Dedicated to
Professor Jakub Gutenbaum
on his 70th birthday

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Second order sufficient conditions and sensitivity analysis for optimal multiprocess control problems

by

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Abstract: Second order sufficient optimality conditions (SSC) are derived for optimal multiprocess control problems. For that purpose the multiprocess control problem is transformed into a single stage control problem with augmented state variables which comprise the state variables of all individual stages as well as the switching times as choice variables. This transformation allows to apply the known SSC for single stage control problems. A numerical test of SSC involves the solution of an associated Riccati equation together with boundary conditions adapted to the multiprocess. Sensitivity analysis of parametric multiprocess problems can be based on SSC. A numerical example of the optimal two–stage control of a robot illustrates both SSC and sensitivity analysis.

Keywords: multiprocess control systems, second order sufficient conditions, Riccati equations, sensitivity analysis, robot control.

1. Introduction

Multiprocess control problems and their great potential for practical applications were first drawn to the attention of the control community through the pioneering work of Gutenbaum (1977, 1979, 1988, 1996), Clarke and Vinter (1989A, B), Tomiyama (1985), Tomiyama and Rossana (1989). We adopt the definition of Clarke and Vinter (1989B) that "optimal multiprocess problems are dynamic optimization problems involving a collection of control systems, coupled through constraints in the endpoints of the constituent state trajectories and the cost function". A unified theory of necessary optimality conditions for a very general optimal multiprocess control problem has been developed in Clarke and Vinter (1989B) using techniques from Nonsmooth Analysis.

Optimization algorithms for control problems are usually based on necessary conditions. Clarke and Vinter (1989B) discuss a number of applications to illustrate the use of necessary conditions, in particular those conditions that arise from the concatenation of stages. However, numerical examples are mostly restricted to cases where optimal solutions can be computed explicitly. Gutenbaum (1996) describes a general methodology for solving multiprocess control problems using the dynamic programming principle. Solution methods for solving the boundary value problem associated with the maximum principle are presented in Chudej (1994,1996) where a complicated problem from aerospace engineering is solved.

The purpose of this paper is to supplement the first order necessary conditions by the second order sufficient conditions (SSC) and methods for sensitivity analysis. In recent years, SSC and sensitivity analysis have been extensively studied for single stage problems; see, e.g. Augustin and Maurer (2000), Malanowski (1995), Malanowski and Maurer (1996), Maurer (1995), Maurer and Pesch (1994), Maurer and Pickenhain (1995), Zeidan (1994). In these papers, the numerical check of SSC and the computation of sensitivity differentials of optimal solutions with respect to perturbations are linked to boundary value methods. A numerical test for SSC requires that a Riccati equation associated with the nominal solution have a bounded solution. Multiprocess problems can benefit from these results for single stage problems by a transformation which allows to view the multistage control problem as an augmented single stage problem. The augmented state comprises the state variables of all individual stages as well as the switching points as choice variables.

In Section 3, we review SSC for single stage control problems in the presence of general mixed boundary conditions. Section 4 discusses the reduction of a multiprocess to a single stage process and evaluates the Riccati equation and boundary condition for the multiprocess control problem. The novel feature is an augmented Riccati equation exhibiting additional components associated with the unspecified switching times. In Section 5, a two-stage robot control problem is solved and SSC are checked numerically. A sensitivity analysis of optimal solutions is conducted with respect to the load mass as parameter.

2. Multistage optimal control problems

The multiprocess optimal control problem (MCP) is defined on a time interval $[0, t_f]$ with unspecified final time t_f. The process is divided into a given number $N > 1$ of stages which are considered on time intervals $[t_{j-1}, t_j]$, $j = 1, ..., N$, forming a partition $0 = t_0 < t_1 < ... < t_{j-1} < t_j < ... < t_N = t_f$ of the total time interval $[0, t_f]$. The switching points t_j, $j = 1, ..., N-1$, are not specified and hence will be treated as choice variables. The dynamics for the absolutely continuous state variable $x \in W^{1,\infty}(0,t_f;\mathbb{R}^n)$ and the essentially bounded control variable $u \in L^\infty(0,t_f,\mathbb{R}^m)$ is given by

$$\dot{x}(t) = f^{(j)}(x(t),u(t)), \quad t \in [t_{j-1},t_j], \quad j = 1,...,N. \tag{1}$$

The functions $f^{(j)}$ are assumed to be C^2–functions on suitable open sets since we intend to derive second order sufficient conditions. The derivative $\dot{x}(t)$ at points $t_j, j = 1,...,N-1$, is understood as left, respectively as right derivative. For simplicity, the boundary conditions for the state variables are given in the special form that some components are specified at each point t_j,

$$x_i(t_j) = a_{ij} \quad \forall\, i \in I_j \subset \{1,...n\}, \quad j = 0,1,...N\,, \tag{2}$$

with given index sets I_j. Then, the optimal multiprocess control problem (MCP) is stated as follows: determine a control function $u \in L^\infty(0,t_f;\mathbb{R}^m)$, a state function $x \in W^{1,\infty}(0,t_f;\mathbb{R}^n)$ and switching times $t_j, j = 1,...,N$, which minimize the cost functional

$$F(x,t_1,...,t_N,u) = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} L^{(j)}(x(t),u(t))\,dt \tag{3}$$

subject to dynamics (1) and boundary conditions (2).

In this formulation we have assumed that the state variable is *continuous* across state junction times. However, the techniques developed in this paper would allow to relax this continuity assumption. Namely, instead of the simple boundary condition (2) more general boundary constraints

$$\psi(x(0),x(t_1^-),x(t_1^+),...,x(t_j^-),x(t_j^+),...,x(t_N)) = 0 \tag{4}$$

are tractable as well as a cost functional of the form

$$\begin{aligned} F(x,t_1,...,t_N,u) = \; & g(x(0),x(t_1^-),x(t_1^+),t_1,...,x(t_j^-),x(t_j^+),t_j,...,x(t_N),t_N) \\ & + \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} L^{(j)}(x(t),u(t))\,dt\,. \end{aligned} \tag{5}$$

Such problems include certain types of impulsive control problems (see Rempala and Zabczyk, 1988, Silva and Vinter, 1997) but these generalisations would lead to a rather complicated form of transversality and boundary conditions. Moreover, the following approach is also suited to include mixed control–state inequality constraints

$$C^{(j)}(x(t),u(t)) \le 0, \quad t \in [t_{j-1},t_j], \quad j = 1,...,N. \tag{6}$$

However, to simplify the presentation we refrain from these extensions and confine ourselves to the discussion of the unconstrained MCP.

3. Review of SSC for single stage optimal control problems with fixed final time

We briefly review second order sufficient optimality conditions (SSC) for a *single stage* optimal control problem with fixed final time. To avoid notational conflicts with the multiprocess control problem (MCP) we denote the state variable by $y \in \mathbb{R}^{n_y}$ and the control variable by $v \in \mathbb{R}^{m_v}$, where the dimensions n_y and m_v will be adapted to the MCP in (1)–(3). Furthermore, we use the time variable s to distinguish it from the time variable t in the MCP. The following *autonomous* control problem in the fixed time interval $[0,1]$ will be denoted by CP: determine a control function $v \in L^\infty(0,1;\mathbb{R}^{m_v})$ that minimizes the functional

$$F(y,v) = \int_0^1 L(y(s),v(s))\,ds \tag{7}$$

subject to

$$\dot{y}(s) = f(y(s),v(s)) \quad \text{for a.e. } s \in [0,1], \tag{8}$$

$$\varphi(y(0),y(1)) = 0\,. \tag{9}$$

It is assumed that the functions $L : \mathbb{R}^{n_y} \times \mathbb{R}^{m_v} \to \mathbb{R}$, $f : \mathbb{R}^{n_y} \times \mathbb{R}^{m_v} \to \mathbb{R}^{n_y}$ and $\varphi : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \to \mathbb{R}^r$, $0 \le r \le 2n_y$, are C^2-functions on appropriate open sets. We suppose further that there exists a feasible pair of functions $(y,v) \in W^{1,\infty}(0,1;\mathbb{R}^{n_y}) \times L^\infty(0,1;\mathbb{R}^{m_v})$ satisfying the constraints (8) and (9).

The first order necessary optimality conditions for an optimal pair (y,v) are well known in the literature. The Hamiltonian function H is defined by

$$H(y,v,\lambda) = L(y,v) + \lambda^* f(y,v)\,, \quad \lambda \in \mathbb{R}^{n_y}, \tag{10}$$

where λ is the adjoint variable and the asterisk denotes the transpose. Henceforth, partial first and second order derivatives are denoted either by D, respectively D^2, or by subscripts. In the following, the hypothesis will be made that first order necessary conditions are satisfied in *normal form* with a non-zero cost multiplier. Hence, we assume that there exist Lagrange–multipliers

$$(\lambda,\rho) \in W^{1,\infty}(0,1;\mathbb{R}^{n_y}) \times \mathbb{R}^r$$

such that the following conditions hold for a.e. $s \in [0,1]$:

$$\dot{\lambda}(s) = -H_y(y(s),v(s),\lambda(s))\,, \tag{11}$$

$$(-\lambda(0),\lambda(1)) = D_{(y(0),y(1))}[\rho^*\varphi](y(0),y(1))\,, \tag{12}$$

$$H_v(y(s),v(s),\lambda(s)) = 0, \tag{13}$$

$$H(y(s),v(s),\lambda(s)) \equiv const\,. \tag{14}$$

In the sequel, the notation $[s]$ will be used to denote arguments of functions at a reference solution $y(s), v(s), \lambda(s)$ and ρ.

SSC can be obtained by studying the behavior of the second variation on the variational system associated with equations (8) and (9). One basic assumption for SSC is the strict Legendre–Clebsch condition

$$H_{vv}[s] \geq c \cdot I_{m_v} \quad \text{for all } s \in [0,1], \quad c > 0, \quad I_{m_v} \text{ unity matrix}. \tag{15}$$

Then, SSC follow from the property that the quadratic form of the second variation is positive definite on the variational system associated with equations (8) and (9). Instead of discussing the second variation explicitly we shall resort to another sufficient condition which guarantees positive definiteness of the second variation. This condition is based on *Riccati equations* and turns out to be helpful for the numerical verification of SSC.

Let $Q \in W^{1,\infty}(0,1;M_{n_y,n_y})$ be a *symmetric* (n_y,n_y)–matrix function for which we consider the Riccati equation

$$\dot{Q} = -Qf_y[s] - f_y[s]^*Q - H_{yy}[s] + (H_{yv}[s] + Qf_v[s])H_{vv}[s]^{-1}(H_{yv}[s] + Qf_v[s])^* . \tag{16}$$

Boundary conditions for $Q(0)$ and $Q(1)$ are imposed by the requirement that

$$(h_0,h_1)^* \left[D^2_{(y(0),y(1))}[\rho^*\varphi](y(0),y(1)) + \begin{pmatrix} Q(0) & \mathbf{0} \\ \mathbf{0} & -Q(1) \end{pmatrix} \right] (h_0,h_1) \; > 0 \tag{17}$$

hold for all $(h_0,h_1) \in \mathbb{R}^{n_y} \times \mathbb{R}^{n_y}$, $(h_0,h_1) \neq 0$, satisfying the linearized boundary conditions

$$D_{y(0)}\,\varphi(y(0),y(1))\,h_0 + D_{y(1)}\,\varphi(y(0),y(1))\,h_1 = 0 . \tag{18}$$

The following SSC have been developed in Maurer and Pickenhain (1995) and Zeidan (1994).

THEOREM 3.1 *(SSC for optimal control problems with fixed final time)*
Let (y,v) be admissible for problem CP. Suppose that there exist multipliers $(\lambda,\rho) \in W^{1,\infty}(0,1;\mathbb{R}^{n_y}) \times \mathbb{R}^r$ such that the following conditions hold:

(1) the first order necessary conditions (11)-(14) are satisfied;
(2) the strict Legendre-Clebsch condition (15) holds;
(3) there exists a bounded solution $Q(t)$ of the Riccati equation (16) such that the boundary conditions (17) and (18) are fulfilled.

Then, there exists $c > 0$ and $\alpha > 0$ such that the cost functional can be estimated from below as

$$F(\tilde{y},\tilde{v}) \geq F(y,v) + c\,[\,||\tilde{y}-y||^2_{1,2} + ||\tilde{v}-v||^2_2\,]$$

for all admissible $(\tilde{y},\tilde{v})$ with $||\tilde{y}-y||_{1,\infty} + ||\tilde{v}-v||_\infty \leq \alpha$. In particular, (y,v) provides a strict weak local minimum for problem (CP).

4. SSC for multiprocess optimal control problems

We shall obtain necessary and sufficient conditions for the multiprocess control problem (MCP) in (1)–(3) by transforming it into a *single stage* control problem of the form CP described in (7)–(9). This approach requires augmented state variables and a transformation of the time variable in each stage. Recall the partition $0 = t_0 < t_1 < ... < t_{j-1} < t_j < ... < t_N = t_f$ of the MCP with unspecified time points t_j, $j = 1,..,N$. The time interval $[t_{j-1}, t_j]$ is mapped to the *fixed* time interval $[0, 1]$ by the transformation

$$t = t_{j-1} + s \cdot \tau_j, \quad s \in [0,1], \quad \tau_j := t_j - t_{j-1}, \quad j = 1, ..., N. \tag{19}$$

Such time transformations have often been used in the literature; see, e.g. Hestenes (1966, p.297), and Tomiyama and Rossana (1989). The state variables x, respectively control variables u on the interval $[t_{j-1}, t_j]$, $j = 1, ..., N$, are considered as functions of the new time variable $s \in [0,1]$ according to

$$x^{(j)}(s) := x(t_{j-1} + s \cdot \tau_j), \quad u^{(j)}(s) := u(t_{j-1} + s \cdot \tau_j), \quad s \in [0,1]. \tag{20}$$

Then, the MCP becomes equivalent to a single stage control problem of the type (7)–(9) by defining *augmented* state and control variables

$$y := (x^{(1)}, \tau_1, x^{(2)}, \tau_2, ..., x^{(N)}, \tau_N) \in \mathbb{R}^{n_y}, \quad n_y = N \cdot (n+1), \tag{21}$$

$$v := (u^{(1)}, u^{(2)}, ..., u^{(N)}) \in \mathbb{R}^{m_v}, \quad m_v = N \cdot m, \tag{22}$$

where $\tau_j = t_j - t_{j-1}$ are treated as choice variables. Using the transformation (20) we obtain the dynamical system

$$\dot{x}^{(j)}(s) = \frac{dx^{(j)}}{ds} = \tau_j \cdot f^{(j)}(x^{(j)}(s), u^{(j)}(s)), \quad \frac{d\tau_j}{ds} \equiv 0, \quad s \in [0,1]. \tag{23}$$

This dynamical system and the boundary conditions (2), i.e.,

$$x_i(t_j) = a_{ij} \quad \forall\, i \in I_j \subset \{1, ..., n\}, \quad j = 0, 1, ..., N,$$

can be written in condensed form as

$$\dot{y}(s) = f(y(s), v(s)), \quad s \in [0,1], \quad \varphi(y(0), y(1)) = 0, \tag{24}$$

where the functions $f : \mathbb{R}^{n_y} \times \mathbb{R}^{m_v} \to \mathbb{R}^{n_y}$ and $\varphi : \mathbb{R}^{2n_y} \to \mathbb{R}^r$, $r := \sum_{i=0}^{N} \mathrm{card}(I_j) + (N-1)n$, are given by

$$f(y,v) = \begin{bmatrix} \tau_1 \cdot f^{(1)}(x^{(1)}, v^{(1)}) \\ 0 \\ \cdots\cdots \\ \cdots\cdots \\ \tau_N \cdot f^{(N)}(x^{(N)}, v^{(N)}) \\ 0 \end{bmatrix}, \tag{25}$$

$$\varphi(y(0),y(1)) = \begin{bmatrix} (x_i^{(1)}(0) - a_{i0})_{i\in I_0} \\ \begin{pmatrix} (x_i^{(j)}(1) - a_{ij})_{i\in I_j} \\ x^{(j+1)}(0) - x^{(j)}(1) \end{pmatrix}_{j=1,\dots,N-1} \\ (x_i^{(N)}(1) - a_{iN})_{i\in I_N} \end{bmatrix}. \tag{26}$$

The terms $x^{(j+1)}(0) - x^{(j)}(1)$ appearing in the function φ reflect the fact that the original state variable $x(t)$ is *continuous* at the switching times t_j. Finally, the cost functional (7) is to minimize

$$F(y,v) = \int_0^1 L(y(s),v(s))\,ds\,, \quad L(y,v) := \sum_{j=1}^{N} \tau_j \cdot L^{(j)}(x^{(j)},u^{(j)})\,. \tag{27}$$

First, we discuss the necessary conditions (11)–(14) and consider the adjoint variable in the partitioned form

$$\begin{aligned} &\lambda = (\lambda^{(1)}, \lambda_f^{(1)}, \lambda^{(2)}, \lambda_f^{(2)}, \dots, \lambda^{(N)}, \lambda_f^{(N)}) \in \mathbb{R}^{n_y}, \\ &\lambda^{(j)} \in \mathbb{R}^n, \quad \lambda_f^{(j)} \in \mathbb{R}, \quad j = 1,\dots,N\,. \end{aligned} \tag{28}$$

The associated Hamiltonian functions are

$$\begin{aligned} &H(y,v,\lambda) = \textstyle\sum_{j=1}^N \tau_j \cdot H^{(j)}(x^{(j)},u^{(j)},\lambda^{(j)})\,, \\ &H^{(j)}(x^{(j)},u^{(j)},\lambda^{(j)}) := L^{(j)}(x^{(j)},u^{(j)}) + (\lambda^{(j)})^* f^{(j)}(x^{(j)},u^{(j)})\,. \end{aligned} \tag{29}$$

The adjoint equations (11) split into equations for $j = 1,\dots,N$ with $s \in [0,1]$,

$$\dot\lambda^{(j)}(s) \;=\; -H_{x^{(j)}}[s] = -\tau_j \cdot H^{(j)}_{x^{(j)}}[s]\,, \tag{30}$$

$$\dot\lambda_f^{(j)}(s) \;=\; -H_{\tau_j}[s] = -H^{(j)}[s]\,. \tag{31}$$

Now we insert the function φ in (26) into the transversality condition (12) which is repeated here for convenience with a multiplier $\rho = (\rho_{ij}) \in \mathbb{R}^r$,

$$(-\lambda(0), \lambda(1)) = D_{(y(0),y(1))}[\rho^* \varphi](y(0),y(1))\,.$$

Exploiting these conditions with respect to the state variables $x^{(j)}$ we find the following boundary and junction conditions

$$\lambda_i^{(1)}(0) = 0 \quad \forall\, i \notin I_0\,, \qquad \lambda_i^{(N)}(1) = 0 \quad \forall\, i \notin I_N\,, \tag{32}$$

$$\lambda_i^{(j+1)}(0) = \lambda_i^{(j)}(1) \qquad \forall\, i \notin I_j, \quad j = 1,\dots,N-1\,, \tag{33}$$

$$\lambda_i^{(j+1)}(0) = \lambda_i^{(j)}(1) + \rho_{ij} \quad \forall\, i \in I_j\,, \quad j = 1,\dots,N-1\,. \tag{34}$$

Evaluating the transversality condition with respect to the free time variables τ_j we get

$$\lambda_f^{(j)}(0) = 0, \quad \lambda_f^{(j)}(1) = 0, \quad j = 1,\dots,N\,. \tag{35}$$

From (14) one can derive the constancy of the Hamiltonians, see also Clarke and Vinter (1989A, Corollary 3.1),

$$H^{(j)}[s] \equiv const., \quad s \in [0,1], \quad j = 1,...,N.$$

Combining the adjoint equation (31) with the boundary condition (35) we conclude that

$$H^{(j)}[s] \equiv 0, \quad s \in [0,1], \quad j = 1,...,N. \tag{36}$$

Finally, the control variables are determined by the conditions

$$H^{(j)}_{u^{(j)}}[s] = 0, \quad s \in [0,1], \quad j = 1,...,N. \tag{37}$$

Our next task is to derive SSC on the basis of Theorem 3.1 involving the relations (15)–(18). The strict Legendre–Clebsch condition (15) holds if the corresponding condition is satisfied on each stage,

$$H^{(j)}_{u^{(j)}u^{(j)}}[s] \geq c \cdot I_m, \quad s \in [0,1], \quad j = 1,...,N, \quad \text{for some } c > 0. \tag{38}$$

Now we turn our attention to the Riccati equation (16). Observing the special structure of the state variable y in (21) and the dynamics (23) we set up the symmetric (n_y, n_y)–matrix $\tilde{Q}$ in the form

$$\tilde{Q} = \begin{pmatrix} \tilde{Q}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \tilde{Q}^{(2)} & \mathbf{0} & \vdots \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \tilde{Q}^{(N)} \end{pmatrix}, \qquad \tilde{Q}^{(j)} = \begin{pmatrix} Q^{(j)} & R^{(j)} \\ (R^{(j)})^* & q_f^{(j)} \end{pmatrix}, \quad j = 1,..,N. \tag{39}$$

The matrices $\tilde{Q}^{(j)}$ are composed of (n,n)–matrices $Q^{(j)} = (q_{ik}^{(j)})_{1 \leq i,k \leq n}$, n–vectors $R^{(j)} = (r_i^{(j)})_{i=1,..,n}$, and a scalar $q_f^{(j)}$, which is associated with the free time points τ_j. We insert the matrix $\tilde{Q}$ into the Riccati equation (16) and use the augmented functions (25) and the Hamiltonians (29). This procedure yields the following three equations where the time argument is suppressed for convenience:

$$\begin{aligned} \dot{Q}^{(j)} = \tau_j \cdot [&-Q^{(j)} f^{(j)}_{x^{(j)}} - (f^{(j)}_{x^{(j)}})^* Q^{(j)} - H^{(j)}_{x^{(j)}x^{(j)}} + \\ &+ (H^{(j)}_{x^{(j)}u^{(j)}} + Q^{(j)} f^{(j)}_{u^{(j)}})(H^{(j)}_{u^{(j)}u^{(j)}})^{-1}(H^{(j)}_{x^{(j)}u^{(j)}} + Q^{(j)} f^{(j)}_{u^{(j)}})^*], \end{aligned} \tag{40}$$

$$\begin{aligned} \dot{R}^{(j)} = &-Q^{(j)} f^{(j)} - \tau_j (f^{(j)}_{x^{(j)}})^* R^{(j)} - (H^{(j)}_{x^{(j)}})^* \\ &+ \tau_j \cdot [H^{(j)}_{x^{(j)}u^{(j)}} + Q^{(j)} f^{(j)}_{u^{(j)}}](H^{(j)}_{u^{(j)}u^{(j)}})^{-1}(f^{(j)}_{u^{(j)}})^* R^{(j)}, \end{aligned} \tag{41}$$

$$\dot{q}_f^{(j)} = -2(R^{(j)})^* f^{(j)} + \tau_j (R^{(j)})^* f^{(j)}_{u^{(j)}} (H^{(j)}_{u^{(j)}u^{(j)}})^{-1} (f^{(j)}_{u^{(j)}})^* R^{(j)}. \tag{42}$$

These equations comprise a Riccati equation for $Q^{(j)}$, a linear equation for $R^{(j)}$ and a direct integration for $q_f^{(j)}$ on the interval $[0,1]$. They generalize similar relations for single stage problems in Maurer (1995), Maurer and Oberle (2000). Note that for multistage *linear-quadratic* control problems, the Riccati equation (40) for $Q^{(j)}$ reduces to a well known Riccati equation with coefficients not depending on the nominal solution; see, e.g., Tomiyama (1985, Section 5).

The final step is to translate the boundary conditions (17) and (18) into the multiprocess setting. In view of the linearity of the function φ in (26), the condition (17) on positive definiteness reduces to the condition that

$$(\tilde{h}_0, \tilde{h}_f)^* \begin{pmatrix} \tilde{Q}(0) & \mathbf{0} \\ \mathbf{0} & -\tilde{Q}(1) \end{pmatrix} (\tilde{h}_0, \tilde{h}_f) > 0 \tag{43}$$

holds for all $(\tilde{h}_0, \tilde{h}_f) \in \mathbb{R}^{n_y} \times \mathbb{R}^{n_y}$, $(\tilde{h}_0, \tilde{h}_f) \neq 0$, which satisfy the linearized boundary conditions

$$D_{y(0)}\, \varphi(y(0), y(1))\, \tilde{h}_0 + D_{y(1)}\, \varphi(y(0), y(1))\, \tilde{h}_f = 0\,. \tag{44}$$

THEOREM 4.1 *(SSC for multiprocess control problems)*
Let (x,u) and the final time t_f, respectively the switching times $t_j, j = 1, ..., N-1$, be admissible for the MCP problem in (1)-(3). Define the transformed functions $x^{(j)}$, $u^{(j)}$, τ_j, $j = 1, ..., N$, through (19) and (20). Suppose that there exist multipliers $\lambda^{(j)} \in W^{1,\infty}(0,1;\mathbb{R}^n)$, $j = 1, ..., N$, and $\rho \in \mathbb{R}^r$ such that

(1) the necessary conditions (24), (30), (32)-(34) and (36) are satisfied;
(2) the strict Legendre–Clebsch condition (38) holds;
(3) there exists a bounded solution $\widetilde{Q}(t)$ of the Riccati equations (40)-(42) for which the boundary conditions (43) and (44) are fulfilled.

Then, there exists $c > 0$ and $\alpha > 0$ such that the cost functional can be estimated from below as

$$\begin{aligned} F(\tilde{x}, \tilde{u}, \tilde{t}_1, ..., \tilde{t}_N) \geq\ & F(x, u, t_1, ..., t_N) \\ & + c \sum_{j=1}^{N} \{ \|\tilde{x}^{(j)} - x^{(j)}\|_{1,2}^2 + |\tilde{\tau}_j - \tau_j|^2 + \|\tilde{u}^{(j)} - u^{(j)}\|_2^2 \} \end{aligned}$$

for all admissible $(\tilde{x}, \tilde{u})$ with $\|\tilde{x}^{(j)} - x^{(j)}\|_{1,\infty} + |\tilde{\tau}_j - \tau_j| + \|\tilde{u}^{(j)} - u^{(j)}\|_\infty \leq \alpha$, $j = 1, ..., N$. In particular, (x,u) provides a strict weak local minimum for the MCP problem.

More effort is needed to bring the boundary conditions (43) and (44) into a form suited for practical applications. Writing the variational vectors $\tilde{h}_0, \tilde{h}_f$ as

$$\begin{aligned} \tilde{h}_0 &= (h_0^{(j)}, k_0^{(j)})_{j=1,..,N}\,, \quad h_0^{(j)} \in \mathbb{R}^n,\ k_0^{(j)} \in \mathbb{R}\,, \\ \tilde{h}_f &= (h_f^{(j)}, k_f^{(j)})_{j=1,..,N}\,, \quad h_f^{(j)} \in \mathbb{R}^n,\ k_f^{(j)} \in \mathbb{R}\,, \end{aligned} \tag{45}$$

the linearized boundary conditions (44) yield the relations

$$\begin{aligned} & h_{0,i}^{(1)} = 0 \quad \forall\, i \in I_0\,, \quad h_{f,i}^{(j)} = 0 \quad \forall\, i \in I_j\,, \quad j = 1, ..., N, \\ & h_0^{(j+1)} = h_f^{(j)}\,, \quad j = 1, ..., N-1\,. \end{aligned} \tag{46}$$

We derive a first set of conditions by observing that the variational equations (44) are satisfied for $h_0^{(j)} = 0$ and $h_f^{(j)} = 0$ and arbitrary $k_0^{(j)}, k_f^{(j)} \in \mathbb{R}$. Then, condition (43) immediately provides the following sign conditions, see also Maurer (1995), Maurer and Oberle (2000):

$$q_f^{(j)}(0) > 0, \quad q_f^{(j)}(1) < 0, \quad j = 1, ..., N. \tag{47}$$

Since it is rather tedious to evaluate the definiteness condition (43) in full generality, we restrict the discussion to some cases of practical interest.

Case 1: The state variables $x(t_j)$ are completely specified for all indices $j = 0, 1, ..., N$. Then, obviously, conditions (47) are equivalent to the condition (43).

Case 2: Suppose that *one* component $x_i^{(j)}(1)$ is *unspecified* for some $i \in \{1, ..., n\}$, and $j \in \{1, ..., N-1\}$. Then, condition (43) is equivalent to the statement that relations (47) hold and that, in addition, the relevant components in the solutions of the Riccati equations satisfy

$$\begin{aligned} &d_{ij} := q_{ii}^{(j+1)}(0) - q_{ii}^{(j)}(1) > 0, \quad r_i^{(j)}(1)^2 < -d_{ij}\, q_f^{(j)}(1), \\ &r_i^{(j+1)}(0)^2 < d_{ij}\, q_f^{(j+1)}(0). \end{aligned} \tag{48}$$

An application of the boundary test (47) and (48) to the optimal control of a robot will be discussed in the next section.

Case 3: Assume that *two* components $x_i^{(j)}(1)$ and $x_k^{(j)}(1)$ are *unspecified* for two indices $i < k$ and some $j \in \{1, ..., N-1\}$. After some manipulations in (43) we find that in addition to relations (47) and (48) the following relation must hold:

$$\left(q_{ik}^{(j+1)}(0) - q_{ik}^{(j)}(1)\right)^2 < d_{ij}\, d_{kj}. \tag{49}$$

5. Sensitivity analysis for multiprocess control problems

Sensitivity analysis for parametric (perturbed) single stage optimal control problems has been the subject of intensive research in recent years. Malanowski (1995), Malanowski and Maurer (1996,1998), Maurer and Pesch (1994) provide conditions for the Fréchet differentiability of optimal solutions with respect to perturbation parameters. These conditions are based on SSC for the nominal solution. We briefly sketch how these results carry over to the multiprocess problem (MCP). Let $p \in \mathbb{R}^l$ be a perturbation parameter which is introduced into the multiprocess (1)–(3) such that the following parametric multiprocess problem (MCP(p)) arises:

$$\text{Minimize} \quad F(x, t_1, ..., t_N, u, p) = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} L^{(j)}(x(t), u(t), p)\, dt \tag{50}$$

subject to the dynamics

$$\dot{x}(t) = f^{(j)}(x(t), u(t), p), \quad t \in [\,t_{j-1}, t_j], \quad j = 1, ..., N, \tag{51}$$

and boundary conditions

$$x_i(t_j) = a_{ij}(p) \quad \forall\, i \in I_j \subset \{1, ...n\}, \quad j = 0, 1, ...N\,. \tag{52}$$

The functions $L^{(j)}, f^{(j)}, a_{ij}$ are of appropriate dimensions and are assumed to be C^2–functions. Let p_0 be the *nominal parameter* and let $x_0^{(j)}$, $u_0^{(j)}$, $(\tau_0)_j$, $j = 1, ..., N$, be the nominal solution, defined on the interval $[0, 1]$ according to (23). The sensitivity result in Maurer and Malanowski (1996) applies to control problems with control–state constraints. For unconstrained control problems this result simplifies and leads to the following theorem on the basis of the transformations in the last section.

THEOREM 5.1 *(Frećhet differentiability for parametric multiprocess problems) Let $p_0 \in \mathbb{R}^l$ and let $x_0^{(j)}, u_0^{(j)}, (\tau_0)_j$, $j = 1, ..., N$, be admissible for the nominal problem MCP(p_0). Suppose that the following conditions are satisfied:*

(1) the control functions $u_0^{(j)}$, $j = 1, ..., N$, are continuous:
(2) the system (24) is completely controllable;
(3) there exist adjoint functions $\lambda_0^{(j)} \in W^{1,\infty}(0, 1; \mathbb{R}^n)$, $j = 1, ..., N$, and a multiplier $\rho_0 \in \mathbb{R}^r$ such that the SSC in Theorem 4.1 hold.

Then, the nominal controls $u_0^{(j)}$, $j = 1, ..., N$, are C^1-functions and the nominal solution $x_0^{(j)}, u_0^{(j)}, \lambda_0^{(j)}, (\tau_0)_j$, $j = 1, ..., N$, can be embedded into a Fréchet differentiable family of optimal solutions $x^{(j)}(\cdot, p)$, $u^{(j)}(\cdot, p)$, $\lambda^{(j)}(\cdot, p), \tau_j(p), \rho(p)$ to the perturbed problem MCP(p) in a neighborhood of p_0. The sensitivity differentials

$$\begin{aligned} z^{(j)}(s) &:= \tfrac{\partial x^{(j)}}{\partial p}(s; p_0), \quad & \gamma^{(j)}(s) &:= \tfrac{\partial \lambda^{(j)}}{\partial p}(s; p_0), \\ u_p^{(j)}(s) &:= \tfrac{\partial u^{(j)}}{\partial p}(s; p_0), \quad & \sigma_j &:= \tfrac{d\tau_j}{dp}(p_0)\,, \end{aligned} \tag{53}$$

exist for all $s \in [0, 1]$ and satisfy a BVP which is obtained by formal differentiation of the necessary conditions with respect to the parameter p.

We dispense with the precise form of the BVP for the sensitivity differentials (53) and refer as an illustration to the practical example in the next section; see (78)–(81). Condition (2) of the theorem requires the complete controllability of the dynamical system (24). We point out that a practical verification of this condition can be organized as a byproduct of solving the BVP for the nominal solution.

6. Optimal two – stage control of a robot

The problem is to control a robot so that in the first stage the robot arm transfers an object to a prescribed endposition where it drops its load, whereas in the second stage the robot arm returns to its initial position. Time optimal solutions for this type of robot control have been discussed in Clarke and Vinter (1989A) where explicit bang–bang solutions are given. In our example we modify the cost function and combine the time optimal with the energy optimal solutions. The cost of energy involves a quadratic control term which is indispensable here since the strict Legendre–Clebsch condition (38) is required. Consider the following two–stage LQ–problem:

$$\text{Minimize} \qquad F(x,t_1,t_f,u) = t_f + \int_0^{t_f} u(t)^2 dt = \int_0^{t_f} (1+u(t)^2)\,dt \tag{54}$$

$$\text{subject to} \qquad \dot{x}_1 = x_2, \quad \dot{x}_2 = \left\{ \begin{array}{rl} (m+M)^{-1}u \ , & 0 \le t < t_1\,, \\ m^{-1}u \ , & t_1 < t \le t_f\,, \end{array} \right\} \tag{55}$$

$$x_1(0) = x_2(0) = 0, \quad x_1(1) = x_2(1) = 0, \tag{56}$$

$$x_1(t_1) = 2, \qquad x_2(t_1^+) = x_2(t_1^-). \tag{57}$$

The state x_1 represents the position of the end of the robot arm and x_2 its velocity, m is the mass of the robot arm and M is the mass of the load. Both the switching time t_1 and the final time t_f are choice variables. We point out that this two–stage LQ–problem differs slightly from the class of LQ–problems considered in Tomiyama (1985) since some state components in the above problem are specified at the switching and final time.

To convert the two–stage problem (54)–(57) into a single stage problem we use the time transformation (19) with $\tau_1 = t_1$ and $\tau_2 = t_f - t_1$. Then the state and control transformations (20)–(27) yield the following single–stage control problem CP on the fixed time interval $[0,1]$:

$$\text{Minimize} \qquad \int_0^1 \tau_1 \left(1 + u^{(1)}(s)^2\right) ds + \int_0^1 \tau_2 \left(1 + u^{(2)}(s)^2\right) ds \tag{58}$$

$$\text{subject to} \qquad \dot{x}_1^{(1)} = \tau_1 \cdot x_2^{(1)}, \quad \dot{x}_2^{(1)} = \tau_1 \cdot (m+M)^{-1} u^{(1)}, \quad \dot{\tau}_1 = 0, \tag{59}$$

$$\dot{x}_1^{(2)} = \tau_2 \cdot x_2^{(2)}, \quad \dot{x}_2^{(2)} = \tau_2 \cdot m^{-1} u^{(2)}, \quad \dot{\tau}_2 = 0, \tag{60}$$

$$x_1^{(1)}(0) = x_2^{(1)}(0) = 0, \quad x_1^{(2)}(1) = x_2^{(2)}(1) = 0, \tag{61}$$

$$x_1^{(1)}(1) = x_1^{(2)}(0) = 2, \quad x_2^{(2)}(0) = x_2^{(1)}(1)\,. \tag{62}$$

6.1. Solution of the boundary value problem

The Hamiltonians (29) for each stage are given by

$$H^{(1)} = \tau_1 \cdot \left(1 + \left(u^{(1)}\right)^2 + \lambda_1^{(1)} x_2^{(1)} + \frac{\lambda_2^{(1)} u^{(1)}}{m+M}\right), \tag{63}$$

$$H^{(2)} = \tau_2 \cdot \left(1 + \left(u^{(2)}\right)^2 + \lambda_1^{(2)} x_2^{(2)} + \frac{\lambda_2^{(2)} u^{(2)}}{m}\right). \tag{64}$$

The adjoint equations (30) yield

$$\dot{\lambda}_1^{(1)} = 0, \quad \dot{\lambda}_2^{(1)} = -\tau_1 \lambda_1^{(1)}, \tag{65}$$

$$\dot{\lambda}_1^{(2)} = 0, \quad \dot{\lambda}_2^{(2)} = -\tau_2 \lambda_1^{(2)}. \tag{66}$$

The transversality conditions (32) and (33) reduce to the continuity condition

$$\lambda_2^{(2)}(0) = \lambda_2^{(1)}(1). \tag{67}$$

In addition, the formal jump condition (34) holds,

$$\lambda_1^{(2)}(0) = \lambda_1^{(1)}(1) + \rho, \ \ \rho \geq 0. \tag{68}$$

The stationarity conditions (37) give

$$H_{u^{(1)}}^{(1)} = \tau_1 \left(2u^{(1)} + \frac{\lambda_2^{(1)}}{m+M}\right) = 0, \quad H_{u^{(2)}}^{(2)} = \tau_2 \left(2u^{(2)} + \frac{\lambda_2^{(2)}}{m}\right) = 0, \tag{69}$$

so that the control variables are evaluated as

$$u^{(1)} = -\frac{\lambda_2^{(1)}}{2(m+M)}, \quad u^{(2)} = -\frac{\lambda_2^{(2)}}{2m}. \tag{70}$$

Finally, by combining the transversality condition (36) with the control law (70) we get the relations

$$0 = H^{(1)}[0] = \tau_1 \left(1 - \frac{\lambda_2^{(1)}(0)^2}{4(m+M)^2}\right), \quad 0 = H^{(2)}[1] = \tau_2 \left(1 - \frac{\lambda_2^{(2)}(1)^2}{4m^2}\right),$$

from which we obtain the boundary conditions

$$\lambda_2^{(1)}(0)^2 = 4(m+M)^2, \quad \lambda_2^{(2)}(1)^2 = 4m^2. \tag{71}$$

In summary, we have to solve the boundary value problem (BVP) comprising the equations (59)–(62), (65)–(67) and (71) where the controls are substituted from (70). This type of BVP can be efficiently solved with shooting methods which are implemented in the routine BNDSCO of Oberle and Grimm (1989).

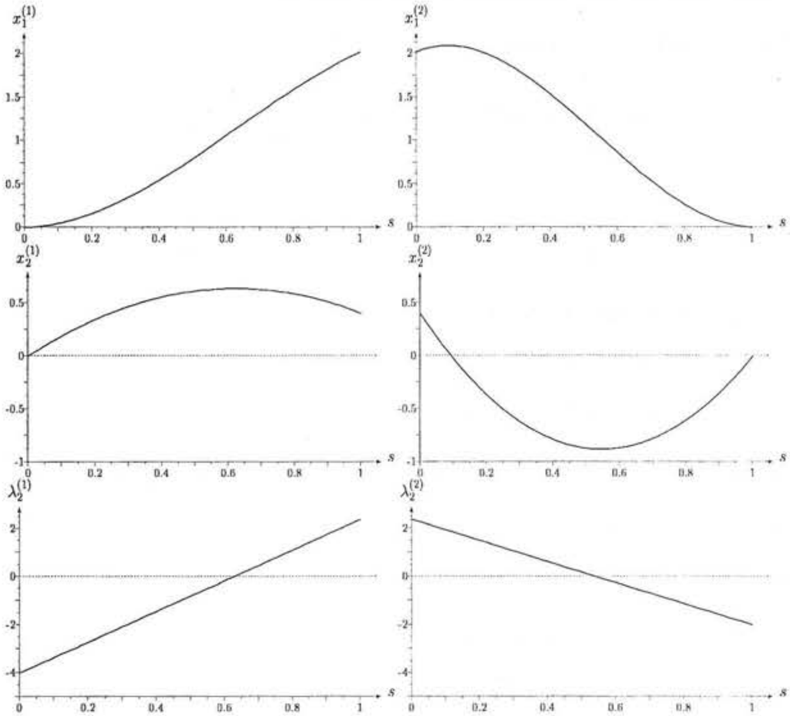

Figure 1. Optimal state and adjoint variables for $m = M = 1$ with $j = 1$ (left) and $j = 2$ (right).

Choosing the nominal values $m = 1$ and $M = 1$ for the mass of the robot and load, we obtain the following numerical results:

$$\begin{array}{lll} \tau_1 = t_1 = 4.14790960, & \tau_2 = t_f - t_1 = 3.89776004, & t_f = 8.04566964, \\ x_2^{(1)}(1) = 0.40953436, & x_2^{(2)}(0) = x_2^{(1)}(1), & \\ \lambda_1^{(1)}(s) \equiv -1.54783427, & \lambda_1^{(2)}(s) \equiv 1.13405561, & \\ \lambda_2^{(1)}(0) = -4, & \lambda_2^{(2)}(1) = -2, & \\ \lambda_2^{(1)}(1) = 2.42027663, & \lambda_2^{(2)}(0) = \lambda_2^{(1)}(1). & \end{array} \tag{72}$$

These data completely specify the solution together with the boundary conditions (61), (62), (67) and (71). The jump multiplier in (34) is $\rho_{11} = \lambda_1^{(2)}(0) - \lambda_1^{(1)}(1) = 2.68188988$. The corresponding state, adjoint and control variables are displayed in Figs. 1 and 2.

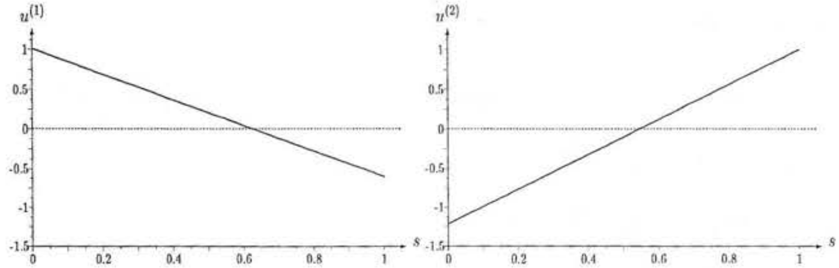

Figure 2. Optimal control for $m = M = 1$ and $j = 1, 2$.

6.2. Verification of SSC

We are going to verify that the solution characterized by the data (72) provides a local minimum. Since the Legendre–Clebsch condition (38) trivially holds in view of (58), it suffices to find a bounded solution of the Riccati equations (40)–(42) such that the boundary conditions (47) and (48) are satisfied. In this example we have dimensions $n = 2, N = 2$, and hence consider the symmetric matrix (39) in the form

$$\widetilde{Q} = \begin{pmatrix} \widetilde{Q}^{(1)} & \mathbf{0} \\ \mathbf{0} & \widetilde{Q}^{(2)} \end{pmatrix}, \quad \widetilde{Q}^{(j)} = \begin{pmatrix} q_{11}^{(j)} & q_{12}^{(j)} & r_1^{(j)} \\ q_{12}^{(j)} & q_{22}^{(j)} & r_2^{(j)} \\ r_1^{(j)} & r_2^{(j)} & q_f^{(j)} \end{pmatrix}, \quad j = 1, 2. \tag{73}$$

For $\mathbf{j=1}$ we insert $m = M = 1$ and obtain from (40)–(42) the following Riccati equations in $[0, 1]$,

$$\begin{aligned} \dot{q}_{11}^{(1)} &= \tau_1 \frac{\left(q_{12}^{(1)}\right)^2}{8}, \quad \dot{q}_{12}^{(1)} = \tau_1 [\, \tfrac{q_{12}^{(1)} q_{22}^{(1)}}{8} - q_{11}^{(1)} \,], \quad \dot{q}_{22}^{(1)} = \tau_1 [\, \tfrac{\left(q_{22}^{(1)}\right)^2}{8} - 2 q_{12}^{(1)} \,], \\ \dot{r}_1^{(1)} &= \tau_1 \tfrac{q_{12}^{(1)} r_2^{(1)}}{8} - q_{11}^{(1)} x_2^{(1)} - \tfrac{q_{12}^{(1)} u^{(1)}}{2}, \\ \dot{r}_2^{(1)} &= \tau_1 \tfrac{q_{22}^{(1)} r_2^{(1)}}{8} - q_{12}^{(1)} x_2^{(1)} - \tfrac{q_{22}^{(1)} u^{(1)}}{2} - \tau_1 r_1^{(1)} - \lambda_1^{(1)}, \\ \dot{q}_f^{(1)} &= \tau_1 \frac{\left(r_2^{(1)}\right)^2}{8} - 2 r_1^{(1)} x_2^{(1)} - r_2^{(1)} u^{(1)}. \end{aligned} \tag{74}$$

For $\mathbf{j}=\mathbf{2}$ we put $m=1$ and get

$$\begin{aligned}
\dot{q}_{11}^{(2)} &= \tau_2 \frac{\left(q_{12}^{(2)}\right)^2}{2}, \quad \dot{q}_{12}^{(2)} = \tau_2 [\, \frac{q_{12}^{(2)} q_{22}^{(2)}}{2} - q_{11}^{(2)} \,], \quad \dot{q}_{22}^{(2)} = \tau_2 [\, \frac{\left(q_{22}^{(2)}\right)^2}{2} - 2 q_{12}^{(2)} \,], \\
\dot{r}_1^{(2)} &= \tau_2 \frac{q_{12}^{(2)} r_2^{(2)}}{2} - q_{11}^{(2)} x_2^{(2)} - q_{12}^{(2)} u^{(2)}, \\
\dot{r}_2^{(2)} &= \tau_2 \frac{q_{22}^{(2)} r_2^{(2)}}{2} - q_{12}^{(2)} x_2^{(2)} - q_{22}^{(2)} u^{(2)} - \tau_2 r_1^{(2)} - \lambda_1^{(2)}, \\
\dot{q}_f^{(2)} &= \tau_2 \frac{\left(r_2^{(2)}\right)^2}{2} - 2 r_1^{(2)} x_2^{(2)} - 2 r_2^{(2)} u^{(2)} .
\end{aligned} \tag{75}$$

Note that the coefficients $x_i^{(j)}$, $u^{(j)}$ in these equations represent the solutions corresponding to the data (72).

Next we evaluate the boundary condition (47) for the indices $j = 1, 2$ and the condition (48) for indices $i = 2$ and $j = 1$. This results in the following set of inequalities,

$$\left\{ \begin{array}{l}
q_f^{(1)}(0) > 0, \quad q_f^{(1)}(1) < 0, \quad q_f^{(2)}(0) > 0, \quad q_f^{(2)}(1) < 0, \\
d_{21} := q_{22}^{(2)}(0) - q_{22}^{(1)}(1) > 0, \quad \left(r_2^{(1)}(1)\right)^2 < -d_{21}\, q_f^{(1)}(1), \\
\left(r_2^{(2)}(0)\right)^2 < d_{21}\, q_f^{(2)}(0).
\end{array} \right\} \tag{76}$$

Choosing the set of boundary values

$$\begin{aligned}
& q_{11}^{(1)}(1) = q_{12}^{(1)}(1) = r_2^{(1)}(1) = q_{11}^{(2)}(0) = q_{12}^{(2)}(0) = r_2^{(2)}(0) = 0, \\
& r_1^{(1)}(1) = 1, \quad r_1^{(2)}(0) = -0.5, \; q_{22}^{(1)}(1) = 0 < 0.1 = q_{22}^{(2)}(0), \\
& q_f^{(1)}(1) = -0.2 < 0, \quad q_f^{(2)}(1) = -0.1 < 0,
\end{aligned}$$

we obtain the following solution of the Riccati equations (74) and (75),

$$\begin{aligned}
& q_{11}^{(1)}[s] \equiv q_{12}^{(1)}[s] \equiv q_{22}^{(1)}[s] \equiv q_{11}^{(2)}[s] \equiv q_{12}^{(2)}[s] \equiv 0, \; r_1^{(1)}[s] \equiv 1, \; r_1^{(2)}[s] \equiv -0.5, \\
& r_2^{(1)}(0) = 2.600075, \; q_{22}^{(2)}(1) = 0.12420632, \; r_2^{(2)}(1) = 0.92649418, \\
& q_f^{(1)}(0) = 0.20043315 > 0, \; q_f^{(2)}(0) = 0.09593715 > 0,
\end{aligned}$$

which satisfies the boundary conditions (76). The functions $q_j^{(j)}$, $j = 1, 2$, are shown in Fig. 3 to illustrate the first set of sign conditions in (76). In summary, we have arrived at the conclusion that the computed solution characterized by the data (72) is indeed a local minimum.

6.3. Sensitivity analysis

We perform now a sensitivity analysis of the optimal solution with respect to a perturbation in the load mass M. We have chosen the mass M as a sensitivity

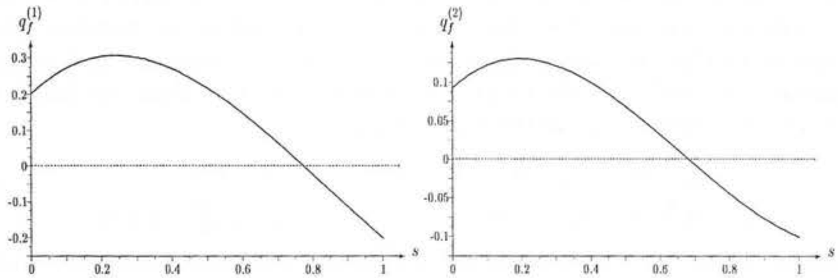

Figure 3. Solutions $q_f^{(j)}$, $j = 1, 2$ of the Riccati equations (74) and (75).

(perturbation) parameter because in practice it often happens that the load has not exactly the desired size. We take $m = 1$ and the nominal paramter $M_0 = 1$. The results in the preceding section show that all assumptions in Theorem 5.1 for Fréchet differentiability of optimal solutions are satisfied. Hence there exist optimal solutions

$$x^{(j)}(s; M), \;\; u^{(j)}(s; M), \;\; \lambda^{(j)}(s; M), \;\; \tau_j(M), \quad j = 1, 2,$$

for all M in a neighborhood of M_0 and, moreover, the following sensitivity differentials exist for $j = 1, 2$:

$$\begin{aligned} z^{(j)}(s) &:= \frac{\partial x^{(j)}}{\partial M}(s; M_0), \quad \gamma^{(j)}(t) := \frac{\partial \lambda^{(j)}}{\partial M}(s; M_0), \\ u_M^{(j)}(t) &:= \frac{\partial u^{(j)}}{\partial M}(s; M_0), \quad \sigma_j := \frac{d\tau_j}{dM}(M_0). \end{aligned}$$

Relations between $u_M^{(j)}(s)$ and $\gamma^{(j)}(s)$ are established by differentiating the control law (70) with respect to M and setting $M_0 = 1$:

$$u_M^{(1)}(s) = -\frac{\gamma_2^{(1)}(s)}{4} + \frac{\lambda_2^{(1)}(s)}{8}, \quad u_M^{(2)}(s) = -\frac{\gamma_2^{(2)}(s)}{2}. \tag{77}$$

ODEs for the other sensitivity differentials are obtained through the process of formal differentiation of the dynamical system (59), (60) and (65), (66) with respect to M. For convenience we suppress the time argument and find for $\mathbf{j = 1}$,

$$\begin{aligned} \dot{z}_1^{(1)} &= \tau_1 z_2^{(1)} + \sigma_1 x_2^{(1)}, \quad & \dot{z}_2^{(1)} &= \tfrac{1}{2}(\sigma_1 u^{(1)} + \tau_1 u_M^{(1)}) - \tfrac{1}{4}\tau_1 u^{(1)}, \\ \dot{\gamma}_1^{(1)} &= 0, & \dot{\gamma}_2^{(1)} &= -\sigma_1 \lambda_1^{(1)} - \tau_1 \gamma_1^{(1)}, \end{aligned} \tag{78}$$

while for $\mathbf{j = 2}$ we get

$$\begin{aligned} \dot{z}_1^{(2)} &= \tau_2 z_2^{(2)} + \sigma_2 x_2^{(2)}, \quad & \dot{z}_2^{(2)} &= \tau_2 u_M^{(2)} + \sigma_2 u^{(2)}, \\ \dot{\gamma}_1^{(2)} &= 0, & \dot{\gamma}_2^{(2)} &= -\sigma_2 \lambda_1^{(2)} - \tau_2 \gamma_1^{(2)}. \end{aligned} \tag{79}$$

In these equations, the control sensitivities have to be substituted from relations (77). Note that the sensitivity derivatives σ_j are treated as free variables which are determined by the following boundary conditions. Namely, the boundary conditions (61), (62) and the continuity of the state at τ_1 yield conditions for the sensitivity differentials of the state variables:

$$\begin{array}{lll} j=1: & z_1^{(1)}(0)=z_2^{(1)}(0)=0, & z_1^{(1)}(1)=0 \\ j=2: & z_1^{(2)}(0)=0,\ z_2^{(2)}(0)=z_2^{(1)}(1), & z_1^{(2)}(1)=z_2^{(2)}(1)=0. \end{array} \tag{80}$$

Boundary values for the sensitivities of adjoint variables are deduced from a differentiation of relations (67) and (71):

$$\gamma_2^{(1)}(0)=-2, \quad \gamma_2^{(2)}(0)=\gamma_2^{(1)}(1), \quad \gamma_2^{(2)}(1)=0. \tag{81}$$

Note that for the computation of $\gamma_2^{(1)}(0)=-2$ we have used the value $\lambda_2^{(1)}(0)=-4$ from (72). Again, the routine BNDSCO of Oberle and Grimm (1989) is a convenient method to solve the BVP in (77)–(81). The solution is characterized by the initial and final values complementary to (80) and (81) and by the free variables σ_1,σ_2 :

$$\begin{array}{ll} z_2^{(1)}(1)=z_2^{(2)}(0)=0.13892468, & \sigma_1=0.63394569, \quad \sigma_2=0.15523510, \\ \gamma_1^{(1)}(0)=\gamma_1^{(1)}(1)=-0.27594324, & \gamma_2^{(2)}(0)=\gamma_2^{(1)}(1)=0.12583048, \\ \gamma_1^{(2)}(0)=\gamma_1^{(2)}(1)=-0.01288298. & \end{array} \tag{82}$$

The sensitivity differentials are shown in Fig. 4.

6.4. Real time control

In practice, sensitivity differentials can be used to approximate perturbed solutions by first order Taylor approximations which are computable in real–time. Details of this real–time approach are to be found in, e.g., Büskens and Maurer (2000), Maurer and Pesch (1994). Let us demonstrate the quality of such an approach by selecting the switching times $\tau_1(M),\tau_2(M)$ as candidates for real–time approximation. Assume that the nominal load $M_0=1$ is perturbed to $M=1.05$. We wish to compare the approximation with the exact solution for the perturbed value $M=1.05$ and compute:

$$\begin{array}{lll} M_0=1 & : & \tau_1(M_0)=4.14790960, \quad \tau_2(M_0)=3.89776004, \\ M=1.05 & : & \tau_1(M)=4.17956227, \quad \tau_2(M)=3.90516036\,. \end{array}$$

Using the sensitivities σ_1,σ_2 from (82) we compute the Taylor approximation for $M=1.05$ as

$$\tau_1(M)\approx\tau_1(M_0)+\frac{\partial\tau_1}{\partial M}(M_0)(M-M_0)=\tau_1(M_0)+0.05\,\sigma_1=4.17960688,$$

$$\tau_2(M)\approx\tau_2(M_0)+\frac{\partial\tau_2}{\partial M}(M_0)(M-M_0)=\tau_2(M_0)+0.05\,\sigma_2=3.90552180\,.$$

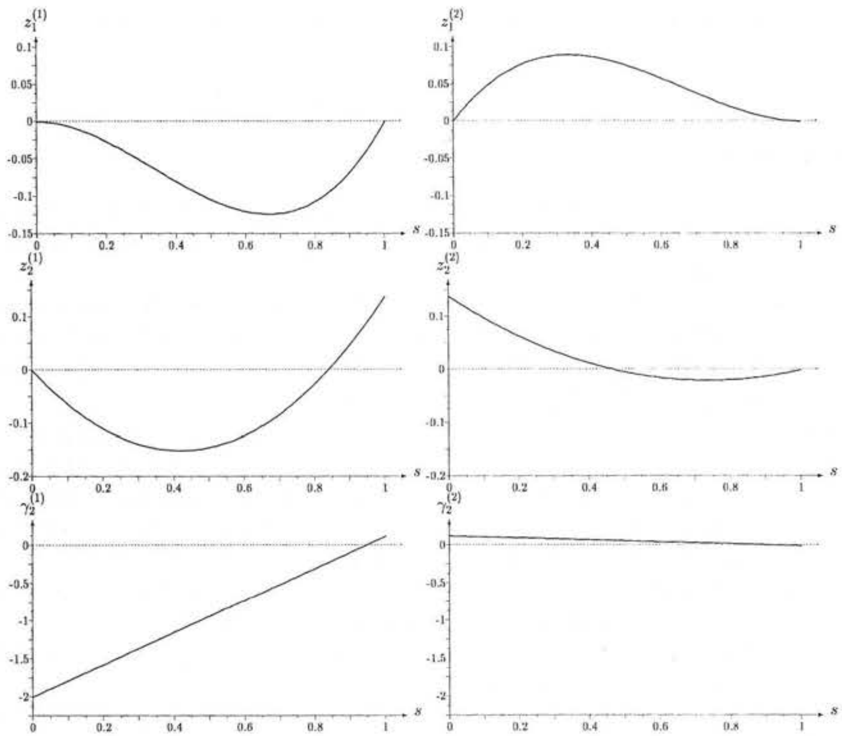

Figure 4. Sensitivity differentials of state and adjoint variables for $m = 1, M_0 = 1$ with $j = 1$ (left) and $j = 2$ (right).

The reader may verify that these values are indeed very good estimates for the exact values. The following estimates for the control approximations confirm also the quality of the real–time approximation for all $s \in [0,1]$:

$$\max_{s\in[0,1]} |\widetilde{u}^{(1)}(s;M) - u^{(1)}(s;M)| = 2.5497 \cdot 10^{-4}\,,$$

$$\max_{s\in[0,1]} |\widetilde{u}^{(2)}(s;M) - u^{(2)}(s;M)| = 1.5501 \cdot 10^{-4}\,,$$

$$\widetilde{u}^{(j)}(s;M) := u^{(j)}(s;M_0) + (M - M_0)\frac{\partial u^{(j)}}{\partial M}(s;M_0),\ j = 1,2\,.$$

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