Dedicated to
Professor Jakub Gutenbaum
on his 70th birthday

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Remarks on the steady-state accuracy of a feedback control system

by

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Abstract: The paper deals with the steady-state response of a feedback control system to the canonical inputs. For its characterization it seems useful to introduce the notion of accuracy index μ beside the standard notion of loop type ν. This index is assumed to be equal to the power of t in the analytic expression of the canonical input that leads to a finite nonzero deviation between the actual and the desired responses. When applied in the control design procedure, the accuracy index μ allows to achieve a steady-state performance that is more satisfactory than the one obtainable with reference to the loop type only. The conditions under which a single-loop feedback control system exhibits a prescribed value of μ, given the value of ν, are derived and discussed with particular regard to their robustness.

Keywords: feedback control systems, steady-state response, robustness.

1. Introduction and problem statement

The concepts of type, error coefficients and steady-state error have long been in use in the study of the asymptotic performance of feedback control systems (see, e.g., Horowitz, 1963, Melsa and Schultz, 1969, D'Azzo and Houpis, 1988, Sinha, 1994). Nevertheless, they have been reconsidered with attention in the recent literature, see, e.g., Weiss (1995). In the following, some further remarks about the *loop error* (*actuating signal* in the ANSI nomenclature, ANSI, 1963) and the difference between the desired output and the actual output (*deviation*, ANSI, (1963) will be made. The two quantities obviously coincide when reference is made to the elementary control configuration of Fig. 1, where the controlled variable $c(t)$ is expected to track the reference input $r(t)$.

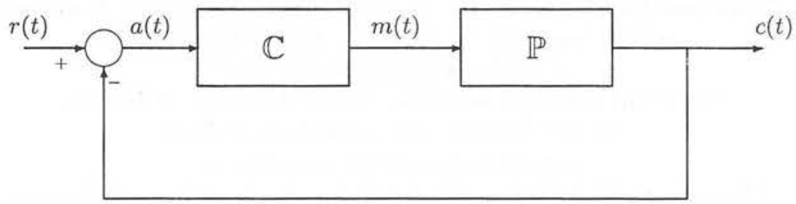

Figure 1. Unity-feedback control system: $\mathbb{C}$ and $\mathbb{P}$ represent, respectively, the controller and the process; $a(t)$ and $m(t)$ are, respectively, the actuating signal and the manipulated variable.

Instead, the deviation differs from the actuating signal $a(t)$ in the more general configuration of Fig. 2, where, besides the forward-path controller $\mathbb{C}$, a (stable) prefilter $\mathbb{F}$ and a feedback-path element $\mathbb{H}$ are present; the parallel dashed-line path in the same figure accounts for the desired relationship $\mathbb{D}$ between the reference signal and the controlled variable; sometimes, $\mathbb{D}$ does not correspond to a constant equal to 1.

This paper deals with linear time-invariant models of the system component parts, which can then be described by the transfer functions relating their input and output Laplace transforms. Specifically, the transfer functions associated with blocks $\mathbb{F}$, $\mathbb{C}$, $\mathbb{P}$ and $\mathbb{H}$ will be denoted by $F(s)$, $G_c(s)$, $G_p(s)$ and $H(s)$. Therefore, by letting:

$$G(s) := G_c(s)G_p(s), \tag{1}$$

the overall transfer function becomes:

$$W(s) := \frac{F(s)G(s)}{1 + G(s)H(s)}. \tag{2}$$

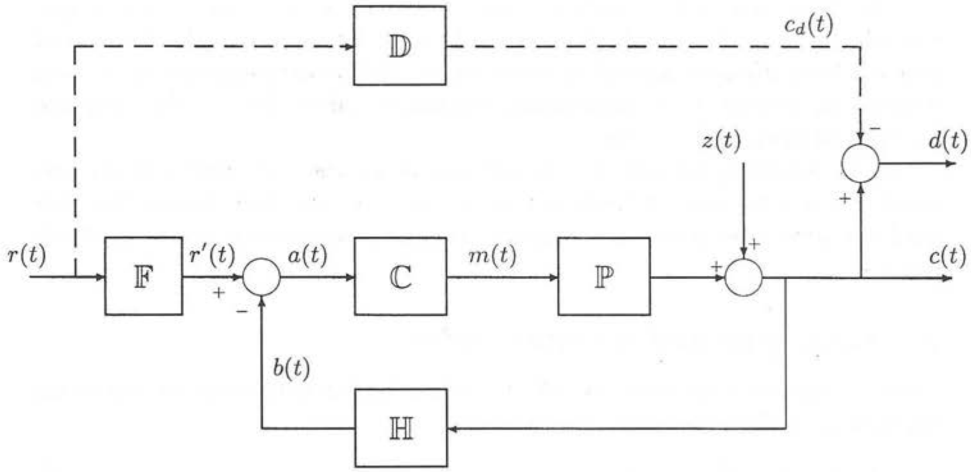

Figure 2. Nonunity-feedback control system with prefilter: $\mathbb{F}$ and $\mathbb{H}$ represent, respectively, the prefilter and the feedback-path element; $\mathbb{D}$ accounts for the desired relation between $r(t)$ and $c(t)$, and $d(t)$ is the deviation according to the ANSI terminology; $z(t)$ is an (equivalent) disturbance acting on the process output.

The actual controlled output $c(t)$ is influenced both by the reference signal $r(t)$ and by the (equivalent) disturbance $z(t)$. The controlled output does not coincide, in general, with the output $c_d(t)$ of block $\mathbb{D}$, which is related to $r(t)$ via the "desired" transfer function:

$$W_d(s) := \frac{C_d(s)}{R(s)}, \tag{3}$$

where Laplace transforms are indicated, as usual, by the capitals of the symbols representing time functions. The transform $D(s)$ of the deviation $d(t)$ is obviously:

$$D(s) = C(s) - C_d(s). \tag{4}$$

In fact, the design specifications usually refer to $D(s)$ and not to $A(s)$, which differs from $D(s)$ due to the presence of a prefilter and that of a feedback element accounting both for the output transducer and for the feedback part of the controller. Note, by the way, that some authors include in the process transfer function those of the power amplifiers and actuators on the one side and those of the measuring devices on the other: in this case, the controlled output is actually the transducer output whose physical nature is the same as that of the reference signal $r(t)$. Clearly, the gain of the transducer transfer function depends on the choice of the units of measure.

The above viewpoint naturally suggests to contrast the steady-state properties related to the loop with those related to the entire system. In this regard, type is a loop property depending solely on the number of integrators in the loop, whereas the possibility of annihilating the output offset ($\lim_{t\to\infty} d(t)$) depends on the overall control system.

In the following sections, the relations between the two properties are analyzed. To this purpose, it turns out to be useful to introduce, beside the standard notion of type, a new index characterizing the asymptotic system accuracy (a.s.a.).

2. Loop type and accuracy index

From the linearity assumption, the controlled output is the sum of two terms, depending on $R(s)$ and $Z(s)$, respectively:

$$C(s) = C_R(s) + C_Z(s) \tag{5}$$

with

$$C_R(s) := W(s)R(s) \tag{6}$$

and

$$C_Z(s) := W_Z(s)Z(s), \tag{7}$$

where

$$W_Z(s) = \frac{1}{1 + G(s)H(s)}. \tag{8}$$

Therefore, by taking (3) into account, $D(s)$ in (4) can be expressed as:

$$D(s) = D_R(s) + C_Z(s) \tag{9}$$

with

$$D_R(s) := [W(s) - W_d(s)]\ R(s). \tag{10}$$

It is usual to develop the a.s.a. analysis with regard to the canonical inputs and disturbances, whose transforms are

$$\frac{1}{s^{q+1}}, \quad q = 0, 1, 2, \ldots. \tag{11}$$

By limiting first attention to the component $D_R(s)$ of $D(s)$, and assuming that $W(s)$ and $W_d(s)$ are BIBO stable (so that the (unique) pole of $R(s)$ in the origin is not a pole of $W(s) - W_d(s)$), then $D_R(s)$ can be decomposed into a steady-state (or asymptotic) component $D_{R_{ss}}(s)$ and a transient component $D_{R_{tr}}(s)$ as:

$$D_R(s) = D_{R_{ss}}(s) + D_{R_{tr}}(s), \tag{12}$$

where the denominator of $D_{R_{ss}}(s)$ is s^{q+1}, Dorato, Lepschy and Viaro (1994).

In order to analyse the steady-state performance, it is particularly useful to consider the MacLaurin series expansions of the relevant transfer functions because of the link between the behavior at $t = \infty$ of time functions and that at $s = 0$ of their transforms (final-value theorem). In this way, it is very easy to derive conditions regarding the system accuracy; they will then be converted into conditions on parameters of greater interest for the designer, like Bode gains and time constants.

To this purpose, let us denote the MacLaurin expansions of $W(s)$ and $W_d(s)$ as:

$$W(s) = \sum_{i=0}^{\infty} a_i s^i, \tag{13}$$

$$W_d(s) = \sum_{i=0}^{\infty} d_i s^i. \tag{14}$$

From (10) with $R(s) = 1/s^{q+1}$ we get:

$$D_R(s) = \sum_{i=0}^{\infty} (a_i - d_i) s^{i-q-1} \tag{15}$$

and, thus, the asymptotic component corresponds to:

$$D_{R_{ss}}(s) = \sum_{i=0}^{q} \frac{a_i - d_i}{s^{q+1-i}} = \frac{a_0 - d_0}{s^{q+1}} + \frac{a_1 - d_1}{s^q} + \cdots + \frac{a_q - d_q}{s}, \tag{16}$$

whose inverse transform is the polynomial:

$$d_{R_{ss}}(t) = (a_0 - d_0)\frac{t^q}{q!} + (a_1 - d_1)\frac{t^{q-1}}{(q-1)!} + \cdots + (a_q - d_q), \quad t > 0. \tag{17}$$

It follows that, in order for $d_{R_{ss}}(t)$ to be finite, the following q conditions must be satisfied:

$$a_i = d_i, \qquad i = 0, 1, \ldots, q-1. \tag{18}$$

In this case, the *offset* in the response to $r(t) = \frac{t^q}{q!}$, $t \geq 0$, is given by:

$$d_{R_{ss}} := d_R(\infty) = a_q - d_q \tag{19}$$

which is zero if and only if, in addition to (18), $a_q = d_q$.

When $W_d(s)$ is a constant d_0, the above conditions on the expansion coefficients of $W(s)$ can easily be converted into conditions on the numerator and denominator coefficients of the expression of $W(s)$ as a ratio of polynomials:

$$W(s) = \frac{\sum_{i=0}^{n-1} p_i s^i}{\sum_{i=0}^{n} q_i s^i}. \tag{20}$$

In fact, equations (18) and (19) become, respectively:

$$p_i = d_0 q_i, \qquad i = 0, 1, \ldots, q-1, \tag{21}$$

and

$$d_{R_{ss}} = \frac{p_q - d_0 q_q}{q_0}. \tag{22}$$

Similarly, concerning the disturbance-dependent component $C_Z(s)$ of $D(s)$, by expressing $W_Z(s)$ as

$$W_Z(s) = \sum_{i=0}^{\infty} z_i s^i, \tag{23}$$

and setting $Z(s) = 1/s^{q+1}$, from (7) we get

$$C_Z(s) = \sum_{i=0}^{\infty} z_i s^{i-q-1}, \tag{24}$$

so that the conditions for its asymptotic component:

$$c_{Z_{ss}}(t) = \mathcal{L}^{-1}\left[\sum_{i=0}^{q} \frac{z_i}{s^{q+1-i}}\right] \tag{25}$$

to be finite and different from zero are:

$$z_i = 0, \qquad i = 0, 1, \ldots, q-1, \tag{26}$$

and

$$c_{Z_{ss}} := c_Z(\infty) = z_q \neq 0. \tag{27}$$

This clearly implies that $W_Z(s)$ has a zero of multiplicity q in the origin.

On the basis of the previous considerations concerning the offset $d_{R_{ss}}$ in the response to canonical reference inputs, it seems useful to introduce the following notion of *accuracy* (or *a.s.a.*) *index*:

DEFINITION 2.1 *The a.s.a. index μ is the power of t in the time-domain expression:*

$$\delta_{-(\mu+1)}(t) := \frac{t^{\mu}}{\mu!}, \qquad t > 0, \tag{28}$$

of the canonical reference signal that gives rise to a finite nonzero offset $d_{R_{ss}}$.

According to (18) and (19), the index μ coincides with the number of consecutive coefficients a_i that are equal to the corresponding coefficients d_i.

Obviously, for canonical input signals characterized by powers of t less than μ the steady-state deviation is (identically) zero, whereas for powers greater than μ the deviation tends to infinity.

Let us recall that the *looptype* is usually defined as the multiplicity ν of the pole at $s = 0$ of the loop transfer function. Since in control systems the feedback-path transfer function $H(s)$ does not contain poles (or zeros) in the origin, the above definition concerns the forward-path transfer function $G(s)$ only.

It follows that the type coincides with:
- the multiplicity of the zero at $s = 0$ of the transfer function between the input to the loop (the reference signal $r(t)$ in Fig. 1 or the prefilter output $r'(t)$ in Fig. 2) and the comparator output $a(t)$ (the actuating signal), and
- the power of t in the analytic expression of the canonical loop input giving rise to a finite nonzero steady-state value of $a(t)$.

Indices μ and ν are equal in the case of the unity-feedback system of Fig. 1, where it is implicitly assumed that $W_d(s) = d_0 = 1$. They may not coincide for the system represented in Fig. 2, because $F(s)$, $H(s)$ and $W_d(s)$ are not necessarily equal to 1.

As far as the design problem is concerned, μ can be considered as a specification and ν as a tool for achieving the desired value of μ. Therefore, the case of $\mu < \nu$ is not of interest. On the other hand, stability considerations often limit the value of ν, which motivates the search for methods to obtain $\mu > \nu$.

For instance, by assuming $W_d(s) = 1$, index μ can be made equal to 1 with $\nu = 0$ by setting:

$$F(0) = \frac{1 + G(0)}{G(0)}, \qquad H(0) = 1, \tag{29}$$

or:

$$H(0) = \frac{G(0) - 1}{G(0)}, \qquad F(0) = 1, \tag{30}$$

since, according to the final value theorem, in both cases the asymptotic value $w_{-1}(\infty)$ of the step response is:

$$w_{-1}(\infty) = W(0) = \frac{F(0)G(0)}{1 + G(0)H(0)} = 1 = W_d(0). \tag{31}$$

Obviously, not all methods for achieving the desired value of μ are equally robust. For example, (29) or (30) are no longer satisfied if the Bode gain $G_p(0)$ of the process (and thus $G(0)$) varies whereas, if $G_c(s)$ introduces a pole at $s = 0$ and $F(0) = H(0) = 1$, then we have $\mu = \nu = 1$ independently of $G_p(0)$.

In this regard, however, it might be argued that index μ becomes zero if the considered pole of $G_c(s)$ is not exactly in the origin. This happens in practice due to the limitations in the physical realizability of controllers, Dorato et al. (1999), Keel and Bhattacharyya (1997), and the necessity of meeting other design requirements. Specifically, if the forward-path transfer function of a unity-feedback system without prefilter is:

$$G_\varepsilon(s) = \frac{K}{s+\varepsilon} G'(s), \qquad G'(0) = 1, \quad \varepsilon > 0, \tag{32}$$

instead of

$$G(s) = \frac{K}{s} G'(s), \tag{33}$$

then the output offset for a unit step input becomes:

$$d_{R_{ss}}(\varepsilon) = -\frac{\varepsilon}{\varepsilon + K}, \tag{34}$$

whose diagram has a slope equal to $-\frac{1}{K}$ at $\varepsilon = 0$.

This situation does not substantially differ from that occurring when the *nominal* forward-path transfer function is

$$G(s) = KG'(s), \qquad G'(0) = 1, \tag{35}$$

(so that $\nu = 0$) if $F(0) = 1$ and the feedback-path transfer function gain is set to

$$H(0) = \frac{K-1}{K} \tag{36}$$

(so that, according to (30), $\mu = 1$).

A variation of the forward-path transfer function gain from K to $K(1+\varrho)$, with $H(0)$ still given by (36), causes the output offset corresponding to a unit step input to become

$$d_{R_{ss}}(\varrho) = \frac{\varrho}{K + (K-1)\varrho}, \tag{37}$$

whose slope at $\varrho = 0$ is $\frac{1}{K}$, so that the effects of *small* gain variations are equivalent to those of small variations of in the position of the pole considered.

Similar considerations could be made with reference to the disturbance-dependent component of the steady-state deviation.

3. Conditions to achieve the desired accuracy index

The purpose of this section is to give conditions on $F(s)$, $G(s)$ and $H(s)$ under which a system of type ν exhibits accuracy index $\mu \geq \nu$. These conditions immediately follow from the relations:

$$a_i = d_i, \quad i = 0, 1, \ldots, \mu - 1, \tag{38}$$

$$a_\mu \neq d_\mu, \tag{39}$$

derived in the previous section.

We shall refer to the power series expansions:

$$F(s) = \sum_{i=0}^{\infty} f_i s^i, \tag{40}$$

$$G(s) = \frac{1}{s^\nu}\hat{G}(s) = \frac{1}{s^\nu}\sum_{i=0}^{\infty} g_i s^i, \tag{41}$$

$$H(s) = \sum_{i=0}^{\infty} h_i s^i, \tag{42}$$

where $F(0) = f_0 \neq 0$, $\hat{G}(0) = g_0 \neq 0$ and $H(0) = h_0 \neq 0$, because this choice minimizes the number of parameters involved; in the next section, however, the conditions will be referred to parameters that are more meaningful from the designer's viewpoint, like time constants and Bode gains.

The expressions of the expansion coefficients a_i of (13) in terms of the expansion coefficients f_i, g_i and h_i can easily be obtained, e.g. by resorting to the Padé technique (see Appendix A).

Here, we only consider the most interesting cases, i.e., the case of $W_d(s) = 1$ and $F(s) = 1$ and the case of $W_d(s) = 1$ and $H(s) = 1$, with $\nu = 0, 1, 2$.

Table 1 provides the constraints relating parameters h_i to parameters g_i for obtaining the desired value of $\mu \geq \nu$ when $F(s) = 1$.

	$\nu = 0$	$\nu = 1$	$\nu = 2$
$\mu = 0$	$\left(h_0 \neq 1 - \frac{1}{g_0}\right)$	–	–
$\mu = 1$	$h_0 = 1 - \frac{1}{g_0}$ $\left(h_1 \neq \frac{g_1}{g_0^2}\right)$	$h_0 = 1$ $\left(h_1 \neq -\frac{1}{g_0}\right)$	–
$\mu \geq 2$	$h_0 = 1 - \frac{1}{g_0}$ $h_1 = \frac{g_1}{g_0^2}$	$h_0 = 1$ $h_1 = -\frac{1}{g_0}$	$h_0 = 1$ $h_1 = 0$

Table 1. Conditions on parameters h_i to achieve the desired accuracy index μ, given the loop type ν.

Note that, if the inequalities between round brackets are not satisfied, the accuracy index increases. Therefore, the number of "true" constraints is equal to μ (independently of the value of ν) as well as to the number of parameters h_i that are present in the constraint equations.

	$\nu = 0$	$\nu = 1$	$\nu = 2$
$\mu = 0$	$\frac{1-\left(\frac{1}{g_0}+h_0\right)}{\frac{1}{g_0}+h_0}$	–	–
$\mu = 1$	$\frac{\frac{g_1}{g_0^2}-h_1}{\left(\frac{1}{g_0}+h_0\right)^2}$	$-\left(\frac{1}{g_0}+h_1\right)$	–
$\mu = 2$	$\frac{\frac{g_2}{g_0^2}-\frac{g_1^2}{g_0^3}-h_2}{\left(\frac{1}{g_0}+h_0\right)^3}$	$\frac{g_1}{g_0^2}-h_2$	$-\left(\frac{1}{g_0}+h_2\right)$

Table 2. Steady-state deviations $d_{R_{ss}}$ for the case of $F(s) = 1$.

The nonzero offset $d_{R_{ss}}$ in the response to the relevant input (i.e., the step for $\mu = 0$, the ramp for $\mu = 1$, etc.) is given in Table 2.

Similarly, Table 3 provides the conditions on the parameters f_i of the prefilter transfer function $F(s)$ when $H(s) = 1$. The corresponding offsets are given in Table 4.

	$\nu = 0$	$\nu = 1$	$\nu = 2$
$\mu = 0$	$\left(f_0 \neq 1+\frac{1}{g_0}\right)$	–	–
$\mu = 1$	$f_0 = 1+\frac{1}{g_0}$ $\left(f_1 \neq -\frac{g_1}{g_0^2}\right)$	$f_0 = 1$ $\left(f_1 \neq -\frac{1}{g_0}\right)$	–
$\mu = 2$	$f_0 = 1+\frac{1}{g_0}$ $f_1 = -\frac{g_1}{g_0^2}$	$f_0 = 1$ $f_1 = -\frac{1}{g_0}$	$f_0 = 1$ $f_1 = 0$

Table 3. Conditions on parameters f_i to achieve the desired accuracy index μ given the loop type ν.

Concerning the robustness of the above conditions, it should be noticed that it is usually possible to assign precise values to the filter and controller parameters f_i and h_i, whereas parameters g_i depend on the process and their values are often uncertain. If, however, the process is time-invariant, the controller parameters can be calibrated once for all.

	$\nu = 0$	$\nu = 1$	$\nu = 2$
$\mu = 0$	$\frac{f_0 g_0 - g_0 - 1}{1 + g_0}$	–	–
$\mu = 1$	$\frac{g_1 + f_1 g_0^2}{g_0(1 + g_0)}$	$f_1 - \frac{1}{g_0}$	–
$\mu = 2$	$\frac{g_0 g_2 - g_1^2 + f_2 g_0^3}{g_0^2(1 + g_0)}$	$f_2 + \frac{g_1}{g_0^2}$	$f_2 - \frac{1}{g_0}$

Table 4. Steady-state deviations $d_{R_{ss}}$ for the case of $H(s) = 1$.

The previous considerations concern the situation in which $W_d(s) = 1$. If this is not the case, from the power series expansion of $W_d(s)$ and relations (38), it is also possible to obtain conditions similar to those in Tables 1 and 3, and the corresponding offset values.

For instance, if it is required to attenuate high-frequency noise, reference could be made (see, e.g., Netushil, 1978) to:

$$W_d(s) = \frac{1}{1 + \tau_d s} \tag{43}$$

whose 3 dB pass-band is $B = \frac{1}{2\pi\tau_d}$. For such a $W_d(s)$, we have:

$$d_i = (-\tau_d)^i, \quad i = 0, 1, 2, \dots \tag{44}$$

so that, if $F(s) = 1$, the conditions for obtaining $\mu \geq 2$ with $\nu = 1$ are:

$$h_0 = 1, \tag{45}$$

$$h_1 = \tau_d - \frac{1}{g_0}, \tag{46}$$

instead of those given in Table 1.

4. Constraints on parameters of more practical interest

As already said, in most practical cases it is preferable to translate the conditions on the power series expansion coefficients of the relevant transfer functions into conditions on parameters having a more direct physical meaning, e.g., time constants and Bode gains.

To this purpose, reference will be made to the control scheme of Fig. 2 with $W_d(s) = F(s) = 1$. Let the process transfer function be approximated by:

$$G_p(s) = K_p \frac{e^{-\Delta s}}{1 + Ts}, \tag{47}$$

as is often done in standard design procedures, and assume that the forward-path controller transfer function is:

$$G_c(s) = K_c \frac{(1+T_1 s)(1+T_2 s)}{s^\nu} \tag{48}$$

and the feedback-path transfer function is:

$$H(s) = K_H \frac{1+T_n s}{1+T_d s}. \tag{49}$$

Expression (48) particularizes to:
the transfer function of a P (proportional) controller for $\nu = 0$ and $T_1 = T_2 = 0$, that of a PD controller for $\nu = 0$, $T_1 \neq 0$ and $T_2 = 0$,
that of a PI controller for $\nu = 1$, $T_1 \neq 0$ and $T_2 = 0$, and to that of a PID controller for $\nu = 1$, $T_1 \neq 0$ and $T_2 \neq 0$.

Table 5 provides the values of the steady-state deviation $d_{R_{ss}}$ for the combinations of practical interest of indices μ and ν; K_G represents the Bode gain of the forward-path transfer function, i.e.:

$$K_G = K_c K_p. \tag{50}$$

	$\nu = 0$	$\nu = 1$
$\mu = 0$	$\frac{K_G - 1 - K_G K_H}{1 + K_G K_H}$	–
$\mu = 1$	$\frac{T_1 + T_2 - (\Delta + T) + (T_d - T_n)(K_G - 1)}{K_G}$	$\frac{K_G K_H (T_d - T_n) - 1}{K_G}$

Table 5. Steady-state deviations $d_{R_{ss}}$ in terms of the parameters characterizing functions $G_p(s)$, $G_c(s)$ and $H(s)$.

From Table 5 the conditions for obtaining the desired value of the accuracy index μ, given ν, can also be derived by setting to zero the relevant expressions. For example, to obtain $\mu = 1$ with $\nu = 0$, it is necessary to annihilate the numerator of the offset corresponding to $\mu = 0$ and $\nu = 0$:

$$K_G - 1 - K_G K_H = 0, \tag{51}$$

from which:

$$K_H \doteq \frac{K_G - 1}{K_G}. \tag{52}$$

To obtain $\mu = 2$ with $\nu = 0$, besides (51), the following condition must be satisfied:

$$T_1 + T_2 - (\Delta + T) + (T_d + T_n)(K_G - 1) = 0, \tag{53}$$

from which

$$T_d - T_n = \frac{\Delta + T - (T_1 + T_2)}{K_G - 1}. \tag{54}$$

Since usually $K_G >> 1$, if $\Delta + T > T_1 + T_2$ we can choose $T_n = 0$ so that $H(s)$ is simply a first-order low-pass filter.

When, instead, $\nu = 1$, the accuracy index can take the value 2 if:

$$K_G K_H (T_d - T_n) - 1 = 0. \tag{55}$$

This condition can be satisfied, e.g., by either:

$$K_H = \frac{1}{K_G}, \quad T_n = 0, \quad T_d = 1, \tag{56}$$

or:

$$K_H = 1, \quad T_n = 0, \quad T_d = \frac{1}{K_G}. \tag{57}$$

Of course, the above conditions hold in practice if the first expansion coefficients of (47) coincide with the corresponding coefficients of the actual process transfer function. If this is not true, the accuracy index decreases but the deviation is usually much smaller than that corresponding to $H(s) = 1$, as shown in the following section; moreover, it can still be annihilated by slightly modifying (calibrating) the relevant controller parameters.

5. Sensitivity considerations

Let us refer to Fig. 2 with $F(s) = W_d(s) = 1$ and $\nu = 0$.

When $H(s) = 1$ and the forward-path gain K_G changes from $\hat{K}_G$ to $\hat{K}_G(1 + \varrho)$, the deviation $d_{R_{ss}}$ becomes

$$\hat{d}_{R_{ss}}(\varrho) = -\frac{1}{1 + \hat{K}_G + \varrho \hat{K}_G}, \tag{58}$$

instead of

$$\hat{d}_{R_{ss}} = -\frac{1}{1 + \hat{K}_G}, \tag{59}$$

which is not appreciably different form (58) for small values of ϱ.

If, instead:

$$H(s) = \frac{\hat{K}_G - 1}{\hat{K}_G}, \tag{60}$$

the deviation, which for $K_G = \hat{K}_G$ is equal to zero, for $K_G = \hat{K}_G(1+\varrho)$ becomes:

$$\tilde{d}_{R_{ss}}(\varrho) = -\frac{\varrho}{-\varrho + \hat{K}_G + \varrho\hat{K}_G}. \tag{61}$$

Since K_G is usually large, the value (61) is about ϱ times the value (58). For instance, if $\hat{K}_G = 10$ and $\varrho = -0.2$, we get:

$$\hat{d}_{R_{ss}} = -\frac{1}{11}, \tag{62}$$

$$\hat{d}_{R_{ss}}(-0.2) = -\frac{1}{9}, \tag{63}$$

$$\tilde{d}_{R_{ss}}(-0.2) = -\frac{1}{41}. \tag{64}$$

Value (64) is about 0.2 times values (62) and (63).

It follows that, by keeping $H(s)$ equal to (60), the accuracy index μ is no longer 1 if K_G changes, but the resulting deviation is much smaller than the one with $H(s) = 1$.

Similar considerations can be made to obtain $\mu = 2$ with $\nu = 1$, even if in this case $H(s)$ cannot be a constant, i.e., $T_d \neq T_n$ in (55).

More demanding constraints must be met when $\mu - \nu > 1$. For instance, to obtain $\mu = 2$ with $\nu = 0$, the parameters of $H(s)$ must be linked not only to K_G but also to the parameters characterizing the dynamics of the forward path. Among these, the controller parameters T_1 and T_2 can be realized accurately, whereas the process parameters Δ and T are usually uncertain.

6. Improving the accuracy index by gain adjustment

As previously seen, the value of the a.s.a. index of a type-0 system can be brought to 1 by properly calibrating either the prefilter gain K_F or the gain K_H of the feedback element. However, if the forward path gain $K_G = K_c K_p$ is, or becomes, different from its nominal value, index μ decreases to 0 (even if the resulting system offset is small compared to that of the system without prefilter).

This suggests resorting to an adaptive scheme for suitably modifying the prefilter gain. Such a solution, however, would imply process identification, which is generally not convenient, Ilchmann (1993).

Here, we outline a different solution based on the knowledge of the instantaneous values of $a(t)$ and $c(t)$, whose ratio tends asymptotically to the actual forward path gain (assuming stability).

To illustrate this approach, we refer to the case of $W_d(s) = 1$ and to a unity feedback system with $\nu = 0$ preceded by a static block whose gain can continuously be adjusted. The same procedure can easily be applied to the feedback element K_H of a nonunity feedback system without prefilter; less simple is its extension to the situation in which $\mu = 2$ with $\nu = 0$.

By recalling that, if $K_F = 1$, in the considered case we have $d(t) = c(t) - r(t) = -a(t)$, so that:

$$d_{R_{ss}} = -\frac{1}{1 + K_G}, \tag{65}$$

it is clear that the offset cannot be annihilated with K_c finite, whereas by setting:

$$K_F = 1 + \frac{1}{K_G}, \tag{66}$$

the overall transfer function $W(s)$ equals 1 for $s = 0$, so that $d_{R_{ss}} = 0$.

Since the steady-state value $c_{ss} := \lim_{t \to \infty} c(t)$ in the step response is given by:

$$c_{ss} = K_G a_{ss}, \tag{67}$$

where $a_{ss} := \lim_{t \to \infty} a(t)$, it seems reasonable to replace the constant K_G in the expression of K_F by its "current approximation":

$$k(t) := \frac{c(t)}{a(t)}. \tag{68}$$

In this way the gain of the block preceding the feedback loop becames:

$$K_F(t) = 1 + \frac{1}{k(t)} \tag{69}$$

which, obviously, leads to a *nonlinear* relation.

By taking into account that $c(0) = 0$ (and $c(t)$ is small when t is small), it is necessary to modify the previous equation, e.g. by setting:

$$K_F(t) = 1 + \frac{1}{\bar{k}}, \quad c(t) < \bar{k}a(t), \tag{70}$$

$$K_F(t) = 1 + \frac{a(t)}{c(t)}, \quad c(t) > \bar{k}a(t), \tag{71}$$

with a suitable $\bar{k}$.

As an alternative, one may set:

$$K_F(t) = 1 + \frac{a(t)}{r(t)}, \tag{72}$$

so that:

$$r'(t) = K_F(t)r(t) = r(t) + a(t), \tag{73}$$

which gives rise to a *linear* relationship between the system variables: the substitution of $c(t)$ with $r(t)$ is based on the consideration that $c(\infty) = r(\infty)$ in the step response with the chosen system structure. In fact, the last relation entails an algebraic loop. If it is replaced by suitably fast dynamic loop, solution (72) becomes practically equivalent to integrating $a(t)$; if the stability margin is large enough, the stability of the overall system may be guaranteed.

7. Conclusions

The loop type ν does not always account in a proper way for the steady-state performance of a feedback control system with respect to the canonical inputs.

It has therefore been suggested to consider, besides the loop type ν, the accuracy index μ, which corresponds to the power of t in the analytic expression of the canonical input that produces a finite nonzero deviation between the actual response and the desired response.

The conditions for obtaining a prescribed value of $\mu \geq \nu$, given the value of ν, have been given, together with the expressions of the corresponding deviations for feedback systems with either a prefilter or a nonunity feedback-path transfer function.

It has been shown that, even if the conditions for obtaining a value of μ greater than that of ν depend on the parameters of the controlled process (which are often not known precisely or are subject to variations), they are somehow robust in that the deviation of the actual response to $t^\mu/\mu!$, $t \geq 0$, from the desired response when such conditions are not exactly satisfied, is appreciably smaller than the deviation afforded by the system with $\mu = \nu$.

Finally, some considerations have been made concerning the possibility of obtaining the desired value of μ by means of a non-identifier-based adaptive scheme.

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A. Appendix

From expressions (2), (41), (40) and (42) the transfer function $W(s)$ can be written as:

$$W(s) = \frac{\sum_{i=0}^{\infty} \beta_i s^i}{\sum_{i=0}^{\infty} \alpha_i s^i}. \tag{74}$$

with:

$$\beta_i = \sum_{k=0}^{i} g_k f_{i-k}, \tag{75}$$

$$\alpha_i = \delta_{i,\nu} + \sum_{k=0}^{i} g_k h_{i-k}, \tag{76}$$

where $\delta_{i,\nu}$ equals 1 if $i = \nu$ and 0 otherwise. By equating the expressions (74) and (13) and employing the classical Padé procedure, the following relations are easily obtained for the coefficients of lower order:

$$a_0 = \frac{\beta_0}{\alpha_0}, \tag{77}$$

$$a_1 = \frac{\beta_1}{\alpha_0} - \frac{\beta_0\alpha_1}{\alpha_0^2}, \tag{78}$$

$$a_2 = \frac{\beta_2}{\alpha_0} - \frac{\beta_0\alpha_2 + \beta_1\alpha_1}{\alpha_0^2} + \frac{\alpha_1^2\beta_0}{\alpha_0^3}, \tag{79}$$

$$a_3 = \frac{\beta_3}{\alpha_0} - \frac{\beta_0\alpha_3 + \beta_1\alpha_2 + \beta_2\alpha_1}{\alpha_0^2} + \frac{\alpha_1^2\beta_1 + 2\alpha_1\alpha_2\beta_0}{\alpha_0^3} - \frac{\beta_0\alpha_1^3}{\alpha_0^4}. \tag{80}$$

When all the relevant transfer functions are rational, the same procedure can be used to express coefficients a_i in terms of the numerator and denominator polynomial coefficients of $G(s)$, $F(s)$ and $H(s)$.