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Professor Jakub Gutenbaum
on his 70th birthday

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Reachability and controllability of 2D positive linear systems with state feedbacks

by

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Abstract: It is shown that the reachability and controllability of positive 2D linear systems are not invariant under the state-feedbacks. By suitable choice of the state-feedbacks the unreachable positive 2D Roesser model can be made reachable and the controllable positive 2D Roesser model can be made uncontrollable.

Keywords:reachability, controllability, positive 2D Roesser model, state-feedback.

1. Introduction

The reachability and controllability are the basic concepts of modern control theory (Kaczorek, 1985, 1993, Klamka, 1991, 1993). The overviews of recent developments in reachability and controllability of 2D linear systems can be found in Klamka (1988, 1991, 1999), Kaczorek (1998). The positive (non-negative) 2D Roesser type model has been introduced in Kaczorek (1996) and its reachability and controllability have been considered in Kaczorek (1996, 1998). The spectral and combinatorial structure and asymptotic behaviour of 2D positive system has been investigated in Fornasini and Valcher (1995, 1996) and recent developments in 2D positive system theory are given in Fornasini and Valcher (1997). It is well-known, Kaczorek (1993), that the reachability and controllability of the standard linear systems are invariant under the state-feedbacks. Similar results are also valid for standard 2D linear systems (Kaczorek, 1993). It has been shown (Kaczorek, 1999) that the reachability and controllability of positive linear 1D systems are not invariant under the state-feedbacks. To the best author's knowledge the reachability and controllability of positive 2D linear systems with state feedbacks have not been considered yet. In this paper it will be shown that the reachability and controllability of the positive 2D linear systems described by the Roesser type model are not invariant under the state-feedbacks.

2. Necessary and sufficient conditions for the reachability and controllability of positive 2D linear systems

Let $Z_+ := \{0, 1, 2, \dots\}$ and $R_+^{n \times m}$ be the set of real matrices of the dimensions $n \times m$ with nonnegative entries ($R_+^n := R_+^{n \times l}$). Consider the 2D Roesser model (Roesser, 1975),

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^\nu \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^\nu \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij} \tag{1}$$

$$y_{ij} = [C_1 C_2] \begin{bmatrix} x_{ij}^h \\ x_{ij}^\nu \end{bmatrix} + D u_{ij}, \ i, j \in Z_+ \tag{2}$$

where $x_{ij}^h \in R^{n_1}$ and $x_{ij}^\nu \in R^{n_2}$ are the horizontal and vertical state vectors at point (i, j), respectively, $u_{ij} \in R^m$ is the input vector, $y_{ij} \in R^p$ is the output vector and $A_{kl} \in R^{n_t \times n_l}, B_k \in R^{n_k \times m}, C_k \in R^{p \times n_k}, k, l = 1, 2, D \in R^{p \times m}$. The model (1) is called internally positive (shortly positive) if for all boundary conditions

$$x_{0j}^h \in R_+^{n_1}, j \in Z_+ \quad \text{and} \quad x_{i0}^{n_2} \in R_+^{n_2}, i \in Z_+ \tag{3}$$

and all $u_{ij} \in R_+^m, i, j \in Z_+$ we have $x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^\nu \end{bmatrix} \in R_+^n, n = n_1 + n_2$ and $y_{ij} \in R_+^p$ for all $i, j \in Z_+$. It is easy to show, Kaczorek (1996) that the model (1) is positive if and only if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in R_+^{n \times n}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in R_+^{n \times m},$$

$$C = [C_1 C_2] \in R_+^{p \times n}, D \in R_+^{p \times m} \tag{4}$$

The transition matrix T_{ij} for (1) is defined as follows, Roesser (1975), Kaczorek (1985)

$$T_{ij} = \begin{cases} I_n \text{ (the identity matrix)} & \text{for} \quad i = j = 0 \\ T_{10} T_{i-1,j} + T_{01} T_{i,j-1} & \text{for} \quad i, j \geq 0 (i + j \neq 0) \\ T_{ij} = 0 \text{ (the zero matrix)} & \text{for} \quad i < 0 \text{ or/and } j < 0 \end{cases} \tag{5}$$

where

$$T_{10} := \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, T_{01} := \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

From (4) it follows that the transition matrix T_{ij} of the positive model (1) is a positive matrix, $T_{ij} \in R_+^{n\times n}$ for all $i, j \in Z_+$.

DEFINITION 2.1 *The positive model (1) is called reachable for zero boundary conditions (2) (ZBC) at point* $(h,k), (h, k \in Z_+, h, k > 0)$ *if for every* $x_f \in R_+^n$ *there exists a sequence of inputs* $u_{ij} \in R_+^m$ *for* $(i,j) \in D_{hk}$ *such that* $x_{hk} = x_f$, *where*

$$D_{hk} := \{(i,j) \in Z_+ \times Z_+ : 0 \le i \le h, 0 \le j \le k \ \ and \ \ i + j \ne h + k\} \quad (6)$$

DEFINITION 2.2 *The positive model (1) is called controllable to zero (shortly controllable) at point* $(h,k), (h, k \in Z_+, h, k > 0)$ *if for any nonzero boundary conditions*

$$x_{0j}^h \in R_+^{n_1}, 0 \le j \le k \ \ and \ \ x_{i0}^\nu \in R_+^{n_2}, 0 \le i \le h \quad (7)$$

there exists a sequence of inputs $u_{ij} \in R_+^m$ *for* $(i,j) \in D_{hk}$ *such that* $x_{hk} = 0$.

A matrix $A \in R^{n\times n}$ is called the generalised positive permutation matrix (GPPM) or monomial matrix if and only if it has only one positive entry in each row and column and the remaining entries are equal zero. In Kaczorek (1996, 1998a, b) the following necessary and sufficient conditions for reachability and controllability have been proved:

THEOREM 2.1 *The positive model (1) is reachable for ZBC at point* h, k *if and only if there exists a GPPM* R_n *consisting of* n *linearly independent columns of the reachability matrix*

$$R_{hk} := [M_{hk}, M_{h-1,k}, M_{h,k-1,\dots}, M_{01}, M_{10}] \quad (8)$$

where

$$M_{ij} := T_{i-1,j} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{i,j-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad (9)$$

and T_{ij} *is defined by (5).*

THEOREM 2.2 *The positive model (1) is controllable if and only if the matrix* A *is a nilpotent matrix, i.e.*

$$det \begin{bmatrix} I_{n_1} z_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} z_2 - A_{22} \end{bmatrix} = z_1^{n_1} z_2^{n_2} \quad (10)$$

3. Reachability of positive linear systems with feedbacks

To simplify the notation we assume that $m = 1$ (the single-input systems) and the matrices A and B of the positive model (1) have the canonical form, Kaczorek, 1985, 1998,

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_{n_1} \end{bmatrix}, \; A_{12} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, A_{21} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1_{n_1}} \\ a_{21} & a_{22} & \dots & a_{2_{n_1}} \\ \dots & \dots & \dots & \dots \\ a_{n_2 1} & a_{n_2 2} & \dots & a_{n_2 n_1} \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ b_2 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n_2-1} & 0 & 0 & \dots & 0 & 1 \\ b_{n_2} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_2-1} \\ b_{n_2} \end{bmatrix} \tag{11}$$

where $a_l \geq 0, a_{kl} \geq 0, b_k \geq 0$ for $k = 1, \dots, n_2, l = 1, \dots, n_1$.
Consider the system (1) with the state-feedback

$$u_{ij} = \nu_{ij} + K \begin{bmatrix} x_{ij}^h \\ x_{ij}^\nu \end{bmatrix}, i, j \in Z_+. \tag{12}$$

where $K = [K_1, K_2], K_1 \in R^{1 \times n_1}, K_2 \in R^{1 \times n_2}$ and $\nu_{ij} \in R^m$ is a new input vector.
Substitution of (12) into (1) yields

$$\begin{bmatrix} x_{1+1,j}^h \\ x_{i,j+1}^\nu \end{bmatrix} = A_c \begin{bmatrix} x_{ij}^h \\ x_{ij}^\nu \end{bmatrix} + B\nu_{ij} \tag{13}$$

where

$$A_c = A + BK = \begin{bmatrix} A_{11} + B_1K_1, & A_{12} + B_1K_2 \\ A_{21} + B_2K_1, & A_{22} + B_2K_2 \end{bmatrix} \tag{14}$$

The standard closed-loop system (13) is reachable (controllable) if and only if the standard 2D Roesser model (1) is reachable (controllable), Kaczorek (1993). It is easy to show that if at least one of $a_l \neq 0, l = 1, \dots, n_1$ or $b_k \neq 0, k = 1, \dots, n_2$, then the condition of Theorem 2.1 is not satisfied and the positive model (1) is not reachable at point (n_1, n_2). To simplify the calculations let us assume that $n_1 = 3$ and $n_2 = 2$. In this case using (11), (5) and (8) we obtain

$$T_{10} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a_1 & a_2 & a_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, T_{01} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & b_1 & 1 \\ a_{21} & a_{22} & a_{23} & b_2 & 0 \end{bmatrix},$$

$$T_{11} = T_{10}T_{01} + T_{01}T_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & a_{13} & b_1 & 1 \\ a_1 a_{13} & a_{11} + a_{13}a_2 & a_{12} + a_{13}a_3 & a_{13} & 0 \\ a_1 a_{23} & a_{21} + a_{23}a_2 & a_{22} + a_{23}a_3 & a_{23} & 0 \end{bmatrix}, \ldots$$

$$M_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, M_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix},$$

$$M_{11} = T_{01}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{10}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b_1 \\ a_{13} \\ a_{23} \end{bmatrix},$$

$$M_{20} = T_{10}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ a_3 \\ 0 \\ 0 \end{bmatrix}, M_{02} = T_{01}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1^2 + b_2 \\ b_1 b_2 \end{bmatrix}, \ldots$$

$$[M_{10}, M_{01}, M_{11}, M_{20}, M_{02}, \ldots] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ 1 & 0 & b_1 & a_3 & 0 & \ldots \\ 0 & b_1 & a_{13} & 0 & b_1^2 + b_2 & \ldots \\ 0 & b_2 & a_{23} & 0 & b_1 b_2 & \ldots \end{bmatrix} \quad (15)$$

It is easy to see that the matrix (15) does not satisfy the condition of Theorem 2.1 if $a_l \neq 0, l = 1, 2, 3$. Let the positive system (1) with (11) be unreachable at point (n_1, n_2). It will be shown that there exists a state-feedback gain matrix K such the closed-loop system (13) is reachable at point (n_1, n_2).
Let

$$K = [-a_1, -a_2, \ldots, -a_{n_1}, -1, 0, \ldots, 0] \quad (16)$$

For (11) and (16) the matrix (14) has the form

$$A_c = A + BK = \left[\begin{array}{c} \overline{A}_{11}, \overline{A}_{12} \\ \overline{A}_{21}, \overline{A}_{22} \end{array} \right] \tag{17}$$

where

$$\overline{A}_{11} = A_{11} + B_1 K_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n_1} \end{bmatrix} +$$

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [-a_0, -a_1, \dots, -a_{n_1}] = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\overline{A}_{12} = A_{12} + B_1 K_2 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [-1 \quad 0 \dots 0] = \tag{18}$$

$$\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}; \quad \overline{A}_{21} = A_{21} + B_2 K_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1_{n_1}} \\ a_{21} & a_{22} & \dots & a_{2_{n_1}} \\ \dots & \dots & \dots & \dots \\ a_{n_2 1} & a_{n_2 2} & \dots & a_{n_2 n_1} \end{bmatrix} +$$

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_2} \end{bmatrix} [-a_0, -a_1, \dots, -a_{n_1}] = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \dots & \overline{a}_{1_{n_1}} \\ \overline{a}_{21} & \overline{a}_{22} & \dots & \overline{a}_{2_{n_1}} \\ \dots & \dots & \dots & \dots \\ \overline{a}_{n_2 1} & \overline{a}_{n_2 2} & \dots & \overline{a}_{n_2 n_1} \end{bmatrix}$$

$$\overline{A}_{22} = A_{22} + B_2 K_2 = \begin{bmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ b_2 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n_2-1} & 0 & 0 & \dots & 0 & 1 \\ b_{n_2} & 0 & 0 & \dots & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_2} \end{bmatrix} [-1 \quad 0 \dots 0] = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

If the assumptions of the canonical form are satisfied, Kaczorek (1985, 1998), then it can be shown that $\overline{a}_{kl} \geq 0$ for $k = 1, \dots, n_2, l = 1, \dots, n_1$. Now we shall show that the closed-loop system with (18) and $b_1 = b_2 = \dots = b_{n_2-1} = 0, b_{n_2} \neq 0$ is reachable at point(n_1, n_2). Using (17), (5) and (9) we obtain

$$M_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = e_{n_1} \text{ (n_1- th column of the $n \times n$ identity matrix)}$$

$$M_{20} = T_{10} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = e_{n_1-1}, \dots, M_{n_1 0} = T_{10}^{n_1-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = e_1,$$

$$M_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = b_{n_2} e_2, M_{02} = T_{01} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = b_{n_2} e_{n-1}, \cdots, \tag{19}$$

$$M_{0n_2} = T_{01}^{n_2-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = b_{n_2} e_{n_1+1}$$

Note that in this case the matrix

$$[M_{10}, M_{20}, \dots, M_{n_1 0}, M_{01}, M_{02}, \dots, M_{0n_2}] =$$

$$[e_{n_1}, e_{n_1-1}, \dots, e_1, b_{n_2} e_n, b_{n_2} e_{n-1}, \dots, b_{n_2} e_{n_1+1}]$$

is GPPM and by Theorem 2.1 the positive system (1) with (17) and $b_1 = b_2 = \dots = b_{n_2-1} = 0, b_{n_2} \neq 0$ is reachable at point (n_1, n_2). In the case when $b_k \neq 0$ for $k = 1, \dots, n_2$ the calculations in the proof are more complicated. Therefore, the following theorem has been proved:

THEOREM 3.1 *Let the positive system (1) with (11) be unreachable at the point (n_1, n_2). Then the closed-loop system (13) with (17) is reachable at the point (n_1, n_2) if the state-feedback gain matrix K has the form (16).*

From Theorem 3.1 we have the following important corollary.

COROLLARY 3.1 *The reachability of positive system (1) with (11) is not invariant under the state-feedback (12).*

Example 1
Consider the positive 2D Roesser model (1) with

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}=\begin{bmatrix} 0 & 1 & \vdots & 0 & 0 \\ 1 & 2 & \vdots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 3 & \vdots & 1 & 0 \\ 3 & 4 & \vdots & 2 & 1 \end{bmatrix}, B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ \dots \\ 1 \\ 2 \end{bmatrix} \quad (20)$$

which is unreachable at point (2,2).
In this case $n_1 = n_2 = 2, m = 1$ and using (16) and (14) we obtain

$$K = [-1, -2, -1, 0] \quad (21)$$

and

$$A_c = A+BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} [-1, -2, -1, 0] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad (22)$$

Using (5), (9) and (20) we calculate

$$M_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, M_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix},$$

$$M_{11} = T_{01} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{10} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$M_{20} = T_{10} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, M_{02} = T_{01} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

Hence the matrix

$$[M_{10}, M_{11}, M_{20}, M_{02}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

is GPPM and by Theorem 2.1 the closed-loop system with (22) is reachable at point $(n_1, n_2) = (2, 2)$.

4. Controllability of positive linear systems with feedbacks

Consider the positive single-input model (1) with (11) and the state-feedback (12). According to Theorem 2.2 the positive system is controllable (to zero) if and only if the matrix A is nilpotent. It is said that the state-feedback (12) violates the nilpotency of A if and only if the closed-loop matrix (14) is not nilpotent. From Theorem 2.2 the following theorem follows:

THEOREM 4.1 *The closed-loop system (13) is uncontrollable at point* (n_1, n_2) *if the state-feedback (12) violates the nilpotency of* A.

COROLLARY 4.1 *The controllability of the positive system (1) is not invariant under the state-feedback (12).*

Example 2. Consider the positive model (1) with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & \vdots & 1 \\ 0 & 0 & \vdots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \vdots & 0 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \dots \\ 1 \end{bmatrix}. \tag{23}$$

In this case $n_1 = 2, n_2 = 1$ and

$$det \begin{bmatrix} I_{n_1} z_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} z_2 - A_{22} \end{bmatrix} = \begin{vmatrix} z_1 & -1 & -1 \\ 0 & z_1 & -1 \\ 0 & 0 & z_2 \end{vmatrix} = z_1^2 z_2 \ .$$

Using (5) we obtain

$$T_{10} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, T_{01} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, T_{20} = T_{10}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$T_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{for} \begin{cases} i > 2 & \text{and} \quad j = 0 \\ i = 0 & \text{and} \quad j > 1 \\ i > 0 & \text{and} \quad j > 0 \end{cases}$$

and the component of x_{ij} caused by nonzero boundary conditions (3) is, Roesser (1975), Kaczorek (1985), Klamka (1991),

$$x_{bc}(i,j) = \sum_{k=0}^{i} T_{i-k,j} \begin{bmatrix} 0 \\ x_{k0}^{\nu} \end{bmatrix} + \sum_{l=0}^{j} T_{i,j-1} \begin{bmatrix} x_{0l}^{h} \\ 0 \end{bmatrix} = 0 \text{ for } i > 2, \ j > 1$$

and any x_{k0}^{ν} and x_{0l}^{h}.

Therefore, the system can be transferred to zero by zero input sequence for any boundary conditions (3) and arbitrary matrix B. Note that if the matrix B has the form (23) then any nonzero gain matrix $K = [k_1, k_2, k_3]$ violates the nilpotency of the matrix A given by (23) since

$$A + BK = \begin{bmatrix} 0 & 1 & 1 \\ k_1 & k_2 & k_3 + 1 \\ k_1 & k_2 & k_3 \end{bmatrix}.$$

If $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ then we have

$$A + BK = \begin{bmatrix} k_1 & 1 + k_2 & 1 + k_3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and for $k_1 = 0$ the nilpotency of A is not violated.

5. Extensions and concluding remarks

It has been shown that the reachability and controllability of positive 2D Roesser type model are not invariant under the state-feedbacks. By suitable choice of the state-feedbacks the unreachable positive 2D Roesser type model can be made reachable and the controllable positive 2D Roesser model can be made uncontrollable. With slight modifications the considerations presented above can be extended for the multi-input positive 2D Roesser type model and the positive $nD(n > 2)$ Roesser type models. It is well known, Kaczorek (1983), that the first Fornasini-Marchesini model, Fornasini and Marchesini (1976), can be recast in the 2D Roesser model. Therefore, the considerations can be immediately extended for the positive first Fornasini-Marchesini model. Extensions of the considerations for the positive second Fornasini-Marchesini model, Fornasini and Marchesini (1978), and general 2D model, Kurek (1985), are also possible. An open problem is an extension of the considerations for the singular 2D linear systems, Kaczorek (1993).

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