to replace the solution tube $X(t)$ of the differential inclusion by a tube $E(t)$ with values in $\mathcal{E}$. In doing this, one has to ensure at least $X(t) \subset E(t)$, but two more properties are also desired: (i) the Markov property of the evolution of $E(\cdot)$, which, together with $X(t) \subset E(t)$, requires invariance of the tube $E(t)$ with respect to the differential inclusion, and (ii) minimality.

The present paper investigates the existence, uniqueness, and the dependence on the data, of tubes $E(\cdot)$ with the above properties (we call such tubes *solution tubes* of the differential inclusion *in the collection* $\mathcal{E}$). Existence of ellipsoidal-valued solution tubes was previously proven in Chernousko (1988) and, in a more elaborate form, in Kurzhanski and Valyi (1997), but the proofs substantially utilize the linearity of the systems considered there. To our knowledge, Lipschitz dependence of the solution tubes in a given collection has not been previously investigated[2], being in the same time of critical importance for some applications, in particular in the context of the differential games with incomplete/imperfect information.

We stress the fact that the notion of solution tube in a given collection of sets does not have a counterpart in the theory of quasi-differential equations (Panasyuk, 1990), or in the theory of mutational equations (Aubin, 1993, 1999). In general, the notion of solution tube in a given collection $\mathcal{E}$ coincides with the notions of solution in the above publications only in the case of the collection $\mathcal{E}$ consisting of all compact set. For the same reason the Filippov-type theorem obtained in Doyen (1993) cannot be used for obtaining Lipschitz stability of the solution tubes in a more general collection $\mathcal{E}$.

The existence and Lipschitz stability results that we obtain in Sections 2 and 3, respectively, provide the basis for extending (also in the direction of algorithms) a number of existing control theoretic considerations in presence of unobservable uncertainties[3]. The complementary set-analytic tools are devel-

[1]The associated Hamilton–Jacobi–Bellman–Isaacs equation, for example, has the dimension of the state estimators, therefore it should not be too large.

[2]The results in Chernousko (1988) and in Kurzhanski and Valyi (1997) may have some implications in this direction, but for linear systems only.

[3]that is, uncertainties that remain unknown, in contrast to other uncertainties whose

oped in Quincampoix and Veliov (1999). These extensions, however, are beyond the scope of the present paper.

The results are obtained under a number of conditions for the collection $\mathcal{E}$ that are discussed in Section 4 together with some examples.

2. Solution tubes in a collection of sets

We shall use the following standard notations: $\mathcal{B}$ is the Euclidean unit ball in $\mathbf{R}^n$, $\text{comp}(\mathbf{R}^n)$ is the set of all compact subsets of $\mathbf{R}^n$, $H(X,Y)$ is the Hausdorff distance between two sets $X, Y \in \text{comp}(\mathbf{R}^n)$, $e(X,Y) \stackrel{\text{def}}{=} \sup_{y\in Y} \inf_{x\in X} |x-y|$. Multiplication of a set with a scalar, and summation of sets are understood in the usual (Minkowski) sense. With the Hausdorff distance and the Minkowski operations, $\text{comp}(\mathbf{R}^n)$ becomes a complete metric cone. For a set X, $f(X)$ stays for $\{f(x) : x \in X\}$.

Below, $[0,T]$ will be a fixed time interval.

DEFINITION. A set-valued map $E(\cdot) : [0,T] \Rightarrow \mathbf{R}^n$ is called a *tube* if it is nonempty compact valued and has closed graph. A tube is *Lipschitz continuous* if there is a constant L such that

$$H(E(s), E(t)) \leq L|t-s| \quad \text{for every } s,t \in [0,T].$$

We consider a differential inclusion

$$\dot{x} \in F(x,t), \quad x \in \mathbf{R}^n, \quad t \in [0,T], \tag{1}$$

supposing the following:

Condition A. $F : \mathbf{R}^n \times [0,T] \Rightarrow \mathbf{R}^n$ is a set-valued mapping with nonempty convex, compact values, measurable in t for every fixed x, and locally Lipschitz continuous in x uniformly with respect to t. Moreover, F satisfies the linear growth condition

$$|F(x,t)| \leq a(1+|x|), \quad \forall x \in \mathbf{R}^n, \quad t \in [0,T].$$

As usual, a solution to (1) is any absolutely continuous function that satisfies (1) for a.e. t. Given a set of initial states $E_0 \subset \mathbf{R}^n$ the solution tube of (1) on $[0,T]$ is defined as

$$X(t) \stackrel{\text{def}}{=} x[0,E_0](s) \stackrel{\text{def}}{=} \{x(s) : x(\cdot)\text{–solution of (1) on } [0,s], \text{ with } x(0) \in E_0\}.$$

This is the unique *minimal* tube that starts from E_0 at $t=0$ and is *invariant* with respect to (1), the latter meaning that

$$\forall s \in [0,T),\ \forall x(\cdot)\text{–solution of (1) on } [s,T], \text{ with } x(s) \in X(s)$$
$$\Longrightarrow x(t) \in X(t) \quad \forall t \in [s,T].$$

Here and below, "minimal", when applied to sets, means 'minimal with respect partial ordering in which $E_1(\cdot) \prec E_2(\cdot)$ if and only if $E_1(t) \subset E_2(t)$ for every $t \in [0, T]$.

In general, it is difficult to calculate the solution tube numerically, unless the values $X(t)$ belong to a class of simple finitely parametrized family of sets. However, this happens only in rather exceptional cases. For this reason we reverse the issue: we fix in advance a collection of sets $\mathcal{E}$ and want to find a solution tube of (1) that takes values from $\mathcal{E}$ only. To make this concept precise, we give the following definition.

DEFINITION. The tube $E(\cdot) : [0, T] \mapsto \text{comp}(\mathbf{R}^n)$ is called *solution tube of (1) in the collection* $\mathcal{E}$, starting from $E_0 \in \mathcal{E}$, if and only if $E(\cdot)$ is a minimal invariant tube with values in $\mathcal{E}$, for which $E_0 \subset E(0)$.

Clearly, $x[0, E_0](\cdot)$ is the unique solution tube in the collection $\mathcal{E} = \text{comp}(\mathbf{R}^n)$, starting from E_0. To ensure existence of a solution tube in a more general collection $\mathcal{E}$ we introduce the following conditions for $\mathcal{E}$.

Condition B.1.a. The collection $\mathcal{E}$ consists of nonempty compact sets and is closed in the Hausdorff metric. For every compact Z there is some $E \in \mathcal{E}$ containing Z.

Obviously **B.1.a**, together with the Zorn lemma, implies that for every $Z \in \text{comp}(\mathbf{R}^n)$ there exists a minimal element of $\mathcal{E}$ containing Z. We denote by $\mathcal{M}(Z)$ the set of all such minimal elements.

Condition B.1.b. There exist real $\bar{\varepsilon} > 0$ and $L_{\mathcal{E}}$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$ and each $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}$ for which $E + \varepsilon\mathcal{B} \subset E' \subset E + \varepsilon L_{\mathcal{E}}\mathcal{B}$.

The above condition is equivalent to the following one:

Condition B.1.b′. There exist real $\bar{\varepsilon} > 0$ and $L_{\mathcal{E}}$ such that for each $Z \in \text{comp}(\mathbf{R}^n)$ and each $E \in \mathcal{E}$ for which $H(Z, E) \le \bar{\varepsilon}$ there exists $E' \in \mathcal{M}(Z)$ satisfying $e(Z, E') \le L_{\mathcal{E}} H(Z, E)$.

Indeed, if **B.1.b′** holds then one can apply it for $Z = E + \varepsilon\mathcal{B}$ to obtain **B.1.b** with the constant $L_{\mathcal{E}} + 1$. On the other hand, if **B.1.b** is fulfilled, then for Z and E such that $\varepsilon \stackrel{\text{def}}{=} H(Z, E) \le \bar{\varepsilon}$ there is $\widetilde{E} \in \mathcal{E}$ for which $Z \subset E + \varepsilon\mathcal{B} \subset \widetilde{E}$ and $\widetilde{E} \subset E + \varepsilon L_{\mathcal{E}}\mathcal{B}$. From the latter we have

$$e(Z, \widetilde{E}) \le e(Z, E) + e(E, \widetilde{E}) \le e(Z, E) + \varepsilon L_{\mathcal{E}} \le (1 + L_{\mathcal{E}}) H(Z, E).$$

Then, the Zorn lemma, together with **B.1.a**, implies existence of $E' \in \mathcal{M}(Z)$ for which the above inequality still holds. Thus, **B.1.b′** is fulfilled with the constant $L_{\mathcal{E}} + 1$.

It will be convenient to suppose that the constant $L_{\mathcal{E}}$ is fixed so large that both **B.1.b** and **B.1.b′** hold with the same $\bar{\varepsilon} > 0$.

THEOREM 2.1 *Suppose that conditions* **A** *and* **B.1** *are fulfilled. Then for every* $E_0 \in \mathcal{E}$ *inclusion (1) has a solution tube in* $\mathcal{E}$ *starting form* E_0*, which satisfies the growth estimation:*

$$E(t) \subset E_0 + (1 + |E_0|)(e^{2a(L_{\mathcal{E}}+1)t} - 1)\mathcal{B}$$

Moreover, if $\mathcal{M}(Z)$ is a singleton for every $Z \in \operatorname{comp}(\mathbf{R}^n)$, then the solution tube in $\mathcal{E}$ is unique and Lipschitz continuous with Lipschitz constant $2a(L_{\mathcal{E}} + 1)(1 + |E_0|)e^{2a(L_{\mathcal{E}}+1)T}$.

Proof. **1.** For every $s \geq 0$ which is so small that $e^{as} \leq 2$, and for every $E \in \mathcal{E}$, the growth condition implies in a standard way that

$$H(x[t,E](s), E) \leq 2as(1 + |E|). \tag{3}$$

As before, $x[t,E](\cdot)$ denotes the solution tube of (1) in $\operatorname{comp}(\mathbf{R}^n)$, starting from the set E at t.

2. *Construction of a discrete-time tube.* We fix N so large that

$$e^{aT/N} \leq 2 \quad \text{and} \quad 2ah(1 + |E_0|)e^{\lambda T} \leq \bar{\varepsilon}, \tag{4}$$

where $h = T/N$ and $\lambda = 2a(L_{\mathcal{E}} + 1)$. We shall define a sequence of elements $E_k^N \in \mathcal{E}$, $k = 0, 1, \ldots, N$, such that the following relations are fulfilled for $k = 0, \ldots, N-1$:

$$x[kh, E_k^N](h) \subset E_{k+1}^N \subset x[kh, E_k^N](h) + 2ahL_{\mathcal{E}}(1 + |E_k^N|)\mathcal{B}, \tag{5}$$

$$E_{k+1}^N \subset E_0 + (1 + |E_0|)(e^{\lambda(k+1)h} - 1)\mathcal{B}, \tag{6}$$

$$1 + |E_{k+1}^N| \leq (1 + |E_0|)e^{\lambda(k+1)h}. \tag{7}$$

Obviously (7) is a consequence of (6).

Set $E_0^N = E_0$. Since, according to (3) and (4),

$$H(x[0, E_0^N](h), E_0^N) \leq 2ah(1 + |E_0|) \leq \bar{\varepsilon},$$

one can apply **B.1.b′** with $Z = x[0, E_0^N](h)$ and $E = E_0^N$ to chose $E_1^N = E' \in \mathcal{M}(x[0, E_0^N](h))$ such that (5) is satisfied for $k = 0$. From (5) and (3) we have also

$$\begin{aligned} E_1^N &\subset x[0, E_0^N](h) + 2ahL_{\mathcal{E}}(1 + |E_0|)\mathcal{B} \\ &\subset E_0 + 2ah(1 + |E_0|)\mathcal{B} + 2ahL_{\mathcal{E}}(1 + |E_0|)\mathcal{B} \\ &\subset E_0 + h\lambda(1 + |E_0|)\mathcal{B} \subset E_0 + (1 + |E_0|)(e^{\lambda h} - 1)\mathcal{B}. \end{aligned}$$

Suppose inductively that E_k^N, $k < N$ is already defined so that (6) and (7) hold for $k-1$. We proceed similarly as at the first step. Since, according to (3), (7) and (4),

$$H(x[kh, E_k^N](h), E_k^N) \leq 2ah(1 + |E_k^N|) \leq 2ah(1 + |E_0|)e^{\lambda kh} \leq \bar{\varepsilon},$$

one can apply **B.1.b'** with $Z = x[kh, E_k^N](h)$ and $E = E_k^N$ to chose $E_{k+1}^N = E' \in \mathcal{M}(x[kh, E_k^N](h))$ such that (5) is satisfied. From (5),(3),(6) and (7) we have also

$$\begin{aligned}
&E_{k+1}^N \subset x[kh, E_k^N](h) + 2ahL_{\mathcal{E}}(1 + |E_k^N|)\mathcal{B} \\
&\subset E_K^N + 2ah(1 + |E_k^N|)\mathcal{B} + 2ahL_{\mathcal{E}}(1 + |E_k^N|)\mathcal{B} \subset E_k^N + h\lambda(1 + |E_k^N|)\mathcal{B} \\
&\subset E_0 + (1 + |E_0|)(e^{\lambda kh} - 1)\mathcal{B} + h\lambda(1 + |E_0|)e^{\lambda kh}\mathcal{B} \\
&\subset E_0 + (1 + |E_0|)(e^{\lambda kh} - 1 + h\lambda e^{\lambda kh})\mathcal{B} \subset E_0 + (1 + |E_0|)(e^{\lambda(k+1)h} - 1)\mathcal{B}.
\end{aligned}$$

This completes the recursive definition of the sequence $\{E_k^N \in \mathcal{E}\}_{k=0}^N$.

3. *Passing to a limit.* In order to obtain an $\mathcal{E}$-solution of (1) we have to pass to a limit with respect to N, first embedding the discrete-time tubes $\{E_k^N \in \mathcal{E}\}_{k=0}^N$ into continuous-time ones. To do the latter we use the following lemma proven in Artstein (1989).

LEMMA 2.1 *Let $t_0 < \ldots < t_N$ be real numbers. Let $F : \{t_0, \ldots, t_N\} \Rightarrow \mathbf{R}^n$ be a compact-valued Lipschitz continuous mapping, with Lipschitz constant L. Then, there exists an extension of F to $[t_0, t_N]$, which is Lipschitz continuous with the same constant L.*

For all N sufficiently large (as specified in (4)) the mapping $E^N(\cdot)$, defined for $t_k^N = kT/N$, $k = 0, \ldots, N$, as $E^N(t_k^N) \stackrel{\text{def}}{=} E_k^N$, is Lipschitz continuous with a constant L (independent of N), say

$$L = 2a(1 + L_{\mathcal{E}})(1 + |E_0|)e^{\lambda T}, \tag{8}$$

due to (5), (3), and (7). According to Lemma 2.1 it can be extended to a Lipschitz continuous mapping $E^N(\cdot)$ on $[0, T]$, with the same Lipschitz constant L. The mappings $E^N(\cdot)$ are uniformly bounded (due to (7)), therefore the Arzela-Ascoli theorem (applied to the space comp$(\mathbf{R}^n)$ with the Hausdorff metric) implies that there exists a subsequence $\mathbf{N}' \subset \mathbf{N}$ and $E(\cdot) : [0, T] \Rightarrow \text{comp}(\mathbf{R}^n)$ such that

$$\lim_{N \in \mathbf{N}'} E^N(\cdot) = E(\cdot),$$

uniformly in the Hausdorff metric. Moreover, $E(\cdot)$ is Lipschitz continuous and satisfies (2), as $\{E^N(\cdot)\}_k$ does for every N.

Clearly, $E(0) = E_0$. From **B.1.a** it easily follows that $E(t) \in \mathcal{E}$ for every $t \in [0, T]$.

The proof of the invariance of $E(\cdot)$ with respect to (1) uses a standard argument, but we provide it for completeness. Let $s < t$ be two numbers in $[0, T]$. We have to prove that

$$x[s, E(s)](t - s) \subset E(t). \tag{9}$$

Take an arbitrary $\varepsilon \in (0, 0.3(t - s)]$ and then an $N \in \mathbf{N}'$ so large that (4) is fulfilled, and also

Let $k, m \in \{0, \ldots, N\}$, $k < m$, be chosen in such a way that $|kh - s| \leq \varepsilon$ and $|mh-t| \leq \varepsilon$. From the first inclusion in (5) applied successively for $k, k+1, \ldots, m$ we obtain that

$$x[kh, E_k^N]((m-k)h) \subset E_m^N. \tag{10}$$

Let Z be a compact set containing $E(t) + \mathcal{B}$, $t \in [0, T]$. Condition **A** and Filippov's theorem (see Clarke, 1983 or Aubin and Frankowska, 1990) imply in a standard way that there exists a constant C such that

$$H(x[\tau, X](\theta), x[\tau', X'](\theta')) \leq C[|\tau - \tau'| + |\theta - \theta'| + H(X, X')],$$

provided that τ, $\tau + \theta$, τ', $\tau' + \theta' \in [0, T]$, $X, X' \subset Z$, and the quantity in the right-hand side does not exceed one. Then, utilizing (10) we obtain

$$x[s, E(s)](t-s) \subset x[kh, E_k^N]((m-k)h) + C(4\varepsilon + L\varepsilon)\mathcal{B}$$
$$\subset E_m^N + C(4+L)\varepsilon\mathcal{B} \subset E(t) + [C(4+L) + L + 1]\varepsilon\mathcal{B}.$$

Since $\varepsilon > 0$ was arbitrarily chosen, this implies (9).

Thus, we proved that an invariant tube in $\mathcal{E}$ starting from E_0 exists. Notice that the invariant tube $E(\cdot)$, that we constructed, satisfies inclusion (2).

4. To prove existence of a solution tube, that is, of a minimal invariant tube in $\mathcal{E}$, we apply the Zorn lemma with respect to the partial ordering $\prec$, defined above, in the set of all invariant tubes with values in $\mathcal{E}$ and starting from E_0. A pointwise intersection of a linearly ordered family of invariant tubes with values in $\mathcal{E}$ is also an invariant tube. Condition **B.1.a** implies that the latter has values in $\mathcal{E}$. Therefore, the Zorn lemma implies existence of a minimal invariant tube in $\mathcal{E}$. Clearly, (2) is also fulfilled.

5. Now, suppose that $\mathcal{M}(Z)$ is a singleton for every $Z \in \text{comp}(\mathbf{R}^n)$. Let $\widehat{E}(\cdot)$ be an arbitrary solution tube in $\mathcal{E}$. We shall prove that $\widehat{E}(\cdot) = E(\cdot)$ (the latter defined in part 3 of the above prove), which implies the last claim of the theorem since $E(\cdot)$ is Lipschitz with the constant L defined in (8). To do this we first notice that $\widehat{E}(0) = E(0) = E_0^N = E_0$. Suppose that $E_k^N \subset \widehat{E}(kh)$ for some $k < N$. Then from the invariance of $\widehat{E}(\cdot)$ we have $x[kh, E_k^N](h) \subset \widehat{E}((k+1)h)$. Since $E_{k+1}^N \in \mathcal{M}(x[kh, E_k^N](h))$, from the uniqueness assumption we get $E_{k+1}^N \subset \widehat{E}((k+1)h)$. Thus the last inclusion holds for all k. Since $\widehat{E}(\cdot)$ has closed graph, one easily obtains that $E(t) \subset \widehat{E}(t)$ for all $t \in [0, T]$. From the minimality of $\widehat{E}(\cdot)$ we conclude that $E(\cdot) = \widehat{E}(\cdot)$ and the proof is complete. ■

In general, neither uniqueness of the solution tube in a given collection of sets nor Lipschitz continuity hold in the case of multivalued $\mathcal{M}(Z)$.

3. Lipschitz stability of the solution tubes

The usefulness of the concept of solution tubes in a collection of sets critically the so-called quasi-Lipschitz stability (or Aubin property) of the solution tubes with respect to the initial state and the right-hand side.

It might be guessed that (at least in the case of a single-valued $\mathcal{M}(Z)$) Lipschtz dependence of $\mathcal{M}(Z)$ on Z may imply Lipschitz dependence of the solution tube on the initial data. In general, however, this can be proven only if the Lipschitz constant of $\mathcal{M}(\cdot)$ is not greater than one, which is a too strong requirement. It does not hold, for example, for the collection of all boxes $\mathcal{E}_4 = \{[a,b] \times [c,d]\}$ in $\mathbf{R}^2$, if the Euclidean norm is taken. If the Hausdorff distance is taken with respect to the max-norm, however, the Lipschitz constant of $\mathcal{M}(\cdot)$ turns out to be equal to one. That is, the max-norm is in a sense adapted to the collection $\mathcal{E}_4$. In the next theorem we extend this observation using adapted 'pseudo-norms'. Collections with non-single valued $\mathcal{M}(\cdot)$ are included, but we restrict the consideration to collections of convex sets.

Let us denote $I = \{1, 2, \ldots, \sigma\}$, where σ can be also $+\infty$, in which case I consists of all natural numbers. Let P be a subset of $(\partial\mathcal{B})^\sigma \times \mathbf{R}^\sigma$ consisting of elements $(l;q) = (l_1, \ldots, l_\sigma; q_1, \ldots, q_\sigma)$, $l_i \in \mathbf{R}^n$, $|l_i| = 1$. With every $p = (l;q) \in P$ we associate the set

$$\mathbf{E}(p) = \{x \in \mathbf{R}^n : \sup_{i \in I} \langle l_i, x\rangle \leq q_i\}.$$

Further in this section we consider only collections that have the parametric representation

$$\mathcal{E} = \{\mathbf{E}(p) : p \in P\}.$$

The following conditions will be supposed.

B.2.a. The set P is nonempty and closed, $\mathbf{E}(p)$ is nonempty and compact for every $p = (l;q) \in P$, and

$$\sup_{e \in \mathbf{E}(p)} \langle l_i, e\rangle = q_i \quad \forall i \in I.$$

B.2.b. For every $p = (l;q) \in P$ and for every $\varepsilon > 0$, it holds that $(l; q_1 + \varepsilon, \ldots, q_\sigma + \varepsilon) \in P$.

Let us introduce the following notation for $p = (l;q) \in P$ and $x \in \mathbf{R}^n \setminus \mathbf{E}(p)$:

$$\rho(E,x) = \sup_{i \in I}\{\langle l_i, x\rangle - q_i\}.$$

B.2.c. There is a number β with the following property: for every $E = \mathbf{E}(l;q) \in \mathcal{E}$, for every $x \in \mathbf{R}^n \setminus E$ and for every $i \in I$ there exists $e_i \in E$ such that

$$\langle l_i, x - e_i\rangle \leq \rho(E,x) \quad \& \quad |x - e_i| \leq \beta\rho(E,x).$$

Notice that the above conditions do not imply single-valuedness of $\mathcal{M}(Z)$. Moreover, the mapping $p \longrightarrow \mathbf{E}(p)$ need not be one-to-one: $\mathbf{E}(p_1) = \mathbf{E}(p_2)$ does not imply $p_1 = p_2$. This is essential for some of the particular cases considered

LEMMA 3.1 *Conditions* **B.2** *imply* **B.1** *(with $L_{\mathcal{E}} = \beta$).*

Proof. Condition **B.1.a** easily follows from **B.2.a** and **B.2.b**. Now take $E = \mathbf{E}(l;q) \in \mathcal{E}$ and $\varepsilon > 0$. Define $E' = \mathbf{E}(l;q+\varepsilon)$ (ε is added to each component of q). According to **B.2.b** we have $(p;q+\varepsilon) \in P$. Obviously $E + \varepsilon\mathcal{B} \subset E'$. Moreover, for every $x \in E'$ we have $\rho(E,x) \leq \varepsilon$. Then, from **B.2.c**, we have, in particular, $\mathrm{dist}(x,E) \leq \beta\rho(E,x) \leq \beta\varepsilon$. ∎

Along with (1) we consider a second differential inclusion

$$\dot{x} \in \widetilde{F}(x,t), \quad x(0) \in \widetilde{E}_0 \in \mathcal{E}. \tag{11}$$

THEOREM 3.1 *Let $t \longrightarrow E(t) = \mathbf{E}(l(t);q(t))$ be a solution tube in $\mathcal{E}$ of (1) starting from the set E_0. Let Z be a compact set such that for some $\gamma > 0$ the set $E(t) + \gamma\mathcal{B}$ is contained in Z for every $t \in [0,T]$. Suppose that F and $\widetilde{F}$ satisfy conditions* **A** *with a Lipschitz constant L, and that $\mathcal{E}$ satisfies conditions* **B.2**. *Suppose also that the mapping $t \longrightarrow l(t)$ is Lipschitz continuous with a Lipschitz constant L_E.*

Denote $C = \beta(L_E + L)$, $\delta_0 = e(E_0, \widetilde{E}_0)$, and

$$\delta(t) = \sup_{x\in Z} e(F(x,t), \widetilde{F}(x,t)).$$

Suppose, in addition that

$$\Delta(T) \stackrel{\text{def}}{=} \beta\Big[e^{CT}\delta_0 + \int_0^T e^{C(T-s)}\delta(s)\,ds\Big] < \gamma. \tag{12}$$

Then, there exists a solution tube $\widetilde{E}(\cdot)$ in $\mathcal{E}$ of the inclusion (11) such that

$$e(E(t), \widetilde{E}(t)) \leq \Delta(t) \quad \forall t \in [0,T]. \tag{13}$$

Proof. In what follows, notation $\widetilde{x}[s,E](t)$ has the same meaning as $x[s,E](t)$, but for the differential inclusion (11).

Take a natural number N and denote $h = T/N$, $t_k = kh$, $p_k = (l^k;q^k) = p(t_k)$, $E_k = E(t_k) = \mathbf{E}(p_k)$.

For the given $\widetilde{E}_0$ we shall define a sequence $\{\widetilde{E}_k\}_{k=0}^N$ as follows. Suppose that the parameter $\widetilde{p}_k$ is already defined, and that $\widetilde{E}_k \stackrel{\text{def}}{=} \mathbf{E}(\widetilde{p}_k) \subset Z$. Denote

$$\rho_k = \rho(E_k, \widetilde{E}_k) \stackrel{\text{def}}{=} \sup_{\widetilde{e}\in\widetilde{E}_k} \rho(E_k, \widetilde{e}).$$

For arbitrary $i \in I$ and $\widetilde{x} \in \widetilde{x}[t_k, \widetilde{E}_k](h)$ we shall estimate the value $\langle l_i^{k+1}, \widetilde{x}\rangle - q_i^{k+1}$. By a standard argument, for any $\widetilde{x} \in \widetilde{x}[t_k, \widetilde{E}_k](h)$ there exist $\widetilde{e} \in \widetilde{E}_k$ and

$$\widetilde{y} \in \frac{1}{h}\int_{t_k}^{t_{k+1}} \widetilde{F}(s,\widetilde{e})\,ds$$

such that

$$|\widetilde{x} - \widetilde{e} - h\widetilde{y}| \le \frac{LM}{2}h^2,$$

where M is a bound of F and $\widetilde{F}$ on $Z \times [0,T]$.

According to **B.2.c** for the given $i \in I$ there is $e \in E_k$ such that

$$\langle l_i^k, \widetilde{e} - e\rangle \le \rho_k \quad \text{and} \quad |\widetilde{e} - e| \le \beta\rho_k \tag{14}$$

(if $\widetilde{e} \in E_k$ we take $e = \widetilde{e}$). Moreover, there exists

$$y \in \frac{1}{h}\int_{t_k}^{t_{k+1}} F(s,e)\,ds$$

such that

$$|\widetilde{y} - y| \le \delta_k + L|\widetilde{e} - e| \le \delta_k + \beta L\rho_k,$$

where

$$\delta_k = \frac{1}{h}\int_{t_k}^{t_{k+1}} \delta(s)\,ds.$$

Then, there exists $x \in x[t_k, E_k](h)$ such that

$$|x - e - hy| \le \frac{LM}{2}h^2.$$

From the above relations we obtain

$$\begin{aligned}
&\langle l_i^{k+1}, \widetilde{x}\rangle - q_i^{k+1} \le \langle l_i^{k+1}, \widetilde{e} + h\widetilde{y}\rangle - q_i^{k+1} + \frac{LM}{2}h^2 \\
&\le \langle l_i^{k+1}, e + hy\rangle - q_i^{k+1} + \langle l_i^{k+1}, \widetilde{e} - e + h(\widetilde{y} - y)\rangle + \frac{LM}{2}h^2 \\
&\le \langle l_i^{k+1}, x\rangle - q_i^{k+1} + \langle l_i^{k+1}, \widetilde{e} - e + h(\widetilde{y} - y)\rangle + LMh^2 \\
&= \langle l_i^k, \widetilde{e} - e\rangle + \langle l_i^{k+1} - l_i^k, \widetilde{e} - e\rangle + h\langle l_i^{k+1}, \widetilde{y} - y\rangle + LMh^2 \\
&\le \rho_k + \beta L_E h\rho_k + h(\delta_k + \beta L\rho_k) + LMh^2 \\
&= (1 + Ch)\rho_k + \delta_k h + LMh^2.
\end{aligned}$$

In the above equalities we use (14) and the inequality (since $x \in E_{k+1}$)

$$\langle l_i^{k+1}, x\rangle - q_i^{k+1} \le 0.$$

Now we define $\widetilde{E}_{k+1} = \mathbf{E}(\widetilde{l}; \widetilde{q})$ with

$$\widetilde{l} = l^{k+1}, \quad \widetilde{q}_i = q_i^{k+1} + (1 + Ch)\rho_k + h\delta_k + LMh^2.$$

Clearly, $(\widetilde{l}; \widetilde{q}) \in P$, according to **B.2.b**. Moreover, $\widetilde{E}_{k+1}$ contains $\widetilde{x}[t_k, \widetilde{E}_k](h)$, and

From this recurrent relation we estimate in a standard way

$$\rho_m \leq e^{Ct_m}\delta_0 + h\sum_{i=1}^{m} e^{Ct_i}\delta_i + LMe^{Ct_m}h, \quad m = 1, \ldots, k+1.$$

Then, applying Lemma 3.1 we obtain

$$e(E_m, \widetilde{E}_m) \leq \beta\Big[e^{Ct_m}\delta_0 + h\sum_{i=1}^{m} e^{Ct_i}\delta_i\Big] + O(h), \quad m = 1, \ldots, k+1. \quad (15)$$

From (12) (supposing in addition that $O(h)$ is smaller than the difference between γ and $\Delta(T)$) we conclude that the condition $\widetilde{E}_{k+1} \subset Z$ is still fulfilled and we may continue the same procedure till $k = N$. In such a way we have constructed a discrete tube $\widetilde{E}_0, \ldots, \widetilde{E}_N$, which is invariant (in the sense that $\widetilde{x}[t_k, \widetilde{E}_k](h) \subset \widetilde{E}_{k+1}$) and (15) holds for $s = 0, \ldots, N$. Then we apply part 3 from the proof of Theorem 2.1, utilizing (15) instead of (5)–(7) to ensure uniform boundedness. For the limit tube $\widetilde{E}'(\cdot)$ we obtain from (15) that (13) holds. Then we repeat part 4 of the proof of Theorem 2.1 to obtain a solution tube $\widetilde{E}(\cdot)$ of (11), for which (13) still holds, since $\widetilde{E}(t) \subset \widetilde{E}'(t)$. ■

COROLLARY 3.1 *Let the conditions of Theorem 3.1 be fulfilled and let, in addition, $\mathcal{M}(Z)$ be a singleton for every $Z \in \text{comp}(\mathbf{R}^n)$. Suppose, moreover, that the l-part of the inverse mapping $E \to \mathbf{E}^{-1}(E)$ has a Lipschitz selection $E \to l(E)$. Then the (unique) solution tube in $\mathcal{E}$ of (1) depends Lipschitz continuously on the initial set and on the right-hand side of the inclusion.*

Proof. It is enough to notice that the solution tube $E(\cdot)$ is Lipschitz continuous, according to Theorem 2.1. Then one can select a Lipschitz continuous parametrization $l(t)$ and apply Theorem 3.1. ■

4. Particular cases and examples

Every collection of sets $\mathcal{E}$ can be represented as consisting of sub-level sets of a parametric family of Lipschitz functions. Namely, if we define

$$\varphi(E, x) = \text{dist}\,(x, E), \quad E \in \mathcal{E}, \quad x \in \mathbf{R}^n,$$

and denote $P = \mathcal{E}$, then obviously

$$E \in \mathcal{E} \;\Leftrightarrow\; \exists p \in P:\; E = E(p) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^n:\; \varphi(p, x) \leq 0\}. \quad (16)$$

For practical reasons, however, we are mainly interested in collections $\mathcal{E}$ that admit a parameterization as in (16) with a set P being a subset of a finite dimensional space. Conditions **B.2** restrict the considerations to such collections

The following particular case is especially convenient and easy for calculation. Let us fix a finite or countable subset $L = \{l_1, l_2, \ldots\}$ of the unit sphere $\partial\mathcal{B} \subset \mathbf{R}^n$, such that the convex cone spanned by L coincides with $\mathbf{R}^n$. With every $Z \in \mathrm{comp}(\mathbf{R}^n)$ we associate the sequence of numbers $q(Z) = (q_1, q_2, \ldots)$, where $q_i = \max_{z \in Z} \langle l_i, z \rangle$. Denote $Q \stackrel{\mathrm{def}}{=} \{q(Z) : \ Z \in \mathrm{comp}\,(\mathbf{R}^n)\}$. Then we define $P = \{(l_1, l_2, \ldots; q) : \ q \in Q\}$.

It is easy to check that the corresponding collection $\mathcal{E}_L \stackrel{\mathrm{def}}{=} \{\mathbf{E}(p) : \ p \in P\}$ satisfies conditions **B.2**[4]. Moreover, the set of minimal elements $\mathcal{M}(Z)$ is a singleton for every $Z \in \mathrm{comp}(\mathbf{R}^n)$, and the assumptions of Corollary 3.1 are fulfilled since the l-part of $\mathbf{E}^{-1}(E)$ is the constant $(l_1, \ldots, l_\sigma)$. The collection $\mathcal{E}_L$ may consist of all convex compact subsets of $\mathbf{R}^n$ (if L is dense in $\partial\mathcal{B}$), of all "boxes" (if $L = \{\pm e_i\}_{i=1}^n$ with $\{e_i\}_i$ – an orthogonal basis in $\mathbf{R}^n$), or of other classes of polyhedral sets. The set Q can be described by a system of linear inequalities, which are finitely many if L is finite. For example, in the case of boxes, if q_i' and q_i'' correspond to e_i and $-e_i$, respectively, then $Q = \{(q_1', q_1'', \ldots, q_{n_2}', q_{n_2}'') : \ q_i' + q_i'' \geq 0\}$.

We illustrate the solution tubes in the collections $\mathcal{E}_k \stackrel{\mathrm{def}}{=} \mathcal{E}_L$ with the set L consisting of k vectors uniformly situated on $\partial\mathcal{B}$. The numerical solution is obtained by implementing the construction in the proof of Theorem 2.1, with the only difference that a high order discrete (local) approximation is used instead of the exact $x[t, E](h)$. The convergence to a solution tube is implied by the proof of the theorem, since for the collections $\mathcal{E}_k$ the minimal set $\mathcal{M}(Z)$ is a singleton for every Z.

Fig. 1 present two solution tubes in $\mathcal{E}_{16}$ of the Bazykin prey-predator model (Bazykin, 1985)

$$\dot{x}_1 = x_1 - \frac{x_1 x_2}{1 + 0.3x_1} - 0.01(x_1)^2, \quad x_1(0) = 1,$$
$$\dot{x}_2 = -x_2 + \frac{x_1 x_2}{1 + 0.3x_1} - \delta(t)(x_2)^2, \quad x_2(0) = 2.$$

Here the predator's competition rate $\delta(t)$ is taken as a uncertain function of time which can deviate up to 10% from some base value $\widehat{\delta}$. It is known that $\widehat{\delta} = 0.228$ is a Hopf point of the system. For fixed values of δ close to, but smaller than 0.228 the system has a stable limit cycle bifurcating from an equilibrium point which becomes unstable. The qualitative difference can be seen also for the system with uncertain $\delta = \delta(t) \in [0.9\widehat{\delta}, 1.1\widehat{\delta}]$. Fig. 1 (left) corresponds to value $\widehat{\delta} = 0.228$ and the solution tube in $\mathcal{E}_{16}$ has a periodic selection. In Fig. 1 (right) we have taken $\widehat{\delta} = 0.3$. Even if the uncertain system has a periodic solution, the period must be much bigger than that of the unperturbed system

[4]Notice that in this case the mapping $p \to E(p)$ is one-to-one, which is not required, in general. This, however, allows in some cases to pass from evolution equations for sets to those in the parametric space P (see Quincampoix and Veliov, 2002, for a discrete-time consideration

with $\widehat{\delta} = 0.228$.

Fig. 2 (left) shows sections of the solution tubes in $\mathcal{E}_4$, $\mathcal{E}_6$, $\mathcal{E}_{12}$ and $\mathcal{E}_{24}$ of a nonlinear pendulum, where the mass can deviate in a uncertain way up to 10% from its base value. Notice that the solution in $\mathcal{E}_6$ is not always contained in that in $\mathcal{E}_4$.

Fig. 2 (right) presents several sections of four different ball-valued solution tubes (that is, solutions in the collection consisting of all balls in $\mathbf{R}^2$). The intersection ot the four tubes is an invariant tube in the collection consisting of all intersections of up to four balls, but it is not a solution in this collection since it is not minimal. The solution tube in $\mathcal{E}_{20}$ is plotted for comparison.

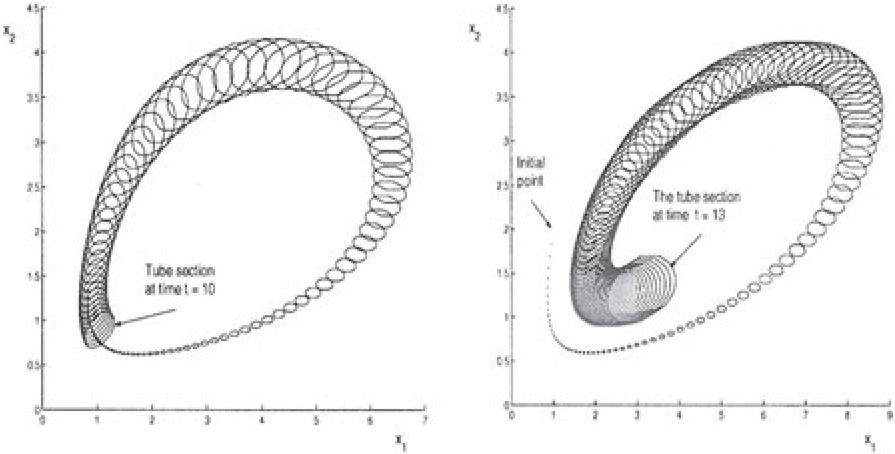

Figure 1. Solution tubes in $\mathcal{E}_{16}$ of the Bazykin system with uncertainty.

As another example we consider the compact polygons in $\mathbf{R}^2$ with not more than m vertices (including degenerate ones, in order to ensure closedness). For $m = 3$ this collection does not satisfy conditions **B.2** (neither **B.1.b**). Indeed, the closer is a triangular to a unit segment, the bigger would be the constant $L_{\mathcal{E}}$ in **B.1.b**. The Lipschitz stability may fail, as in the following example: take $F(x) = \{(0,0)\}$, $\widetilde{F}(x) = \{0\} \times [-\delta, \delta]$, $E_0 = \widetilde{E}_0 = [-1,1] \times \{0\}$, $T = 1$. Here $E(t) = E_0$ is a solution tube in $\mathcal{E}$ of the nonperturbed equation. On the other hand it is fairly easy to estimate that for every solution tube $\widetilde{E}(\cdot)$ of the δ-perturbed inclusion in the collection $\mathcal{E}$ there is

$$e(E(1), \widetilde{E}(1)) \geq \sqrt{\delta},$$

that is, the Lipschitz stability does not take place.

If we exclude from the above collection $\mathcal{E}$ the polygons, which are close to degenerate, however, conditions **B.2** would be satisfied. More precisely, we

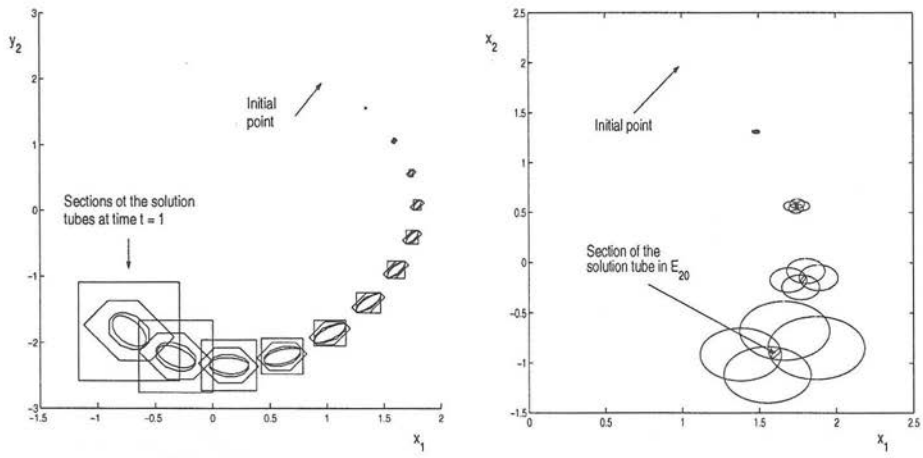

Figure 2. Solution tubes in different collections of the nonlinear pendulum with uncertain mass.

generated by l_1, l_2, l_3, coincides with $\mathbf{R}^2$ and, in addition, $\langle l_i, l_j \rangle \geq -\alpha > -1$, where α is a constant. Then, condition **B.2** is fulfilled with $\beta = \sqrt{2/(1-\alpha)}$.

The existence conditions **B.1** are satisfied also for the collection of all (not necessarily convex) box complexes or simplicial complexes.

Finally we give an example where a solution tube $E(\cdot)$ starting from a set $E_0 \in \mathcal{E}$ exists, for which E_0 is a proper subset of $E(0)$. This happens for the collections of all triangles in $\mathbf{R}^2$ with fixed angles $\pi/2$, $\pi/4$, $\pi/4$ (this collection satisfying even conditions **B.2**). Take as E_0 the triangle $((0,-1),(0,1),(1,0))$. Let the right-hand side of the differential inclusion be just the unit ball. Then it is clear that a minimal solution tube can be obtained from the initial triangle by translating the vertical side to the left with speed 1 and moving the opposite vertex to the right with speed $\sqrt{2}$. Formally, $E(t) = ((-t,-c(t)),(-t,c(t)),(1+\sqrt{2}t,0))$, where $c(t) = 1 + t + \sqrt{2}t$. There is, however, another solution tube, which informally speaking, jumps at the very beginning from E_0 to $\bar{E} = ((-1,0),(1,-2),(1,2))$. Then $\bar{E}$ develops similarly as $E(\cdot)$, but the vertical side translates to the right with speed 1, while the opposite vertex moves to the left with speed $\sqrt{2}$. Formally, $\bar{E}(t) = ((-(1+\sqrt{2}t),0),(1+t,-d(t)),(1+t,d(t)))$, where $d(t) = 2 + t + \sqrt{2}t$. Obviously this is an invariant tube, but it is also a minimal one containing E_0 at $t = 0$. Indeed, this tube solves the minimization problem

$$\min_{E(\cdot)} \max_{(x_1,x_2)\in E(1)} x_1,$$

among all invariant tubes starting from E_0 in the sense of Definition 1, therefore it must be also minimal, that is, a solution tube.

References

ARTSTEIN, Z. (1989) Piecewise linear approximations of set-valued maps. *J. Approx. Theory*, **56** (1), 41–47.

AUBIN, J.-P. (1993) Mutational equations in metric spaces *Set-Valued Analysis*, **1** (1), 3–46.

AUBIN, J.-P. (1999) *Mutational and Morphological Analysis. Tools for Shape Evolution and Morphogenesis.* Systems & Control: Foundations & Applications. Birkhäuser, Boston.

AUBIN, J.-P. and FRANKOWSKA, H. (1990) *Set-valued Analysis.* Birkhäuser, Boston, Basel, Berlin.

BAZYKIN, A. (1985) *Mathematical Biophysics of Interacting Populations.* Nauka, Moscow. (in Russian).

CARDALIAGUET, P., QUINCAMPOIX, M. and SAINT-PIERRE, P. (1999) Set-valued numerical analysis for optimal control and differential games. *Stochastic and differential games*, 177–247, Ann. Internat. Soc. Dynam. Games, **4**, Birkhäuser, Boston.

CHERNOUSKO, F.L. (1998) *Estimation of the Phase State of Dynamical Systems.* Nauka, Moscow. (in Russian).

CHERNOUSKO, F.L and ROKITYANSKII, D.YA. (2000) Ellipsoidal bounds on reachable sets of dynamical systems with matrices subjected to uncertain perturbations. *J. Optim. Theory Appl.*, **104** (1), 1–19.

CLARKE, F. (1983) *Optimization and Nonsmooth Analysis.* John Wiley & Sons, New York.

DONTCHEV, A.L. and LEMPIO, F. (1992) Difference methods for differential inclusions: A survey. *SIAM Review*, **34** (2), 263–294.

DOYEN, L. (1993) Filippov and invariance theorems for mutational equations for tubes. *Set-Valued Analysis*, **1**, 289–303.

HÄCKL, G. (1992-93) Numerical approximation of reachable sets and control sets. *Random & Computational Dynamics*, **1** (4), 371–394.

KURZHANSKI, A.B. and FILIPPOVA, T.F. (1993) On the theory of trajectory tubes – a mathematical formalism for uncertain dynamics, viability and control. In *Advances in Nonlinear Dynamics and Control: a Report from Russia*; **17**, *Progress in Systems and Control Theory*, 122–188. Birkhäuser, Boston.

KURZHANSKI, A.B. and VÁLYI, I. (1997) *Ellipsoidal Calculus for Estimation and Control.* Birkhäuser, Boston.

KURZHANSKI, A.B. and VARAIYA, P. (2000) Ellipsoidal techniques for reachability analysis: internal approximation. *Systems & Control Letters*, **41**, 201–211.

LEMPIO, F. and VELIOV, V.M. (1998) Discrete approximations to differential inclusions, *Mitteilungen der GAMM,* **21** (2), 101–135.

PANASYUK, A. (1990) Equations for attainable set dynamics, Part 1: integral funnel equations. *J. Optim. Theory and Appl.*, **64** (2), 349–366.

QUINCAMPOIX, M. and VELIOV, V.M. (1999) Open-loop viable control under uncertain initial state information. *Set-Valued Analysis*, **7** (1), 55–87.

QUINCAMPOIX, M. and VELIOV, V.M. (2002) Optimal Control of Discrete-Time Uncertain Systems with Imperfect Measurement. *IEEE Trans. Automat. Contr* (to appear).

SAINT-PIERRE, P. (1994) Approximation of the viability kernel. *Applied Mathematics and Optimization*, **29**, 187–209.

USHAKOV, V. and KHRIPUNOV, A. (1994) On the approximation of the integral funnel of a differential inclusion. *J. Vych. Mat. Math. Phis.*, **34** (7), 965–977.