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A silent duel with two kinds of weapon

by

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Abstract: In this paper the silent duel is solved in which Player I has two kinds of weapon, the first with one bullet which he can use when he wants, and the second which he can use only when the distance between players is zero. In distinction to the paper of Trybuła (1998), it is assumed here that the probability of success of the second weapon can be smaller than one. Player II has a weapon with one bullet which he can use when he wants.

Keywords: game of timing, silent duel.

1. Introduction

Consider a game which will be called the $(1, 1)$ game. Two Players, I and II, fight a duel. They move toward each other. They have one bullet each, and this fact is known to both of them. It is also known that Player I has a second weapon which he can use when the distance between the players is zero. The second weapon destroys the opponent with probability p, $0 \leq p \leq 1$. The duel is silent: at a given moment none of the players knows whether or nor his opponent has fired (and missed).

At the beginning of the duel the players are at the distance of 1 from each other. Let $P(s)$ be the probability of succeeding (destroying the opponent) using the first weapon when the distance between players is $1 - s$. The function $P(s)$ will be called the accuracy function. It is the same for both players. It is assumed that the function $P(s)$ is increasing and continuous in $[0, 1]$, has continuous first derivative in $(0, 1)$ and that $P(s) = 0$ for $s \leq 0$, $P(1) = 1$. Time is taken to be equal to s.

A player gains 1 if only he succeeds, "gains" -1 if only his opponent succeeds, and gains 0 in the remaining cases. Thus, the duel is a zero-sum game.

The noisy duel with two kinds of weapon and arbitrary moving was consid-

2. Expected gain

Denote by $K(s,t)$ the expected gain of Player I if he fires his first weapon at time $s \in [0,1]$ and if Player II fires at time $t \in [0,1]$. The function $K(s,t)$ is called the payoff function. We obtain

$$K(s,t) = \begin{cases} P(s) - (1-P(s))P(t) + p(1-P(s))(1-P(t)) & \text{if } s < t < 1, \\ p(1-P(s))^2 & \text{if } s = t, \\ -P(t) + (1-P(t))P(s) + p(1-P(s))(1-P(t)) & \text{if } t < s, \\ P(s) - (1-p)(1-P(s)) & \text{if } s < t = 1. \end{cases} \tag{1}$$

Denote by ξ_a the strategy of Player I in the (1,1) game in which he fires his first weapon at a random time s distributed according to the density

$$f_1(s) = \frac{CP'(s)}{P^3(s)} \tag{2}$$

in the interval $[a,1]$, $0 < a < 1$.

From (2) we obtain

$$\int_a^1 f_1(s)ds. = \frac{C}{2}\left(\frac{1}{P^2(a)} - 1\right) = 1. \tag{3}$$

Similarly, denote by η_a the "strategy" of Player II in the (1,1) game in which he fires at a random time t according to the density

$$f_2(t) = \frac{DP'(t)}{P^3(t)} \tag{4}$$

in the interval $[a,1]$, $0 < a < 1$, and according to the probability q, $0 \le q < 1$, at the "time" $1-$, where $K(s,1-) \stackrel{def}{=} \lim_{\epsilon \to 0+} K(s, 1-\epsilon)$.

From (4) we obtain

$$\int_a^1 f_2(t)dt = \frac{D}{2}\left(\frac{1}{P^2(a)} - 1\right) = 1 - q. \tag{5}$$

Moreover, taking into account the formula (1) we obtain for $t \in [a,1)$

$$K(\xi_a, t) = \int_a^1 K(s,t) f_1(s) ds =$$

$$= \int_a^t (P(s) - (1-P(s))P(t) + p(1-P(s))(1-P(t))) f_1(s) ds +$$

$$+ \int_t^1 (-P(t) + (1-P(t))P(s) + p(1-P(s))(1-P(t))) f_1(s) ds =$$

$$= P(t)C\left(\frac{3}{2} + \frac{1}{P(a)} - \frac{1}{2P^2(a)} - p\left(\frac{1}{2} - \frac{1}{P(a)} + \frac{1}{2P^2(a)}\right)\right) +$$

If

$$\frac{3}{2}+\frac{1}{P(a)}-\frac{1}{2P^2(a)}-p\left(\frac{1}{2}-\frac{1}{P(a)}+\frac{1}{2P^2(a)}\right)=0 \tag{6}$$

then

$$v_1=C\left(\frac{1}{P(a)}-3+p\left(\frac{1}{2}-\frac{1}{P(a)}+\frac{1}{2P^2(a)}\right)\right). \tag{7}$$

Similarly

$$\begin{aligned}
&K(s,\eta_a)=\\
&=\int_a^s(-P(t)+(1-P(t))P(s)+p(1-P(s))(1-P(t))),f_2(t)dt+\\
&\int_s^1(P(s)-(1-P(s))P(t)+p(1-P(s))(1-P(t)))f_2(t)dt+q(2P(s)-1)\\
&=P(s)D\left(\frac{1}{2P^2(a)}-\frac{1}{P(a)}-\frac{3}{2}-p\left(\frac{1}{2P^2(a)}-\frac{1}{P(a)}+\frac{1}{2}\right)\right)+\\
&+D\left(3-\frac{1}{P(a)}+p\left(\frac{1}{2P^2(a)}-\frac{1}{P(a)}+\frac{1}{2}\right)\right)+q(2P(s)-1)=v_2=\text{const.}
\end{aligned}$$

If

$$D\left(\frac{1}{2P^2(a)}-\frac{1}{P(a)}-\frac{3}{2}-p\left(\frac{1}{2P^2(a)}-\frac{1}{P(a)}+\frac{1}{2}\right)\right)+2q=0 \tag{8}$$

then

$$v_2=D\left(3-\frac{1}{P(a)}+p\left(\frac{1}{2P^2(a)}-\frac{1}{P(a)}+\frac{1}{2}\right)\right)-q. \tag{9}$$

Solving the system of equations given by (3), and (5)-(9) with respect to a, C, D, q, v_1, and v_2, we obtain

$$P(a)=\frac{1}{3-p}\left(2\sqrt{1+p}-1-p\right), \tag{10}$$

$$C=\frac{1+p}{2\left(\sqrt{1+p}+1\right)}, \tag{11}$$

$$D=\frac{\sqrt{1+p}}{2\left(\sqrt{1+p}+1\right)}, \tag{12}$$

$$q=\frac{p}{1+p+\sqrt{1+p}}, \tag{13}$$

$$v_1=v_2=\frac{p}{2+p+2\sqrt{1+p}}\stackrel{def}{=}v. \tag{14}$$

3. Proof of optimality

Let η_a^ϵ be a strategy according to which Player II fires with probability density $f_2(t)$ in the interval $[a, 1-\epsilon_0]$ and with probability $q' = 1 - \int_a^{1-\epsilon_0} f_2(t)dt$ at time $1-\epsilon_0$, where ϵ_0 is a positive small number. The strategy η_a^ϵ is an approximation of the "strategy" η_a. We shall prove that *the game (1,1) has a value and, for constants a, C, D and q given by (10)-(13), the strategy ξ_a is maximin and the strategy η_a^ϵ is ϵ-minimax.*

Suppose that Player II fires at time t, $a \leq t < 1$. From Section 2 it follows that

$$K(\xi_a, t) = v.$$

Suppose that Player II fires at time t, $0 \leq t < a$. We have

$$K(\xi_a, t) = \int_a^1 (-P(t) + (1-P(t))P(s) + p(1-P(s))(1-P(t)))f_1(s)ds \geq$$

$$\geq \int_a^1 (-P(a) + (1-P(a))P(s) + p(1-P(s))(1-P(a)))f_1(s)ds = v.$$

Suppose that Player II fires at time $t = 1$. In this case

$$K(\xi_a, t) = \int_a^1 (P(s) - (1-p)(1-P(s)))f_1(s)ds =$$

$$= \frac{C}{2}\left(\frac{1}{P(a)} - 1\right)\left(3 - p - (1-p)\frac{1}{P(a)}\right) \geq$$

$$\geq C\left(\frac{1}{P(a)} - 3 + p\left(\frac{1}{2} - \frac{1}{P(a)} + \frac{1}{2P^2(a)}\right)\right) = v$$

since after multiplying both sides of the above inequality by $\frac{2P^2(a)}{C}$ we obtain

$$3P^2(a) + 2P(a) - 1 \geq 0 \tag{15}$$

which is always satisfied because $P(a)$ is an increasing function of the parameter p (see (10)) and $P(a) = \frac{1}{3}$ when $p = 0$.

On the other hand, suppose that Player I fires at time s, where $a \leq s < 1-\epsilon_0$. In this case

$$K(s, \eta_a) \leq v + \epsilon_1$$

for each $\epsilon_1 > \bar{\epsilon} > 0$. The constant $\bar{\epsilon}$ is chosen with respect to ϵ_0 and tends to 0 if $\epsilon_0 \to 0$.

Suppose that Player I fires at time s, $0 \leq s < a$. We have

$$K(s, \eta_a^\epsilon) \leq \int_a^1 (P(s) - (1-P(s))P(t) + p(1-P(s))(1-P(t)))f_2(t)dt +$$

$$\int_a^1 (-P(t)+p(1-P(t))+P(s)(1+P(t)-p(1-P(t)))f_2(t)dt+$$
$$+q(2P(s)-1)+\epsilon_2 \leq$$
$$\int_a^1 (-P(t)+p(1-P(t))+P(a)(1+P(t)-p(1-P(t)))f_2(t)dt+$$
$$+q(2P(a)-1)+\epsilon_2 = v+\epsilon_2$$

for each $\epsilon_2 > \bar{\epsilon} > 0$.

Suppose that Player I fires at time $1-\epsilon_0$. We obtain

$$K(1-\epsilon_0,\eta_a^\epsilon) \leq \int_a^1 (1-2P(t))f_2(t)dt+\epsilon_3 \overset{(4),(5)}{=}$$
$$= 1-q-2D\left(\frac{1}{P(a)}-1\right)+\epsilon_3 \overset{(5)}{=}$$
$$\frac{D}{2P^2(a)}(1-3P(a))(1-P(a))+\epsilon_3 \leq v+\epsilon_3 \tag{16}$$

for each $\epsilon_3 > \bar{\epsilon} > 0$. The second inequality in (16) follows from the fact that for each $0 \leq p \leq 1$ we have $P(a) \geq \frac{1}{3}$ and $v \geq 0$.

Finally, suppose that Player I fires at time $1-\epsilon'$ where $\epsilon_0 > \epsilon' \geq 0$. We get

$$K(1-\epsilon',\eta_a^\epsilon) \leq \int_a^1 (1-2P(t))f_2(t)dt - q+\epsilon_4 \overset{(16)}{<} v+\epsilon_4$$

for each $\epsilon_4 > \bar{\epsilon} > 0$.

Assume that $\epsilon = \max(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)$. From the above inequalities and the fact that $\bar{\epsilon} \to 0$ when $\epsilon_0 \to 0$ it follows that the game $(1,1)$ has a value and for a, C, D and q given by (10)-(13), the strategy ξ_a is maximin and the strategy η_a^ϵ is ϵ-minimax.

Silent duels are considered in Cegielski (1986A, B), Karlin (1959), Kimeldorf (1983), Radzik and Orłowski (1985A, B), Restrepo (1957), Teraoka (1979), Trybuła (1992, 1998).

For other duels, see Fox and Kimeldorf (1969), Kimeldorf (1983), Styszyński (1974), Trybuła (1990-1995, 1993-1995).

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