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Random approximations in multiobjective programming – with an application to portfolio optimization with shortfall constraints

by

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Abstract: Decision makers often have to deal with a programming problem where some of the quantities are unknown. They will usually estimate these quantities and solve the problem as it then appears - the 'approximate problem'. Thus, there is a need to establish conditions which will ensure that the solutions to the approximate problem will come close to the solutions to the true problem in a suitable manner. The paper summarizes such results for multiobjective programming problems. The results are illustrated by means of the Markowitz model of portfolio optimization. In order to show how probabilistic constraints may be dealt with using this framework, a shortfall constraint is taken into account.

Keywords: multiobjective programming, estimated quantities, stability, Markowitz model, probabilistic constraints

1. Introduction

Decision makers often have to deal with a programming problem where some of the quantities are unknown. Then it is usual to estimate these quantities and to solve the problem as it then appears - the 'approximate problem'. The hope is that the solution to the approximate problem will be a good approximation of the solution to the true problem. Thus there is a need for conditions ensuring that this hope is justified, conditions as to the nature of the true problem and as to the behaviour of the estimates.

Many papers have been published on the approximation of optimization problems. Especially, the stability theory of parametric programming yields many helpful results, both in respect of one objective function and in respect of several objective functions (see Bank et al., 1982, Robinson, 1987, Sawaragi et al., 1985, Penot and Sterna-Karwat, 1986, 1989).

Estimated quantities produce, in effect, a random approximate problem. This means that there must be additional considerations in order to enable the deterministic results to be adapted to this random setting. In the 'almost surely' (a.s.) setting this problem has been mainly dealt with in the context of stochastic programming and Markovian decision processes, where an important role is played by objective functions that are integrals with respect to probability measures. Meanwhile, various quantitative and qualitative results are available for cases with a single objective function (Langen, 1981, Robinson and Wets, 1987, Kall, 1987, Dupačová and Wets, 1988, Wets, 1989, King and Wets, 1991, Römisch and Schultz, 1991, 1993, 1996, Vogel, 1994a).

However, the problems are often such that conditions which guarantee the almost sure convergence of the estimates can hardly be assumed. Then it will be necessary to look for concepts of weaker convergence, for instance convergence 'in probability'. Here a demand for appropriate convergence concepts of random functions and random sets is the first to arise. Salinetti and Wets (1981, 1986) investigated the Kuratowski-Painlevé-convergence of random sets and the epi-convergence of random functions (for the 'a.s.', 'in probability' and 'in distribution' cases). On the basis of these concepts it is possible to derive stability results 'in probability' for the single objective case that are similar to the deterministic form (see Vogel, 1994a, b).

In the multiobjective case there are several stability results for deterministic parametric programming problems (see Naccache, 1979, Papageorgiou, 1985, Sawaragi et al., 1985, Penot and Sterna-Karwat, 1986, Vogel, 1992) which consider the 'semicontinuous' behaviour of the sets of efficient points and the solution sets. These results may be employed to derive statements about the a.s. setting. It is the aim of the first part of the present paper to summarize the results for the a.s. case and to discuss the assumptions. The Markowitz model of portfolio optimization, which is here extended by a shortfall constraint, serves as an illustrative example.

However, the attempt to derive results for the single objective case from the semicontinuous behaviour of the sets of efficient points will only achieve results about the continuity of the optimal value. The reason is that the lower (or upper) semicontinuity of a multifunction reduces to continuity if the multifunction is single-valued. Penot and Sterna-Karwat (1989) filled this gap in the deterministic setting. They introduced and investigated the 'order semicontinuity' of the sets of efficient points and thus obtained multiobjective generalizations for statements on the semicontinuity of the optimal value function. In order to carry over these results to the random setting, the second part of this paper will investigate random versions of the order semicontinuity in more detail. It will offer statements for the a.s. setting and then show how results for the 'in probability' case may be derived. Note that the way of proof which is followed in this part could also be used to derive 'in probability' versions of the assertions made in the first part of the paper. Again, the results will be illustrated by the

The paper is organized as follows. The mathematical model is provided in Section 2. The Markowitz model which is to serve as an illustration is explained in Section 3. In Section 4 the results on the semicontinuous behaviour of the constraint sets, the sets of efficient points, and the solutions sets (in the a.s. setting) are summarized and there is a discussion of conclusions to be drawn for the Markowitz model. Section 5 has as its subject the 'a.s.' and 'in probability' order semicontinuity. Section 6 summarizes the results.

In this paper the focus is on a deterministic original problem because of the applications dealt with. Note that most of the results may also be proved for the original random problems.

Finally, it should be mentioned that for a deterministic original problem convergence in probability and convergence in distribution (for the diverse kinds of convergence considered here) can be shown to coincide.

2. Mathematical model

Suppose that we are given the deterministic multiobjective programming problem

$$(P_o) \quad \min_{x \in \Gamma_o} \mathbf{f}_o(x)$$

where $\Gamma_o \subset R^p$ is a nonempty closed set and $\mathbf{f}_o | R^p \to R^r$. Minimization is understood with respect to the usual partial ordering " $\leq$ " in R^r, which is generated by the cone R^r_+.

We consider random surrogate problems

$$(P_n(\omega)) \quad \min_{x \in \Gamma_n(\omega)} \mathbf{f}_n(x, \omega)$$

where $\Gamma_n, n \in N$, are multifunctions defined on a given complete probability space $[\Omega, \mathcal{A}, P]$ with values in the σ–field of Borel sets Σ^p. $\mathbf{f}_n | R^p \times \Omega \to R^r$ is taken as $(\Sigma^p \otimes \mathcal{A}, \Sigma^r)-$ measurable. (Sufficient conditions for this property are given by Vogel, 1992).

To avoid restricting the model to closed-valued multifunctions we, additionally, assume that the graphs $\Gamma_n, n \in N$, belong to $\mathcal{A} \otimes \Sigma^p$. In our setting multifunctions with measurable graphs are measurable, i.e. $\Gamma_n^{-1}(A) := \{\omega \in \Omega : \Gamma_n(\omega) \cap A \neq \emptyset\} \in \mathcal{A}$ for every closed set $A \in \Sigma^p$.

Γ_o and $\Gamma_n(\omega)$ may be specified by inequality constraints:

$$\Gamma_o := \{x \in R^p : g_o^j(x) \leq 0, j \in J\},$$

$$\Gamma_n(\omega) := \{x \in R^p : g_n^j(x, \omega) \leq 0, j \in J\}$$

where $g_o^j | R^p \to R^1$; $g_n^j | R^p \times \Omega \to R^1$ is $(\Sigma^p \otimes \mathcal{A}, \Sigma^1)$- measurable, and J is a

When a single component of $\mathbf{f}_o$ or $\mathbf{f}_n$ (or other vector-valued functions or elements of R^r) is dealt with, the same letter is used without bold-face and the corresponding index is added: f_n^j denotes the j-th component of $\mathbf{f}_n$.

The sets of efficient points (or efficiency sets) for the original problem (P_o) and the approximate problems $(P_n(\omega))$ are explained by

$$E_o := \{\mathbf{y} \in \mathbf{f}_o(\Gamma_o) : \nexists\, \bar{\mathbf{y}} \in \mathbf{f}_o(\Gamma_o) \text{ with } (\bar{\mathbf{y}} \leq \mathbf{y} \wedge \bar{\mathbf{y}} \neq \mathbf{y})\},$$

$$E_n(\omega) := \{\mathbf{y} \in \mathbf{f}_n(\Gamma_n(\omega), \omega) : \quad \nexists \quad \bar{\mathbf{y}} \in \mathbf{f}_n(\Gamma_n(\omega), \omega) \text{ with } (\bar{\mathbf{y}} \leq \mathbf{y} \wedge \bar{\mathbf{y}} \neq \mathbf{y})\}$$

where $\mathbf{f}_o(\Gamma_o)$ stands for $\{\mathbf{f}_o(x) : x \in \Gamma_o\}$ and $\mathbf{f}_n(\Gamma_n(\omega), \omega)$ for $\{\mathbf{f}_n(x, \omega) : x \in \Gamma_n(\omega)\}$.

By S_o and S_n we denote the corresponding solution sets

$$S_o := \{x \in \Gamma_o : \nexists\, \bar{x} \in \Gamma_o \quad \text{with} \quad (\mathbf{f}_o(\bar{x}) \leq \mathbf{f}_o(x) \wedge \mathbf{f}_o(\bar{x}) \neq \mathbf{f}_o(x))\},$$

$$S_n(\omega) := \{x \in \Gamma_n(\omega) : \nexists\, \bar{x} \in \Gamma_n(\omega) \quad \text{with} \quad (\mathbf{f}_n(\bar{x}, \omega) \leq \mathbf{f}_n(x, \omega) \wedge \mathbf{f}_n(\bar{x}, \omega) \neq \mathbf{f}_n(x, \omega))\}.$$

Moreover, we introduce the sets of weakly efficient points

$$W_o := \{\mathbf{y} \in \mathbf{f}_o(\Gamma_o) : \nexists\, \bar{\mathbf{y}} \in \mathbf{f}_o(\Gamma_o) \text{ with } \bar{\mathbf{y}} < \mathbf{y}\} \quad \text{and}$$

$$W_n(\omega) := \{\mathbf{y} \in \mathbf{f}_n(\Gamma_n(\omega), \omega) : \nexists\, \bar{\mathbf{y}} \in \mathbf{f}_n(\Gamma_n(\omega), \omega) \text{ with } \bar{\mathbf{y}} < \mathbf{y}\},$$

which, in the single objective case, also reduce to the optimal value, and the corresponding 'weak' solution sets

$$S_o^W := \{x \in \Gamma_o : \nexists\, \bar{x} \in \Gamma_o \text{ with } \mathbf{f}_o(\bar{x}) < \mathbf{f}_o(\omega)\},$$

$$S_n^W(\omega) := \{x \in \Gamma_n(\omega) : \nexists\, \bar{x} \in \Gamma_n(\omega) \text{ with } \mathbf{f}_n(\bar{x}, \omega) < \mathbf{f}_n(x, \omega)\}.$$

(By $(a^1 \dots a^r)^T < (b^1 \dots b^r)^T$; $a^i, b^i \in R$; we mean $a^i < b^i\ \forall i \in \{1, \dots, r\}$.)

By definition, the sets of efficient points are contained in the sets of weakly efficient points and the corresponding relation holds for the solution sets.

Eventually, we introduce the multifunctions F_n with

$$F_n(\omega) := \{\mathbf{f}_n(x, \omega) : x \in \Gamma_n(\omega)\} = \mathbf{f}_n(\Gamma_n(\omega), \omega).$$

Firstly, we have to guarantee that the necessary measurability conditions are fulfilled. We start with an auxiliary result.

LEMMA 2.1 *Let Γ_n be closed-valued for P-almost all ω and measurable. Furthermore, let $f_n^j(\cdot, \omega)$ be l.s.c. for P-almost all ω and all j and f_n^j be $(\Sigma^p \otimes \mathcal{A}, \Sigma^r)$−measurable. Then the multifunctions $\tilde{F}_n | \Omega \to 2^{R^r}$, $n \in N$, with $\tilde{F}_n(\omega) = \mathbf{f}_n(\Gamma_n(\omega), \omega) + R_+^r$, are closed-valued for P-almost all ω and measur-*

Proof. We consider a fixed n and omit the index n. Under our assumptions the epigraph multifunction $\omega \to Epi\ \mathbf{f}(\cdot,\omega)$ is measurable and almost surely closed-valued. Hence, because of $\tilde{F}(\omega) = Epi\ \mathbf{f}(\cdot,\omega) \cap (\Gamma(\omega) \times R^r_+)$, the conclusion follows. ■

In a former paper (Vogel, 1992) we have shown that the multifunctions E_n, W_n, S_n and S_n^w have measurable graphs whenever either the functions $\mathbf{f}_n(\cdot,\omega)$ are continuous and the sets $\Gamma_n(\omega)$ are compact for almost all ω or the functions $\mathbf{f}_n(\cdot,\omega)$ take (with probability one) only values in a set with finitely many elements, which is of interest when probabilities occur among the objective functions. This result can be proved in a unified way and extended, supposing that the multifunctions $\tilde{F}_n$ as defined in Lemma 2.1 are closed-valued and measurable.

LEMMA 2.2 *Let* $\mathbf{f}_n$ *be* $(\Sigma^p \otimes \mathcal{A}, \Sigma^r)-$*measurable and Graph* $\Gamma_n \in \mathcal{A} \otimes \Sigma^p$. *Furthermore, assume that* $\tilde{F}_n$ *is closed-valued for* P*-almost all* ω *and measurable. Then the multifunctions* E_n, W_n, S_n *and* S_n^W *have measurable graphs.*

Proof. Again, we consider a fixed n and omit the index n.
Let $\tilde{\Omega} := \{\omega \in \Omega : \tilde{F}(\omega) \text{ is closed}\}, \tilde{\mathcal{A}} := \{A \cap \tilde{\Omega},\ A \in \mathcal{A}\}$ and consider the following multifunctions on $[\tilde{\Omega} \times R^p, \tilde{\mathcal{A}} \otimes \Sigma^p]$. From the definition of efficiency we have

$$S(\omega) = \{x \in \Gamma(\omega) : (\mathbf{f}(x,\omega) - \dot{R}^r_+) \cap \tilde{F}(\omega) = \emptyset\}$$

with

$$\dot{R}^r_+ := R^r_+ \backslash \{0\}.$$

We introduce the multifunction Φ with $\Phi(\omega,x) := (\mathbf{f}(x,\omega) - \dot{R}^r_+) \cap \tilde{F}(\omega)$ and obtain

$$\begin{aligned} \text{Graph}S &= \{(\omega,x) : \omega \in \Omega,\ x \in \Gamma(\omega), \Phi(x,\omega) = \emptyset\} \\ &= \text{Graph}\Gamma \cap (\text{dom}\ \Phi)^c \end{aligned}$$

where $(\text{dom}\ \Phi)^c$ denotes the complement of the domain of Φ.

Furthermore, let $G(\omega,x) := (\mathbf{f}(x,\omega) - R^r_+) \cap \tilde{F}(\omega)$. Then

$$\Phi(\omega,x) = G(\omega,x) \backslash (\{\mathbf{f}(x,\omega)\} \cap \tilde{F}(\omega)).$$

G is closed-valued and measurable, and $\{\mathbf{f}(\cdot,\cdot)\} \cap \tilde{F}(\cdot)$ is empty or single-valued, hence closed and measurable. Consequently, Φ is measurable by Theorem 4.5 and dom Φ is measurable by Proposition 2.2 (Himmelberg, 1975). Thus, Graph $S \in \mathcal{A} \otimes \Sigma^p$.

Eventually, taking into account that

we can proceed in the same way in order to show the measurability of Graph$E$. If we consider S^W and W, we replace $\dot{R}_+^r$ by int R_+^r and Φ by Φ_W with

$$\begin{aligned}\Phi_W(\omega, x) &:= (\mathbf{f}(x,\omega) - \text{ int } R_+^r) \cap \tilde{F}(\omega) \\ &= G(\omega, x) \backslash (bdy(\{\mathbf{f}(x,\omega)\} - R_+^r) \cap \tilde{F}(\omega)).\end{aligned}$$

The boundary multifunction *bdy* is measurable because of Theorem 4.6. by Himmelberg (1975). ■

Lemma 2.1 and Lemma 2.2 together imply the quoted results (Vogel, 1992), because, first, functions which are measurable with respect to ω and continuous in x satisfy the measurability conditions of Lemma 2.1 according to Propositon 2C by Rockafellar (1976) and, second, if there is a finite set V_n with

$$P\{\omega : \mathbf{f}_n(x,\omega) \in V_n \ \forall x \in R^p\} = 1,$$

then $\tilde{F}_n$ is closed-valued and, F_n being measurable, also measurable.

Moreover, lower-semicontinuous objective functions with measurable epigraphs, i.e. normal integrands, are $(\Sigma^p \otimes \mathcal{A}, \Sigma^r)$-measurable (Rockafellar, 1976, Theorem 2A) and can hence be treated in this framework.

3. The Markowitz model of portfolio optimization

Suppose that an investor has a certain amount of money which is assumed to be one unit and he can choose between p different assets $A_1, \ldots, A_p$. Here, ρ^k denotes the random return at the end of the planning period if the whole amount of money would be invested in asset A_k. By x^k we denote the fraction of the money the investor will spend for A_k, short sales are not allowed. Hence $\sum_{i=1}^{p} x^i \leq 1, x^i \geq 0, i = 1, \ldots, p.$

In the classical Markowitz model two objective functions are taken into account. The investor will maximize the expected return and minimize the return's variance, which is used as a surrogate for risk. In order to show how probabilistic constraints fit into our framework, we will, additionally, take a shortfall constraint into account, i.e. the probability that the return does not fall under a given target return α should be not less than η $(\alpha > 0,\ \eta \in (0,1))$.

Thus, we can use the following mathematical model. The returns ρ^i are supposed to be random variables, defined on $[\Omega, \mathcal{A}, \mathcal{P}]$ with values in $[R^1, \Sigma^1]$ and such that $E\rho^i > 0$ and $D^2\rho^i$ exists for all $i \in \{1, \ldots, p\}$. Furthermore, we suppose that the random vector ρ , $\rho = (\rho^1, \ldots, \rho^p)^T$, depends continuously on ω, which is for instance satisfied, if $[\Omega, \mathcal{A}]$ is identified with $[R^p, \Sigma^p]$. In our minimization-framework the objective functions take the form

where $x := (x^1, \ldots, x^p)^T$. E, D^2, and B denote the expectation operator, the variance operator and the covariance matrix of ρ, respectively. The constraints are given by

$$\Gamma_o = \{x \in R^p : \sum_{i=1}^{p} x^i \leq 1, \quad x \geq 0, \quad P(\sum_{i=1}^{p} x^i \rho^i \geq \alpha) \geq \eta\}.$$

Γ_o can be rewritten in the following form, where $g_o(x) \leq 0$ means that the shortfall constraint is satisfied for the decision vector x:

$$\Gamma_o = \Gamma_o^{(1)} \cap \{x \in R^p : g_o(x) \leq 0\} \text{ with}$$

$$\Gamma_o^{(1)} = \{x \in R^p : \sum_{i=1}^{p} x^i \leq 1, x \geq 0\} \text{ and}$$

$$g_o(x) = \eta - \int_{R^p} \chi_{M(x)}(z) dP_o(z),$$

$$M(x) := \{z = (z^1, \ldots, z^p)^T : \sum_{i=1}^{p} x^i z^i \geq \alpha\}.$$

P_o denotes the probability measure which is induced on $[R^p, \Sigma^p]$ by the (true) random vector ρ, and $\chi_{M(x)}$ is the characteristic function of $M(x)$:

$$\chi_{M(x)}(z) = \begin{cases} 1 \text{ if } z \in M(x), \text{i.e., } \sum_{i=1}^{p} x^i z^i \geq \alpha \\ 0 \text{ otherwise.} \end{cases}$$

However, the distribution of ρ, which is needed in the objective function and in the constraints, is usually unknown and has to be approximated. Of course, the approximation of this distribution is a crucial step: there are attempts to exploit technical analysis, fundamental analysis, experts' forecasts, etc. How to obtain good estimates is a separate question and would go beyond the scope of this paper.

To have examples, let us assume that

i) our estimates are based on independent 'forecasts' ρ_j, $j = 1, \ldots, n$, for ρ (forecasts of experts or forecasts based on scenarios) and

ii) the rewards ρ have a nonsingular normal distribution (this assumption is not very realistic, but often used) and only the expectation vector and covariance matrix have to be estimated (for instance relying on technical analysis or scenarios).

In both the cases we will use the surrogate random objective functions

with the sample mean $\bar{\rho}_n = \frac{1}{n}\sum_{j=1}^{n}\rho_j$ and the sample covariance matrix

$$B_n = \frac{1}{n-1}\sum_{j=1}^{n}(\rho_j - \bar{\rho}_n)(\rho_j - \bar{\rho}_n)^T.$$

(Some authors use the biased maximum-likelihood estimate for B, which differs from B_n only by the factor $\frac{n-1}{n}$. For our asymptotic assertions this is meaningless.)

Differences occur in the treatment of the shortfall constraints: in the first case we will deal with

$$g_{n,I}(x) = \eta - \frac{1}{n}\sum_{j=1}^{n}\chi_{M(x)}(\rho_j)$$

which means replacement of P_o by the 'empirical measure' based on $\rho_1, \ldots, \rho_n$.

In the case of normally distributed returns, $x^T\rho$ follows a normal distribution with expectation $x^T E\rho$ and variance $x^T Bx$. By replacing these parameters with the above estimates, we obtain

$$g_{n,II}(x) = \begin{cases} \eta \ + \phi\left(\frac{\alpha - x^T\bar{\rho}_n}{\sqrt{x^T B_n x}}\right) - 1, & \text{if } x^T B_n x > 0 \\ \eta \qquad \text{otherwise} \end{cases}$$

where ϕ denotes the distribution function of a standard normal variable:

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt.$$

Here we did not indicate the dependence on the elements ω of Ω, but, since the ρ_j are random variables, the functions f_n^1, f_n^2, $g_{n,I}$ and $g_{n,II}$ are functions of x and ω. Then, f_n^1 and f_n^2 are continuous in x and measurable, hence they satisfy our measurability conditions.

Because of $\{(x,\omega) \in R^p \otimes \Omega : \chi_{M(x)}(\rho(\omega)) = 1\} = \{(x,\omega) : x^T\rho(\omega) \geq \alpha\}$, the $(\Sigma^p \otimes \mathcal{A}, \Sigma^1)$-measurability of $\chi_{M(\cdot)}(\rho(\cdot))$ and hence of $g_{n,I}$ is guaranteed. B being regular, the approximations B_n are regular for almost all realizations of ρ, and hence $g_{n,II}$ is continuous in x.

We still have to consider $g_{n,I}(\cdot)$ with respect to lower semicontinuity for a fixed sequence $z_1, \ldots, z_n$ of realizations of $\rho_1, \ldots, \rho_n$. Let x_j be a discontinuity point of $\chi_{M(\cdot)}(z_j)$. For $\chi_{M(x_j)}(z_j) = 1$ there is nothing to show. Then, $\chi_{M(x_j)}(z_j) = 0$ means $x_j^T z_j < \alpha$. Hence, there is a neighbourhood $U\{x_j\}$ of x_j with $x^T z_j < \alpha \forall x \in U\{x_j\}$, i.e. $\chi_{M(x)}(z_j) = 0\ \forall x \in U\{x_j\}$.

This implies upper semicontinuity of $\chi_{M(\cdot)}(z_j)$ and, consequently, $g_{n,I}(\cdot)$ is lower semicontinuous for all realizations of the rewards.

Summarizing, in both cases, the assumptions of Lemma 2.1 and Lemma 2.2 are satisfied, hence measurability of the sets under consideration in this paper

4. Convergence of the constraint sets, the sets of efficient points and the solution sets

4.1. Notions of convergence

In the first part of this section the results on the semicontinuous behaviour almost surely will be summarized which may be derived from results for multiobjective deterministic parametric programming problems. We start by explaining the suitable convergence notions. Let $\mathbf{f}_o | R^p \to R^r$ and a closed set $X \subset R^p$ be given.

DEFINITION 4.1 *A sequence* $(\mathbf{f}_n)_{n\in N}$ *of* $(\Sigma^p \otimes \mathcal{A}, \Sigma^r)$*-measurable functions is said to be*

i) a lower semicontinuous approximation almost surely to $\mathbf{f}_o$ *on* X *(abbreviated* $\mathbf{f}_n \xrightarrow[X]{l-a.s.} \mathbf{f}_o$*) if*

$$\forall j \in \{1,\dots,r\}: \quad P\{\omega : \forall x_o \in X \quad \forall (x_n)_{n\in N} \text{ with } x_n \to x_o : \liminf_{n\to\infty} f_n^j(x_n,\omega) \le f_o^j(x_o)\} = 1,$$

ii) an upper semicontinuous approximation almost surely to $\mathbf{f}_o$ *on* X $(\mathbf{f}_n \xrightarrow[X]{u-a.s.} \mathbf{f}_o)$ *if* $-\mathbf{f}_n \xrightarrow[X]{l-a.s.} -\mathbf{f}_o$,

iii) continuously convergent almost surely to $\mathbf{f}_o$ *on* X $(\mathbf{f}_n \xrightarrow[X]{c-a.s.} \mathbf{f}_o)$ *if* $\mathbf{f}_n \xrightarrow[X]{l-a.s.} \mathbf{f}_o \wedge \mathbf{f}_n \xrightarrow[X]{u-a.s.} \mathbf{f}_o$.

Continuous convergence a.s. is the natural generalization of continuous convergence for a sequence of deterministic functions. 'In probability' versions of the above convergence notions are discussed by the author in Vogel (1994a, b) and, applied to multiobjective programming and taking into account a convergence rate, in Vogel (1992). They come into play when only convergence in probability can be guaranteed for the estimates.

At a first glance the notions of Definition 4.1 may seem unwieldy, but for l.s.c. (u.s.c., or continuous) objective functions f_o^j there are pointwise sufficient conditions (Vogel, 1994a, Theorem 5.1) which can be used in several applications (Vogel, 1994b). We recall the 'continuous' version (here C^p denotes the class of compact subsets of R^p).

LEMMA 4.1 *Let* f_o^j *be continuous on* X *for all* $j \in \{1,\dots,r\}$. *Then the condition*

$$\forall j \in \{1,\dots,r\} \ \forall x_o \in X \ \forall \epsilon > 0 \ \exists U\{x_0\} \in C^p :$$
$$P\{\omega : \limsup_{n\to\infty} \sup_{x\in U\{x_o\}} |f_n^j(x,\omega) - f_o^j(x_o)| > \epsilon\} = 0$$

implies $\mathbf{f}_n \xrightarrow[X]{c-a.s.} \mathbf{f}_o$.

As to the multifunctions, in this section we use the following convergence notions, which in the deterministic case reduce to upper semicontinuity and lower semicontinuity in the sense of Sawaragi et al. (1985).

Let G be a closed subset of R^p.

DEFINITION 4.2 *A sequence $(G_n)_{n\in N}$ of multifunctions with measurable graphs is said to be*

i) upper semiconvergent almost surely to G_o $(G_n \xrightarrow{u-a.s.} G_o)$ *if*

$$P\{\omega : \limsup_{n\to\infty} G_n(\omega) \subset G_o\} = 1,$$

ii) lower semiconvergent almost surely to G_o $(G_n \xrightarrow{l-a.s.} G_o)$ *if*

$$P\{\omega : \liminf_{n\to\infty} G_n(\omega) \supset G_o\} = 1,$$

iii) convergent almost surely to G_o if $G_n \xrightarrow{u-a.s.} G_o \wedge G_n \xrightarrow{l-a.s.} G_o$.

The limes inferior and the limes superior in Definition 4.2 are understood in the Kuratowski-Painlevé sense:

$$\limsup_{n\to\infty} \tilde{G}_n := \{x \in R^p : \exists (x_{n_k})_{k\in N} \quad \text{with} \quad \lim_{k\to\infty} x_{n_k} = x \quad \text{and} \quad x_{n_k} \in \tilde{G}_{n_k} \ \forall k \in N\},$$

$$\liminf_{n\to\infty} \tilde{G}_n := \{x \in R^p : \exists (x_n)_{n\in N} \quad \text{with} \quad \lim_{n\to\infty} x_n = x \quad \text{and} \quad x_n \in \tilde{G}_n \ \forall n \geq n_o\}$$

where $(\tilde{G}_n)_{n\in N}$ denotes a sequence of subsets of R^p.

4.2. Constraint sets

Firstly, we consider the constraint sets. The following result may be derived from Theorems 3.1.1 and 3.1.5 by Bank et al. (1982).

THEOREM 4.1 *i) Let $g_n^j \xrightarrow[R^p]{l-a.s.} g_o^j \quad \forall j \in J$. Then $\Gamma_n \xrightarrow{u-a.s.} \Gamma_o$.*

ii) Let the following conditions be satisfied:

a) $g_n^j \xrightarrow[\Gamma_o]{l-a.s.} g_o^j \quad \forall j \in J$,

b) $\Gamma_o \subset cl\{x : g_o^j(x) < 0 \quad \forall j \in J\}$.

Then $\Gamma_n \xrightarrow{l-a.s.} \Gamma_o$.

Before we turn to the efficiency sets, we shall have a look at the Markowitz model. With $\Gamma_o^{(1)}$ being fixed, the interesting point is the behaviour of $g_{n,I}$ and $g_{n,II}$.

We start by investigating $(g_{n,I})_{n\in N}$. Making use of former results (Vogel, 1992, Section 6), we can conclude, that, given a continuous distribution of the rewards, $(g_{n,I})_{n\in N}$ is continuously convergent a.s. to $g_{o,I}$, hence $\Gamma_n \xrightarrow{u-a.s.} \Gamma_o$. For condition b) of ii) a result which is due to J. Wang (see Vogel, 1992, Proposition 10), may be employed. A logarithmic concave probability distribution of ρ and the existence of an $x_o \in \Gamma_o^{(1)}$ with $P({x_o}^T \rho \geq \alpha) > \eta$ imply that also

the assertion also holds for the larger class of quasi-concave measures. Thus, distributions which are of interest when modelling the random returns, e.g. non-singular normal distributions, lognormal distributions, and Pareto distributions, are admitted.

For $(g_{n,II})_{n\in N}$ we proceed in the following way: as already mentioned, $x^T B_n x$ $(x \neq 0)$ is positive for almost all realizations of $(\rho_1, \ldots, \rho_n)$, hence $g_{n,II}$ depends continuously on x and the estimates. This, by the strong consistency of $\bar{\rho}_n$ and B_n, implies continuous convergence. Concerning condition b) of ii), the above considerations hold as well.

4.3. Efficiency sets

The following result may be derived from Theorems 4.2.1 and 4.2.2 by Sawaragi et al. (1985).

THEOREM 4.2 *Let the following assumptions be satisfied:*

i) $\mathbf{f}_n \xrightarrow[R^p]{c-a.s.} \mathbf{f}_o$,

ii) $\Gamma_n \xrightarrow{a.s.} \Gamma_o$,

iii) $\exists K \in C^p : \; P\{\omega : \exists n_o(\omega)\, \forall_n \geq n_o : \; \Gamma_n \subset K\} = 1.$

Then we have $E_n \xrightarrow{u-a.s.} W_o$ *and,*
given that $P(E_n \neq \emptyset) = 1$ *for almost all* n, *also* $E_n \xrightarrow{l-a.s.} E_o$.

Unfortunately, in general, we do not have $W_o = E_o$, thus in the surrogate problems the efficiency set may be 'too big'.

Conditions implying equality are summarized by Vogel (1992, Proposition 2). For the Markowitz model none of them applies immediately. We have, however, the following result:

PROPOSITION 4.1 *Let* P_o *be quasi-concave and assume that* $E\rho^i \neq E\rho^j \; \forall i \neq j$ *and that* B *is a positive definite matrix. If then there is an* $x_o \in \Gamma_o$ *with* $P({x_o}^T \rho \geq \alpha) > \eta$, *the equality* $W_o = E_o$ *holds.*

Proof. Quasiconcave measures and probabilistic constraints in terms of convex functions imply a convex constraint set Γ_o (Wets, 1989, Prékopa, 1995, Vogel, 1992).

Suppose that there is a $\mathbf{y}_o \in W_o$, which does not belong to E_o. Then there exists a $\mathbf{y}_1 \in \mathbf{f}(\Gamma_o)$ such that $y_1^j \leq y_o^j \; \forall_j \in \{1, ..., r\}$ and $y_1^{j_o} < y_o^{j_o}$ for at least one j_o. To $\mathbf{y}_o$ and $\mathbf{y}_1$ we find $x_o \in \Gamma_o$ and $x_1 \in \Gamma_o$ with $\mathbf{y}_o = \mathbf{f}(x_o)$ and $\mathbf{y}_1 = \mathbf{f}(x_1)$ and consider $x_\lambda := \lambda x_o + (1-\lambda)x_1$, $\lambda \in (0,1)$ arbitrary. With B being positive definite, f_o^2 is strictly convex, hence $f_o^2(x_\lambda) < y_o^2$ and $f_o^2(x) < y_o^2$ for all x that belong to a suitable neighbourhood $U\{x_\lambda\}$. If $f_o^1(x_\lambda) < y_o^1$ we have a contradiction to the assumption that $\mathbf{y}_o \in W_o$.

Now, suppose that $f_o^1(x_\lambda) = y_o^1 = y_1^1$. Hence, x_λ, $\lambda \in (0,1)$ belongs to a contour line of f_o^1. If x_λ is an inner point of Γ_o, we can find $\tilde{x}_\lambda \in U\{x_\lambda\} \cap \Gamma_o$ with $f_o^1(\tilde{x}_\lambda) < y_o^1$ and $\ldots$

Eventually, we assume that all elements of $G_\lambda := \{x_\lambda : x_\lambda = \lambda x_o + (1-\lambda)x_1\}$ are boundary points of Γ_o and construct $\tilde{x}_\lambda \in \Gamma_o \cap U\{x_\lambda\}$ with $f_o^1(\tilde{x}_\lambda) < y_o^1$. We distinguish three cases:

i) First, let $x_\lambda^i = 0$ for all $\lambda \in (0,1)$ and all i from an index set $I \subset \{1,...,p\}$. Then $\tilde{x}_\lambda := x_\lambda + \beta \sum_{i \notin I} e_i$, where e_i denotes the ith unit vector $(0...0\ 1\ 0...0)^T$ with 1 at the ith position and $\beta > 1$ a suitable real number.

ii) Suppose that $x_\lambda^i = 0$ for all $\lambda \in (0,1)$ and all $i \in I \subset \{1,...,p\}$ and $\sum_{i=1}^{p} x_\lambda^i = 1$. Then the set $\bar{I} := \{1,...,p\} \backslash I$ contains at least two elements. Let $i_o \in \bar{I}$ be such that $E\rho^{i_o} > E\rho^i \quad \forall i \in \bar{I}$. Then we take (with $i_1 \in I$ and a suitable $\beta > 0$) $\tilde{x}_\lambda = x_\lambda + \beta e_{i_o} - \beta e_{i_1}$.

iii) Finally, suppose that $g_o(x_o) = 0$ and $g_o(x_1) = 0$. Then, because of the convexity of Γ_o, the inequality $g_o(x_\lambda) \leq 0$ holds. Now, choose $\tilde{x}_\lambda := x_\lambda + \beta \sum_{i \in \tilde{I}}^{p} e_i \in \Gamma_o^{(1)}$ for a suitable $\beta > 0$ and a suitable index set $\tilde{I}$. If no such point exists, Γ_o cannot have inner points, which contradicts the assumption. ■

Finally, let us consider the condition $\mathbf{f}_n \xrightarrow[R^p]{c-a.s.} \mathbf{f}_o$ in the Markowitz model. Due to the simple form of f_n^1 and f_n^2, the assumptions that $\bar{\rho}_n$ and B_n are strongly consistent estimates for $E\rho$ and B imply continuous convergence.

4.4. Solution sets

Concerning the solution sets, the following result can be derived from Lemma 4.4.2 (i) by Sawaragi et al. (1985).

THEOREM 4.3 *Let the following assumptions be satisfied:*

i) $\mathbf{f}_n \xrightarrow[\Gamma_o]{c-a.s.} \mathbf{f}_o$,

ii) $\Gamma_n \xrightarrow{u-a.s.} \Gamma_o$

iii) $E_n \xrightarrow{u-a.s.} E_o$.

Then $S_n \xrightarrow{u-a.s.} S_o$.

The crucial point is iii) where, again, the condition $E_o = W_o$ comes into play.

If $\mathbf{f}_o$ is one-to-one, a corresponding result is available for $S_n \xrightarrow{l-a.s.} S_o$; it may be directly derived from Theorem 4.2. As the objective function in the Markowitz model is not one-to-one, we will quote the following result (Vogel, 1990):

THEOREM 4.4 *Let the following assumptions be satisfied:*

i) $\mathbf{f}_n \xrightarrow[\Gamma_o]{c-a.s.} \mathbf{f}_o$,

ii) $\Gamma_n \xrightarrow{a.s.} \Gamma_o$,

iii) $\forall x_o \in S_o\ \forall x \in \Gamma_o$ *with* $x \neq x_o\ \exists_{j_o} \in \{1,\ldots,r\} : f_o^{j_o}(x) > f_o^{j_o}(x_o)$.

Lemma 7 (Vogel, 1990) summarizes sufficient conditions for iii). As these conditions do not directly apply to the Markowitz model, the following Proposition will be proved:

PROPOSITION 4.2 *Let Γ_o be convex, suppose that all $f_o^j, j \in \{1,\dots,r\}$, are convex and that one objective function $f_o^{j_o}$ is strictly convex. Then iii) is satisfied.*

Proof. Suppose that there are an $x_o \in S_o$ and an $x_1 \in \Gamma_o, x_1 \neq x_o$, such that for $j \in \{1,\dots,r\}$ $f_o^j(x_1) \leq f_o^j(x_o)$ holds. Consider x_λ with $x_\lambda := \lambda x_o + (1-\lambda)x_1$ $(\lambda \in (0,1))$. With Γ_o being convex, x_λ belongs to Γ_o. Because of the convexity of f_o^j the relation $f_o^j(x_\lambda) \leq f_o^j(x_o)$ holds. Strict convexity of $f_o^{j_o}$ implies $f_o^{j_o}(x_\lambda) < f_o^{j_o}(x_o)$, hence x_o cannot belong to S_o. ■

Hence, in the Markowitz model for a quasi-concave measure and a positive definite covariance matrix B the assumptions of Proposition 4.2 are satisfied.

5. Order semicontinuity

As mentioned, in the single-valued case lower or upper semiconvergence of multifunctions reduces to convergence. Hence, when specializing Theorem 4.2 to the single objective case, only assertions on the continuity of the optimal value function can be derived. Results that are 'vector-valued' generalizations of assertions on the semicontinuous behaviour of the optimal value functions may be obtained using order semicontinuity as introduced by Penot and Sterna-Karwat (1989). We will consider corresponding random notions and discuss stability results in our setting. Let $G_o \subset R^r$.

DEFINITION 5.1 *A sequence $(G_n)_{n\in N}$ of multifunctions with measurable graphs is said to be an*

i) order upper approximation almost surely to G_o $(G_n \xrightarrow{o-u-a.s.} G_o)$ *if* $P\{\omega : G_o \subset \liminf_{n\to\infty}(G_n(\omega) + R_+^r)\} = 1$,

ii) order lower approximation almost surely to G_o $(G_n \xrightarrow{o-l-a.s.} G_o)$ *if* $P\{\omega : \limsup_{n\to\infty} G_n(\omega) \subset G_o + R_+^r\} = 1$.

If $(G_n)_{n\in N}$ is a sequence of subsets of R^r then, analogously,

$$G_n \xrightarrow{o-u} G_o :\Longleftrightarrow G_o \subset \liminf_{n\to\infty}(G_n + R_+^r),$$

$$G_n \xrightarrow{o-l} G_o :\Longleftrightarrow \limsup_{n\to\infty} G_n \subset G_o + R_+^r.$$

An order lower approximation according to Definition 5.1 corresponds to a 'sup-upper continuous' multifunction in the sense of Penot and Sterna-Karwat. Similarly, an order upper approximation corresponds to an 'inf-lower continuous'

by the fact that one has in each case only one side of the usual upper (lower) semicontinuity. Both properties together in general do not imply convergence of $(G_n)_{n\in N}$ to G_o, unless the multifunctions are single-valued.

In the single-valued case the above properties reduce to upper and lower 'semicontinuity', respectively, and according to this relation the notation 'order upper (lower) approximation' has been chosen in this paper.

It is straightforward to prove the following assertion.

LEMMA 5.1 *Let* $G_n(\omega) = \{\mathbf{y}_n(\omega)\}$ $P-a.s.$, $n \in N$, *and* $G_o = \{\mathbf{y}_o\}$. *Then*

i) $G_n \xrightarrow{o-u-a.s.} G_o \iff \limsup_{n\to\infty} \mathbf{y}_n \le \mathbf{y}_o \quad a.s.$

ii) $G_n \xrightarrow{o-l-a.s.} G_o \iff \liminf_{n\to\infty} \mathbf{y}_n \ge \mathbf{y}_o \quad a.s.$

With the above notions the following results may be obtained.

THEOREM 5.1 *Let the following assumptions be satisfied:*

i) $\mathbf{f}_n \xrightarrow[\Gamma_o]{l-a.s.} \mathbf{f}_o$,

ii) $\Gamma_n \xrightarrow{u-a.s.} \Gamma_o$,

iii) $\exists K \in C^p$: $P\{\omega : \exists n_o(\omega)\ \forall n \ge n_o\ \Gamma_n(\omega) \subset K\} = 1$,

iv) $\mathbf{f}_o(\Gamma_o) \subset E_o + R^r_+$.

Then $E_n \xrightarrow{o-l-a.s.} E_o$.

Proof. This theorem may be derived from Theorem 3.1 by Penot and Sterna-Karwat (1989). As Penot and Sterna-Karwat deal with a more general framework and partly different denotations, for the reader's convenience we present the short proof for our special case.

Let $\Omega' := \{\omega \in \Omega : \mathbf{f}_n(\omega) \xrightarrow[\Gamma_o]{l} \mathbf{f}_o,\ \Gamma_n(\omega) \xrightarrow{u} \Gamma_o$ and $\Gamma_n(\omega) \subset K\ \forall_n \ge n_o(\omega)\}$ and consider an $\omega \in \Omega'$.

Suppose that $\mathbf{y}_m \in E_m(\omega)$ for infinitely many m and $\lim_{n\to\infty} \mathbf{y}_m = \mathbf{y}_o$. To $\mathbf{y}_m$ there is an $x_m \in \Gamma_m(\omega)$ with $\mathbf{y}_m = \mathbf{f}_m(x_m, \omega)$. Because of ii) and iii) there is a subsequence $(x_{n_k})_{k\in N}$ with $\lim_{k\to\infty} x_{n_k} = x_o \in \Gamma_o$. Then, i) implies $f^j_o(x_o) \le \liminf_{k\to\infty} f^j_{n_k}(x_{n_k}, \omega) = y^j_o \quad \forall j \in \{1, ..., r\}$. According to iv) to $\mathbf{f}_o(x_o)$ there is a $\tilde{\mathbf{y}}_o \in E_o$ with $\mathbf{f}_o(x_o) \ge \tilde{\mathbf{y}}_o$, hence $\tilde{\mathbf{y}}_o \le \mathbf{y}_o$. ∎

The following assertion corresponds to Proposition 4.1 by Penot and Sterna-Karwat (1989).

THEOREM 5.2 *Let the following assumptions be satisfied:*

i) $\mathbf{f}_n \xrightarrow[\Gamma_0]{u-a.s.} \mathbf{f}_o$,

ii) $\Gamma_n \xrightarrow{l-a.s.} \Gamma_o$,

iii) $P\{\omega : \exists n_o(\omega)\ \forall_n \ge n_o : \mathbf{f}_n(\Gamma_n(\omega), \omega) \subset E_n(\omega) + R^r_+\} = 1$.

Then $E_n \xrightarrow{o-u-a.s.} E_o$.

Proof. Let

$$\Omega' := \{\omega \in \Omega : \quad \mathbf{f}_n(\omega) \xrightarrow[\Gamma_o]{u} \mathbf{f}_o,\ \Gamma_n(\omega) \xrightarrow{l} \Gamma_o \text{ and}$$

and consider an $\omega \in \Omega'$.

Suppose that $\mathbf{y}_o \in E_o$. Hence, there is an $x_o \in \Gamma_o$ with $\mathbf{y}_o = \mathbf{f}_o(x_o)$. Because of ii), there exists a sequence $(x_n)_{n\in N}$ with $x_n \to x_o$ and $x_n \in \Gamma_n(\omega)\ \forall n \geq n_1$. Consequently, employing i), we obtain $\limsup_{n\to\infty} f_n^j(x_n,\omega) \leq f_o^j(x_o)\,\forall j \in \{1,...,r\}$. Now, $\mathbf{f}_n(x_n,\omega)$ with $n \geq n_o(\omega)$ may be represented as $\mathbf{f}_n(x_n,\omega) = \mathbf{y}_n + \mathbf{w}_n$ with $\mathbf{y}_n \in E_n(\omega)$ and $\mathbf{w}_n \geq 0$. Furthermore, let $\mathbf{v}_n := \mathbf{f}_o(x_o) - \mathbf{f}_n(x_n,\omega)$, hence $\mathbf{y}_o = \mathbf{y}_n + \mathbf{v}_n + \mathbf{w}_n$. Because of $\liminf_{n\to\infty}(\mathbf{v}_n + \mathbf{w}_n) \geq 0$, we find a sequence $(\tilde{\mathbf{w}}_n)_{n\in N}$ with $\tilde{\mathbf{w}}_n \geq 0\ \forall n \in N$ and $\lim_{n\to\infty}(\mathbf{y}_n + \tilde{\mathbf{w}}_n) = \mathbf{y}_o$. ■

Let us consider the illustrating Markowitz model: semicontinuous convergence of the objective functions and the required behaviour of the constraint set was investigated in Section 4. We still have to consider the conditions iv) of Theorem 5.1 and iii) of Theorem 5.2, which are usually called 'external stability'. Sufficient conditions for the external stability are given, for instance, by Sawaragi et al. (1985). We can employ the following result, which is formulated in terms of the original problem (P_o), but holds analogously for $(P_n(\omega))$.

LEMMA 5.2 *Let the following conditions be satisfied:*

i) $\mathbf{f}_o(\Gamma_o) \neq \emptyset$,

ii) $\mathbf{f}_o(\Gamma_o) + R_+^r$ *is closed,*

iii) $\exists \mathbf{y}_o \in R^r : \mathbf{f}_o(\Gamma_o) \subset \mathbf{y}_o + R_+^r$.

Then $E_o \neq \emptyset$ *and* $\mathbf{f}_o(\Gamma_o) \subset E_o + R_+^r$.

It is easy to see that these conditions are satisfied in the Markowitz model for Γ_o and Γ_n as well.

Now we turn to the 'in probability' sense. We propose the following definitions, where U_ϵ denotes an ϵ-neighbourhood.

DEFINITION 5.2 *A sequence* $(G_n)_{n\in N}$ *of multifunctions with measurable graphs is said to be an*

i) order upper approximation in probability to G_o $(G_n \xrightarrow{o-u-prob} G_o)$ *if*
$$\forall \epsilon > 0\ \forall K \in C^p : \lim_{n\to\infty} P\{\omega : [G_o \backslash (U_\epsilon G_n(\omega) + R_+^r)] \cap K \neq \emptyset\} = 0,$$

ii) order lower approximation in probability to G_o $(G_n \xrightarrow{o-l-prob} G_o)$ *if*
$$\forall \epsilon > 0\ \forall K \in C^p : \lim_{n\to\infty} P\{\omega : [G_n(\omega) \backslash (U_\epsilon G_o + R_+^r)] \cap K \neq \emptyset\} = 0.$$

Relying on results by Salinetti and Wets (1981), who proved that convergence almost surely of closed-valued measurable multifunctions implies convergence in probability, and on former considerations (Vogel, 1994a, Section 2), we can conclude that

$$(G_n \xrightarrow{o-l-a.s} G_o) \Rightarrow (G_n \xrightarrow{o-l-prob} G_o) \text{ and}$$

$$(G_n \xrightarrow{o-u-a.s} G_o) \Rightarrow (G_n \xrightarrow{o-u-prob} G_o).$$

THEOREM 5.3 *Let the following assumptions be satisfied:*

i) *The functions* f_o^j, $j \in \{1,\dots,r\}$, *are l.s.c. and*
$\forall j \in \{1,\dots,r\}\ \forall \epsilon > 0\ \forall K \in C^{p+1}$:
$\lim_{n\to\infty} P\{\omega : [Epif_n^j(\cdot,\omega) \backslash U_\epsilon Epif_o^j] \cap K \neq \emptyset\} = 0,$

ii) Γ_o *ist closed and*
$\forall \epsilon > 0\ \forall K \in C^p : \lim_{n\to\infty} P\{\omega : [\Gamma_n(\omega) \backslash U_\epsilon \Gamma_o] \cap K \neq \emptyset\} = 0,$

iii) $\exists \tilde{K} \in C^p\ \forall K \in C^p : \lim_{n\to\infty} P\{\omega : \Gamma_n(\omega) \cap K \subset \tilde{K}\} = 1,$

iv) $\mathbf{f}_o(\Gamma_o) \subset E_o + R_+^r$.

Then $E_n \xrightarrow{o-l-prob} E_o$.

The convergence condition in i) denotes a 'lower semicontinuous approximation in probability' to $\mathbf{f}_o$, and the convergence condition in ii) means that $(\Gamma_n)_{n\in N}$ is 'upper semiconvergent in probability' to Γ_o. Sufficient conditions for the assumption i) that are relatively easy to check and apply to many real life situations are given by the author, Vogel (1994b). 'In probability' versions of the results of subsection 4.2 may be used to decide whether ii) is satisfied.

Before we prove the above theorem, we shall present the corresponding 'order upper' part.

THEOREM 5.4 *Let the following assumptions be satisfied:*

i) *The functions* f_o^j, $j \in \{1,\dots,r\}$, *are u.s.c. and*
$\forall j \in \{1,\dots,r\}\ \forall \epsilon > 0\ \forall K \in C^{p+1}$:
$\lim_{n\to\infty} P\{\omega : [Epi(-f_n^j(\cdot,\omega)) \backslash U_\epsilon Epi(-f_o^j)] \cap K \neq \emptyset = 0\},$

ii) $\forall \epsilon > 0\ \forall K \in C^p : \lim_{n\to\infty} P\{\omega : (G_o \backslash U_\epsilon G_n(\omega)) \cap K \neq \emptyset\} = 0,$

iii) $\forall K \in C^p : \lim_{n\to\infty} P\{\omega : [\mathbf{f}_n(\Gamma_n(\omega),\omega) \backslash (E_n(\omega) + R_+^r)] \cap K \neq \emptyset\} = 0.$

Then $E_n \xrightarrow{o-u-prob} E_o$.

We shall show how these results may be derived from the corresponding a.s. assertions. In a similar way the results of Section 4 may be carried over to the 'in probability' setting. However, using the way of proof mentioned we sometimes need additional closedness conditions for Γ_o or the epigraph (hypograph) of $\mathbf{f}_o$, because otherwise the introduced convergence notions a.s. and in probability for multifunctions do not fulfil desirable relations which are known for sequences of random variables.

We start by proving two auxiliary results. The abbreviations Limsup and Liminf denote the limes superior and limes inferior in the set theoretic sense.

LEMMA 5.3 *Let* $\{G_n, n \in N_o\}$ *be a family of multifunctions* $G_n|\Omega \to R^r$ *with measurable graphs.*

i) *If each subsequence of* $(G_n)_{n\in N}$ *contains a subsequence* $(G_{n_k})_{k\in N}$ *with* $\limsup_{k\to\infty} G_{n_k} \subset G_o \quad P-a.s.$ *then*

holds.

ii) If, additionally, G_o is closed-valued, both conditions in part (i) are equivalent.

iii) If and only if each subsequence of $(G_n)_{n\in N}$ contains a subsequence $(G_{n_k})_{k\in N}$ with $\liminf_{k\to\infty} G_{n_k} \supset G_o \quad P-a.s.$, then

$$\forall \epsilon > 0\ \forall K \in C^r : \lim_{n\to\infty} P\{\omega : (G_o(\omega)\backslash U_\epsilon G_n(\omega)) \cap K \neq \emptyset\} = 0. \quad (2)$$

holds.

Proof. We abbreviate $D_{n,\epsilon}(\omega) := G_n(\omega)\backslash U_\epsilon G_o(\omega)$, and denote by B_k the closed ball in R^k with centre 0 and radius k. Furthermore, we recall that

$$\limsup_{n\to\infty} G_n \subset G_o \ P-a.s.$$

implies

$$\forall \epsilon > 0\ \forall K \in C^r : \lim_{n\to\infty} P\left(\bigcup_{m\geq n} \{\omega : D_{m,\epsilon}(\omega) \cap K \neq \emptyset\}\right) = 0 \quad (3)$$

and that equivalence holds if G_o is closed-valued (Vogel, 1994b, proof of Proposition 2.1). In a similar way it can be proved that

$$\liminf_{n\to\infty} G_n \supset G_o \ P-a.s.$$

is equivalent to

$$\forall \epsilon > 0 \quad \forall K \in C^r : \lim_{n\to\infty} P\left(\bigcup_{m\geq n} \{\omega : (G_o(\omega)\backslash U_\epsilon G_n(\omega)) \cap K \neq 0\}\right) = 0. \quad (4)$$

i) Suppose that (1) is not fulfilled, i.e.

$$\exists \epsilon > 0 \quad \exists K \in C^r \quad \exists (n_k)_{k\in N} \quad \exists \alpha > 0 \quad \forall k \in N : P\{\omega : D_{n_k,\epsilon}(\omega) \cap K \neq \emptyset\} > \alpha.$$

Hence, $P(\operatorname{Limsup}_{k\to\infty}\{\omega : D_{n_k,\epsilon}(\omega) \cap K \neq \emptyset\}) > \alpha$, and further

$$P\left(\bigcup_{k\geq l}\{\omega : D_{n_k,\epsilon}(\omega) \cap K \neq \emptyset\}\right) > \alpha \quad \forall l \in N,$$

which contradicts (3).

ii) Let (1) be satisfied and consider a subsequence $(G_n)_{n\in\tilde{N}\subset N}$ of $(G_n)_{n\in N}$. To every $k \in N$ we find $\tilde{n}_k \in \tilde{N}$ such that for $n \geq \tilde{n}_k, n \in \tilde{N}$:

$$P\{\omega : D_{n,\epsilon}(\omega) \cap B_k \neq \emptyset\} < \frac{1}{k}.$$

Let $n_1 := \tilde{n}_1, n_k := \max\{n_{k-1}+1, \tilde{n}_k\}, \tilde{N}_1 := \{n_1, n_2, ...\}$, and
$A_k := \{\omega : D_{n_k,\frac{1}{2^k}}(\omega) \cap B_k \neq \emptyset\}$.
Making use of $P(A_k) < \frac{1}{2^k}$ we obtain

$$\forall \epsilon > 0\, \forall K \in C^r : \sum_{k=1}^{\infty} P\{\omega : D_{n_k,\epsilon}(\omega) \cap K \neq \emptyset\} < \infty$$

and consequently, by the Borel-Cantelli-Lemma, $\forall \epsilon > 0\, \forall K \in C^r$:

$$\lim_{k\to\infty} P\left(\bigcup_{m\geq n_k, m\in\tilde{N}_1} \{\omega : D_{m,\epsilon}(\omega) \cap K \neq \emptyset\}\right)$$

$$= \lim_{n\to\infty, n\in\tilde{N}_1} P\left(\bigcup_{m\geq n, m\in\tilde{N}_1} \{\omega : D_{m,\epsilon}(\omega) \cap K \neq \emptyset\}\right) = 0.$$

This, for closed-valued G_o, implies $\limsup_{n\to\infty, n\in\tilde{N}_1} G_n \subset G_o\ P-a.s.$

iii) The both directions of this part may be proved like part i) and part ii) replacing $D_{n,\epsilon}$ by $\tilde{D}_{n,\epsilon}$ with $\tilde{D}_{n,\epsilon}(\omega) := G_o(\omega)\backslash U_\epsilon G_n(\omega)$. ■

LEMMA 5.4 *Let $(G_n)_{n\in N}$ be a sequence of multifunctions with measurable graphs. Then*

$$\forall K \in C^p : \lim_{n\to\infty} P\{\omega : G_n(\omega) \cap K \neq \emptyset\} = 0 \tag{5}$$

$\Leftrightarrow$

Each subsequence of $(G_n)_{n\in N}$ contains a subsequence $(G_{n_k})_{k\in N}$ with

$$P(\underset{k\to\infty}{Liminf}\{\omega : G_{n_k}(\omega) = \emptyset\}) = 1. \tag{6}$$

Proof. First, assume that (5) is satisfied. As in the proof of part ii) of Lemma 5.3 we can show that

$$\forall K \in C^r : \sum_{k=1}^{\infty} P\{\omega : G_{n_k}(\omega) \cap K \neq \emptyset\} < \infty,$$

hence, by the Borel-Cantelli-Lemma,

$$\forall K \in C^r : P(\underset{k\to\infty}{\text{Liminf}}\{\omega : G_{n_k}(\omega) \cap K = \emptyset\}) = 1.$$

which implies (6).

Secondly, assume that (5) is not fulfilled. Hence

consequently

$$P(\underset{k\to\infty}{\text{Limsup}}\{\omega : G_{n_k}(\omega) \cap K \neq \emptyset\}) > \alpha \quad \text{and}$$

$$P(\underset{k\to\infty}{\text{Liminf}}\{\omega : G_{n_k}(\omega) \cap K = \emptyset\}) < 1 - \alpha$$

which contradicts (6). ■

Now, we are able to prove Theorem 5.3. Note that Theorem 5.4 may be proved in a similar way.

Proof of Theorem 5.3. We consider a subsequence $(E_n)_{n\in\tilde{N}}$ of $(E_n)_{n\in N}$ and show that it contains a subsequence $(E_{n_k})_{k\in N}$ with

$$E_{n_k} \xrightarrow{o-l-a.s.} E_o \quad (k \to \infty).$$

Then, applying Lemma 5.1 to $G_n = E_n(\omega)$ and $G_o = E_o + R^r_+$, the conclusion follows.

We consider the sequence $(\mathbf{f}_n)_{n\in\tilde{N}}$. By Lemma 2.1 in Lachout and Vogel (1999) it contains a subsequence $(\mathbf{f}_n)_{n\in N_1\subset\tilde{N}}$ with $\mathbf{f}_n \xrightarrow{u-a.s.} \mathbf{f}_o$ $(n \in N_1)$. To $(\Gamma_n)_{n\in N_1}$ we find a subsequence $(\Gamma_n)_{n\in N_2\subset N_1}$ with $\Gamma_n \xrightarrow{u-a.s.} \Gamma_o$ $(n \in N_2)$. Eventually, there is a subsequence $(\Gamma_n)_{n\in N_3\subset N_2}$ with $\Gamma_n(\omega) \subset \tilde{K} \quad P-a.s.\ \forall n \geq n_o$, and it remains to apply Theorem 5.1. ■

6. Conclusions

In this paper, we have considered multiobjective programming problems which are approximated by random problems. We started with conditions that imply desirable measurability properties for the approximate problem. It has been shown that the sets of efficient points and the solution sets which belong to them have measurable graphs if, additionally to the general measurability assumptions throughout the paper, the objective functions are lower semicontinuous with respect to x.

Afterwards, we have summarized conditions which guarantee that the sets of efficient points and the solution sets of the original problem are approximated by the corresponding approximate sets in a suitable 'almost surely' sense. Furthermore, we have investigated how these results may be used to derive statements for the Markowitz model with a shortfall constraint, assuming either that there are i.i.d. forecasts for the returns or that the returns have a normal distribution and strongly consistent estimates for the expectation vector and the covariance matrix are available. It turned out that, given an absolutely continuous quasiconcave probability measure (as for example a nonsingular normal distribution), pairwise different expected returns, a positive definite covariance matrix, and a Slater-type condition for the shortfall constraint in the original problem, for

are continuously approximated (almost surely). The assumptions concerning the distribution are brought into play by the shortfall constraint.

However, it may be doubted whether in real-life situations the i.i.d. condition or only the strong consistency of the estimates can be assumed. Therefore, we have asked for convergence notions which can be proved to hold under weaker assumptions on the behaviour of the estimates.

Weak consistency, which can often be maintained even for dependent samples, leads to convergence in probability. To give an example, what appropriate convergence notions 'in probability' look like and by what means statements on the convergence in probability can be derived from corresponding 'a.s.' statements, we have investigated the 'order' behaviour of the sets of efficient points, i.e. we have considered order lower (and upper) approximations. Roughly spoken, the elements of an order lower approximation may be regarded as approximate upper bounds to subsets of the efficiency set of the original problem. Application of the results to the Markowitz model shows that 'order semicontinuity' can be proved without the assumptions that the expected returns are pairwise different and the covariance matrix is positive definite. This implies, for instance, that in the classical Markowitz model (without shortfall constraints) 'order semicontinuous' behaviour in the a.s. ('in probability') sense is guaranteed if only strong (weak) consistency of the estimates can be assumed.

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