

*Dedicated to  
Professor Jakub Gutenbaum  
on his 70th birthday*

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### Quality of solutions for perturbed combinatorial optimization problems

by

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**Abstract:** We consider a general combinatorial optimization problem in which the set of feasible solutions is defined as a given and fixed family of subsets for some finite ground set. To any element of the ground set the so-called weight is associated. The problem consists in finding a feasible subset for which the sum of weights of its elements is the minimum.

When the weights of elements vary or are estimated with some accuracy, then the solution of the problem obtained for some initial weights may appear non-optimal. In this paper we consider the quality of a given solution in the case of weights perturbation or inaccuracy. Namely, we study the relative error of a given solution as a function of particular weights perturbation. We also calculate the maximum perturbation or estimation errors of weights which preserve the optimality of a given solution of the problem.

**Keywords:** combinatorial optimization, sensitivity analysis, accuracy function, stability function, accuracy radius, stability radius.

## 1. Introduction

Let  $E = \{e_1, \dots, e_n\}$  be a finite set. Assume that for  $e \in E$  the so-called *weight*  $c(e) \geq 0$  of element  $e$  is given. Denote by  $c = (c(e_1), \dots, c(e_n))^T \in \mathbf{R}_+^n$  the vector of weights. For a given vector  $c$  and a subset  $S \subseteq E$ , the weight  $w(c, S)$  of subset  $S$  is defined, where

$$w(c, S) = \sum_{e \in S} c(e). \quad (1)$$

Let  $\mathcal{G}$  be an arbitrary family of subsets of  $E$ . Define

$$m(c, \mathcal{G}) = \min\{w(c, S) : S \in \mathcal{G}\} \quad (2)$$

with a standard convention that  $m(c, \mathcal{G}) = +\infty$  if  $\mathcal{G} = \emptyset$ . Similarly,

$$M(c, \mathcal{G}) = \max\{w(c, S) : S \in \mathcal{G}\} \quad (3)$$

and  $M(c, \mathcal{G}) = -\infty$  if  $\mathcal{G} = \emptyset$ .

Assume that there is a specified family of sets  $\mathcal{F} \subseteq 2^E$ . Elements of  $\mathcal{F}$  are called *feasible subsets* or *feasible solutions*.

For a given set  $\mathcal{F}$  and the vector of weights  $c$  the *combinatorial optimization problem* is defined in the following form:

$$\text{Find } F \in \mathcal{F} \text{ such that } w(c, F) = m(c, \mathcal{F}), \quad (4)$$

or

$$\text{Find } F \in \mathcal{F} \text{ such that } w(c, F) = M(c, \mathcal{F}). \quad (5)$$

Most of the discrete optimization problems can be stated in the above form. In the following we will consider mainly the minimization problem (4) and we will use also a more standard notation for this problem:

$$\min_{F \in \mathcal{F}} w(c, F). \quad (6)$$

It is assumed that the set  $\mathcal{F}$  of feasible solutions of the problem is fixed, but the vector of weights may vary or is estimated with errors. Moreover, it is assumed that for some originally specified vector of weights  $c^o \geq 0$  an optimal solution  $F^o \in \mathcal{F}$  of the problem (4) is known. The main question considered in this paper is the following:

*What is the quality of the solution  $F^o$  when the vector of weights changes?*

The quality of an arbitrary feasible solution  $F \in \mathcal{F}$  for a given vector of weights  $c$  may be measured by the value of the so-called *relative error*  $e(c, F)$  of this solution, where

$$e(c, F) = \frac{w(c, F) - m(c, \mathcal{F})}{m(c, \mathcal{F})}. \quad (7)$$

In fact, we will be interested in the maximum value of this error when the vector of weights  $c$  belongs to some specified set. Two particular cases are considered in the following.

In the first case we assume that the weight of any element  $e \in E$  may be perturbed by no more than some given percentage of its original value  $c^o(e)$ . This leads to a concept of the *accuracy function*. The value of the accuracy function for a given  $\delta \in [0, 1)$  is equal to the maximum relative error of the

solution  $F$  under the assumption that the weights of elements are perturbed by no more than  $\delta \cdot 100\%$  of their original values.

In the second case we are interested in absolute perturbations of weights of elements and then the quality of a given solution is described by the so-called *stability* (or *sensitivity*) *function*. For a given  $\varrho \geq 0$  the value of the stability function is equal to the maximum relative error of a given solution under the assumption that no weight of element is increased or decreased by more than  $\varrho$ .

The accuracy and stability functions are introduced in (Libura, 1999). In this paper we give some extensions of the results presented in (Libura, 1999). Namely, we study the case when changes of weights are restricted to some arbitrary subset  $X \subseteq E$  of elements. We also introduce the concept of the so-called *accuracy radius* of the optimal solution  $F^\circ$ . The value of this radius corresponds to the maximum percentage changes of weights which preserve the optimality of  $F^\circ$ . Similarly, for the stability function we have the so-called *stability radius* which corresponds to the maximum absolute deviation of weights which preserve the optimality of  $F^\circ$ .

The paper is organized as follows. In Section 2 we give general formulae for calculating the accuracy and the stability functions as well as corresponding radii. Section 3 describes a method of approximating these functions and radii using the so-called *k*-best solutions of the problem.

## 2. Accuracy function, stability function and corresponding radii

Let  $c^\circ$  be the original vector of weights for which  $F^\circ$  is an optimal solution of the problem (4), i.e.,  $w(c^\circ, F^\circ) = m(c^\circ, \mathcal{F})$ , and assume that  $m(c^\circ, \mathcal{F}) > 0$ . Denote by  $X$  the set of elements of  $E$  for which weights may change. Let

$$C^\circ(X) = \{c \in \mathbf{R}^n : c(e) = c^\circ(e), e \in E \setminus X\}. \quad (8)$$

For a given  $\delta \in [0, 1)$  and  $X \subseteq E$  we will consider vector of weights restricted to the set

$$T_\delta(c^\circ, X) = \{c \in C^\circ(X) : |c(e) - c^\circ(e)| \leq \delta \cdot c^\circ(e), e \in X\}. \quad (9)$$

This means that elements of the set  $T_\delta(c^\circ, X)$  are obtained from the vector of weights  $c^\circ$  by increasing or decreasing any weight  $c^\circ(e)$ ,  $e \in X$ , by at most  $\delta \cdot 100\%$  and keeping all weights  $c^\circ(e)$ ,  $e \in E \setminus X$ , unchanged.

For an optimal solution  $F^\circ$ , an arbitrary set of elements  $X \subseteq E$ , and  $\delta \in [0, 1)$ , the value of the *accuracy function*  $a(F^\circ, X, \delta)$  is defined as the maximum of the relative error of the solution  $F^\circ$  over the set  $T_\delta(c^\circ, X)$ , i.e.,

$$a(F^\circ, X, \delta) = \max_{c \in T_\delta(c^\circ, X)} e(c, F^\circ). \quad (10)$$

In a similar way the *stability function*  $s(F^o, \varrho)$  is defined. Let  $\varrho(X, c^o) = \min\{c^o(e) : e \in X\}$ . For  $\varrho \in [0, \varrho(X, c^o)]$  and  $X \subseteq E$  define

$$K_\varrho(c^o, X) = \{c \in C^o(X) : |c(e) - c^o(e)| \leq \varrho, e \in X\}.$$

The stability function provides the maximum relative error of the solution  $F^o$  over the set  $K_\varrho(c^o, X)$ , i.e.,

$$s(F^o, X, \varrho) = \max_{c \in K_\varrho(c^o, X)} e(c, F^o). \quad (11)$$

In (Libura, 1999) the general formulae for calculating values of the accuracy and the stability functions in the case of  $X = E$  were given. The following theorem extends these results for an arbitrary subset  $X \subseteq E$ . Denote for  $S', S'' \subseteq E$ ,

$$S' \otimes S'' = (S' \setminus S'') \cup (S'' \setminus S').$$

**Theorem 1** For an optimal solution  $F^o$ ,  $X \subseteq E$ , and  $\delta \in [0, 1)$ ,

$$a(F^o, X, \delta) = \max_{F \in \mathcal{F}} \frac{w(c^o, F^o) - w(c^o, F) + \delta \cdot w(c^o, (F^o \otimes F) \cap X)}{w(c^o, F) - \delta \cdot w(c^o, F \cap X)}. \quad (12)$$

For an optimal solution  $F^o$ ,  $X \subseteq E$ , and  $\varrho \in [0, \varrho(X, c^o))$ ,

$$s(F^o, X, \varrho) = \max_{F \in \mathcal{F}} \frac{w(c^o, F^o) - w(c^o, F) + \varrho \cdot |(F^o \otimes F) \cap X|}{w(c^o, F) - \varrho \cdot |F \cap X|}. \quad (13)$$

**Proof.** We will prove only the first part of the theorem; the proof of formula (13) is analogous and will be omitted. From (10) and (7) we have:

$$\begin{aligned} a(F^o, X, \delta) &= \max_{c \in T_\delta(c^o, X)} \frac{w(c, F^o) - m(c, \mathcal{F})}{m(c, \mathcal{F})} \\ &= \max_{c \in T_\delta(c^o, X)} \left( \frac{w(c, F^o)}{m(c, \mathcal{F})} - 1 \right) \\ &= \max_{c \in T_\delta(c^o, X)} \left( \frac{w(c, F^o)}{\min_{F \in \mathcal{F}} w(c, F)} - 1 \right) \\ &= \max_{c \in T_\delta(c^o, X)} \max_{F \in \mathcal{F}} \left( \frac{w(c, F^o)}{w(c, F)} - 1 \right) \\ &= \max_{F \in \mathcal{F}} \max_{c \in T_\delta(c^o, X)} \frac{w(c, F^o) - w(c, F)}{w(c, F)} \\ &= \max_{F \in \mathcal{F}} \max_{c \in T_\delta(c^o, X)} \frac{l}{w(c, F)}, \end{aligned} \quad (14)$$

where

$$l = w(c, (F^o \setminus F) \cap X) - w(c, (F \setminus F^o) \cap X) + w(c, (F^o \setminus F) \setminus X) - w(c, (F \setminus F^o) \setminus X).$$

The maximum over  $c \in T_\delta(c^\circ, X)$  in (14) is attained when  $c(e) = c^\circ(e) + \delta \cdot c^\circ(e)$  for  $e \in (F^\circ \setminus F) \cap X$ , and  $c(e) = c^\circ(e) - \delta \cdot c^\circ(e)$  for  $e \in F \cap X$ . Observe that for all other elements of the set  $E$  the weights remain unchanged, i.e.,  $c(e) = c^\circ(e)$  for  $e \in E \setminus X$ . Thus, finally, we have

$$a(F^\circ, X, \delta) = \max_{F \in \mathcal{F}} \frac{l'}{w(c^\circ, F) - \delta \cdot w(c^\circ, F \cap X)},$$

where

$$\begin{aligned} l' &= w(c^\circ, (F^\circ \setminus F) \cap X) + \delta \cdot w(c^\circ, (F^\circ \setminus F) \cap X) - w(c^\circ, (F \setminus F^\circ) \cap X) \\ &\quad + \delta \cdot w(c^\circ, (F \setminus F^\circ) \cap X) + w(c^\circ, (F^\circ \setminus F) \setminus X) - w(c^\circ, (F \setminus F^\circ) \setminus X) \\ &= w(c^\circ, F^\circ \setminus F) - w(c^\circ, F \setminus F^\circ) + \delta \cdot w(c^\circ, (F^\circ \setminus F) \cap X) \\ &\quad + \delta \cdot w(c^\circ, (F \setminus F^\circ) \cap X) \\ &= w(c^\circ, F^\circ) - w(c^\circ, F) + \delta \cdot w(c^\circ, (F^\circ \otimes F) \cap X). \end{aligned}$$

■

As a simple corollary to Theorem 1 we obtain the formulae derived in (Libura, 1999) for the accuracy and the stability functions in the case of  $X = E$ :

**Corollary 1** For an optimal solution  $F^\circ$  and  $\delta \in [0, 1)$ ,

$$a(F^\circ, E, \delta) = \max_{F \in \mathcal{F}} \frac{w(c^\circ, F^\circ) - w(c^\circ, F) + \delta \cdot w(c^\circ, F^\circ \otimes F)}{(1 - \delta)w(c^\circ, F)}.$$

For an optimal solution  $F^\circ$  and  $\varrho \in [0, \varrho(X, c^\circ))$ ,

$$s(F^\circ, E, \varrho) = \max_{F \in \mathcal{F}} \frac{w(c^\circ, F^\circ) - w(c^\circ, F) + \varrho \cdot |F^\circ \otimes F|}{w(c^\circ, F) - \varrho \cdot |F|}.$$

Let

$$a_F(F^\circ, X, \delta) = \frac{w(c^\circ, F^\circ) - w(c^\circ, F) + \delta \cdot w(c^\circ, (F^\circ \otimes F) \cap X)}{w(c^\circ, F) - \delta \cdot w(c^\circ, F \cap X)} \quad (15)$$

and

$$s_F(F^\circ, X, \varrho) = \frac{w(c^\circ, F^\circ) - w(c^\circ, F) + \varrho \cdot |(F^\circ \otimes F) \cap X|}{w(c^\circ, F) - \varrho \cdot |F \cap X|}. \quad (16)$$

Theorem 1 states that the accuracy function can be computed for a given  $F^\circ$  and  $X \subseteq E$  as a pointwise maximum of  $|\mathcal{F}|$  functions  $a_F(F^\circ, X, \delta)$ . Observe that due to the fact that  $a_{F^\circ}(F^\circ, X, \delta) = 0$  for  $\delta \in [0, 1)$ , only these functions  $a_F(F^\circ, X, \delta)$  which are nonnegative for a given  $\delta$  must be considered in formula (12). Moreover, it is easy to see that any function  $a_F(F^\circ, X, \delta)$  is a continuous nondecreasing function of  $\delta$  in an interval of  $\delta \in [0, 1)$  in which  $a_F(F^\circ, X, \delta) \geq 0$ , i.e., for these values of  $\delta$  we have  $\frac{\partial a_F(F^\circ, X, \delta)}{\partial \delta} \geq 0$ . One can also easily show that

$$\frac{\partial^2 a_F(F^\circ, X, \delta)}{\partial \delta^2} = \frac{2w(c^\circ, F \cap X)}{w(c^\circ, F) - \delta \cdot w(c^\circ, F \cap X)} \cdot \frac{\partial a_F(F^\circ, X, \delta)}{\partial \delta},$$

which means that if  $a_F(F^o, X, \delta) \geq 0$ , then also  $\frac{\partial^2 a_F(F^o, X, \delta)}{\partial \delta^2} \geq 0$  for  $F \in \mathcal{F}$ , i.e., all functions  $a_F(F^o, X, \delta)$  are convex for any  $\delta \in [0, 1)$  for which  $a_F(F^o, X, \delta) \geq 0$ .

Similar arguments can be used for the stability function  $s(F^o, X, \varrho)$ . It can be expressed as a pointwise maximum of  $|\mathcal{F}|$  functions  $s_F(F^o, X, \varrho)$ . This leads to the following corollary:

**Corollary 2** *For a given optimal solution  $F^o$  and  $X \subseteq E$ , the accuracy function  $a(F^o, X, \delta)$  is a nondecreasing convex function of  $\delta$  with the number of breakpoints not bigger than  $|\mathcal{F}|$  in the interval  $\delta \in [0, 1)$ .*

*For a given optimal solution  $F^o$  and  $X \subseteq E$ , the stability function  $s(F^o, X, \varrho)$  is a nondecreasing convex function of  $\varrho$  with the number of breakpoints not bigger than  $|\mathcal{F}|$  in the interval  $\varrho \in [0, \varrho(X, c^o))$ .*

The formulae (12) and (13) can hardly be regarded as efficient tools to compute exact values of the accuracy and the stability functions for a given solution  $F^o$ , but they appear useful in calculating upper and lower approximations of these functions (see Section 3). They can be also used to introduce the so-called accuracy radius and the stability radius as well as to derive formulae to calculate exact and approximate values of these radii.

Observe that if  $F^o$  is an optimal solution of the problem (4) then, obviously,  $a(F^o, X, 0) = 0$ . It is of special interest to know the maximum value of  $\delta$  for which  $a(F^o, X, \delta) = 0$ . This value is called the *accuracy radius* of the solution  $F^o$  with respect to the set  $X$  and is denoted by  $r^a(F^o, X)$ . Formally

$$r^a(F^o, X) = \sup\{\delta \in [0, 1) : a(F^o, X, \delta) = 0\}. \quad (17)$$

The accuracy radius can be introduced in an alternative way in the framework of the so-called "tolerance approach" described in Wendell (1982, 1984), Labbé et al. (1991).

The practical importance of the accuracy radius consists in the fact that given the value  $r = r^a(F^o, X)$  we know that the weight of any element  $e$  belonging to the set  $X$  may be perturbed (increased or decreased) arbitrarily by  $r \cdot 100\%$  (or less) without destroying the optimality of  $F^o$ . Similarly, if we know that the weights of elements in  $X$  are estimated with the accuracy  $r \cdot 100\%$ , then we can guarantee that the solution  $F^o$ , calculated for the estimated vector of weights  $c^o$ , is also optimal for the actual vector of weights.

In an analogous way the *stability radius*  $r^s(F^o, X)$  of the solution  $F^o$  with respect to the set  $X$  can be defined. Formally,

$$r^s(F^o, X) = \sup\{\delta \in [0, \varrho(X, c^o)) : s(F^o, X, \delta) = 0\}. \quad (18)$$

Observe that the value of stability radius gives the maximum absolute deviation of any weight of element from the set  $X$  which does not destroy the optimality of the solution  $F^o$ . The stability radius is a standard object studied in the sensitivity analysis for combinatorial optimization problems. It is typically derived

from the so-called *stability region* of the optimal solution (see e.g. Sotskov et al., 1995, 1998, Libura, 1996, Greenberg, 1998). Let

$$\mathcal{F}_X = \{F \in \mathcal{F} : w(c^o, (F^o \otimes F) \cap X) \neq 0\}$$

and

$$\mathcal{F}'_X = \{F \in \mathcal{F} : (F^o \otimes F) \cap X \neq \emptyset\}.$$

The following theorem gives the general formulae for calculating the accuracy radius and the stability radius of the solution  $F^o$  with respect to the set  $X$ .

**Theorem 2** For an optimal solution  $F^o$  and  $X \subseteq E$ ,

$$r^a(F^o, X) = \min \left\{ 1, \min_{F \in \mathcal{F}_X} \frac{w(c^o, F) - w(c^o, F^o)}{w(c^o, (F^o \otimes F) \cap X)} \right\} \quad (19)$$

and

$$r^s(F^o, X) = \min \left\{ \varrho(X, c^o), \min_{F \in \mathcal{F}'_X} \frac{w(c^o, F) - w(c^o, F^o)}{|(F \otimes F^o) \cap X|} \right\}. \quad (20)$$

**Proof.** We will prove only (19); the proof of (20) is analogous and will be omitted. Consider first the case  $\mathcal{F}_X = \emptyset$ . Then, according to a standard convention,  $\min_{F \in \mathcal{F}_X} \frac{w(c^o, F) - w(c^o, F^o)}{w(c^o, (F^o \otimes F) \cap X)} = \infty$ , so we have to prove that in this case  $r^a(F^o, X) = 1$ . Indeed, observe that when  $\mathcal{F}_X = \emptyset$ , then for any  $F \in \mathcal{F}$  we have  $a_F(F^o, X, \delta) \leq 0$  for  $\delta \in [0, 1)$ . From (12) it follows that in this case  $a(F^o, X, \delta) = a_{F^o}(F^o, X, \delta) = 0$  for any  $\delta \in [0, 1)$ , and from (17) we have  $r^a(F^o, X) = 1$ .

Assume now that  $\mathcal{F}_X \neq \emptyset$  and for any  $F \in \mathcal{F}_X$  consider the value of ratio  $\delta_F = \frac{w(c^o, F) - w(c^o, F^o)}{w(c^o, (F^o \otimes F) \cap X)}$ . If  $\delta_F < 1$ , then it gives the maximum value of  $\delta \in [0, 1)$  for which  $a_F(F^o, X, \delta) \leq 0$ . Otherwise,  $a_F(F^o, X, \delta) \leq 0$  in the whole interval  $\delta \in [0, 1)$ . From (12) it now follows that if  $\delta_F \geq 1$  for any  $F \in \mathcal{F}_X$ , then  $a(F^o, X, \delta) = 0$  for  $\delta \in [0, 1)$ . In this case we have  $r^a(F^o, X) = 1$ . Otherwise, the maximum value of  $\delta$  for which  $a(F^o, X, \delta) = 0$ , is equal to the minimum value of  $\delta_F$  over the set  $\mathcal{F}_X$ , which proves (19). ■

### Example

As an example of the combinatorial optimization problem (4) we will consider the well known symmetric traveling salesman problem defined on the graph  $G$  shown in Fig. 1. In this case the ground set  $E$  is the set of all edges of the graph, i.e.,  $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}\}$ . Weights of edges are given by respective elements of the vector  $c^o = (3, 4, 3, 2, 2, 6, 4, 0, 3, 5, 1)^T$ .

The set of feasible solutions  $\mathcal{F}$  is equal to the family of all subsets of edges which form the Hamiltonian cycles in the graph  $G$ . There are only 10 such subsets for this small example of the traveling salesman problem. They are listed below together with their weights for the vector  $c^o$ :

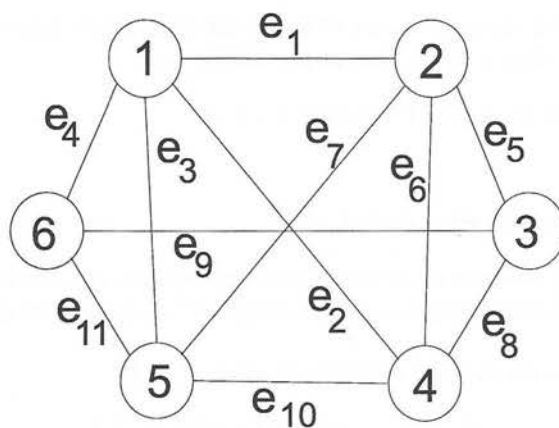


Figure 1. Graph G.

$$\begin{array}{ll}
 F^0 = \{e_1, e_5, e_8, e_{10}, e_{11}, e_4\}, & w(c^o, F^0) = 13, \\
 F^1 = \{e_2, e_8, e_5, e_7, e_{11}, e_4\}, & w(c^o, F^1) = 13, \\
 F^2 = \{e_1, e_7, e_{11}, e_9, e_8, e_2\}, & w(c^o, F^2) = 15, \\
 F^3 = \{e_1, e_6, e_8, e_9, e_{11}, e_3\}, & w(c^o, F^3) = 16, \\
 F^4 = \{e_1, e_7, e_{10}, e_8, e_9, e_4\}, & w(c^o, F^4) = 17, \\
 F^5 = \{e_3, e_7, e_6, e_8, e_9, e_4\}, & w(c^o, F^5) = 18, \\
 F^6 = \{e_1, e_5, e_9, e_{11}, e_{10}, e_2\}, & w(c^o, F^6) = 18, \\
 F^7 = \{e_2, e_6, e_5, e_9, e_{11}, e_3\}, & w(c^o, F^7) = 19, \\
 F^8 = \{e_2, e_{10}, e_7, e_5, e_9, e_4\}, & w(c^o, F^8) = 20, \\
 F^9 = \{e_3, e_{10}, e_6, e_5, e_9, e_4\}, & w(c^o, F^9) = 21.
 \end{array}$$

Consider an optimal solution  $F^o = \{e_1, e_5, e_8, e_{10}, e_{11}, e_4\}$  with  $w(c^o, F^o) = 13$ . Let  $X = \{e_4, e_9, e_{11}\}$ , i.e., the set of edges, for which weights may change, contains all of the edges incident to the vertex 6 of the graph  $G$ .

The accuracy function  $a(F^o, X, \delta)$ , computed according to Theorem 1 and drawn with the *Mathematica* package, is shown in Fig. 2.

From the plot of the accuracy function  $a(F^o, X, \delta)$  one can read, for example, that if weights of edges  $e_4, e_9, e_{11}$  are perturbed by no more than 60% of their original values, then the relative error of the solution  $F^o$  may not exceed 8%, approximately. Moreover, if perturbations of these weights do not exceed 40%, then the solution  $F^o$  remains optimal.

The first breakpoint of the function  $a(F^o, X, \delta)$ , i.e.,  $\delta = 0.4$ , gives the value of the accuracy radius. The same value can be calculated from Theorem 2 without computing the accuracy function in the interval  $[0, 1)$ . To use this theorem it is necessary to find the set  $\mathcal{F}_X$ . One can easily check that  $w(c^o, (F^o \otimes F^i) \cap X) = 0$  for  $i = 1$  and  $w(c^o, (F^o \otimes F^i) \cap X) \neq 0$  for  $i = 2, 3, \dots, 9$ . Thus, we have  $\mathcal{F}_X = \{F^i : i = 2, 3, \dots, 9\}$ . By calculating values of ratios  $\delta_F = \frac{w(c^o, F) - w(c^o, F^o)}{w(c^o, (F^o \otimes F) \cap X)}$  for  $F \in \mathcal{F}_X$  we can verify that the minimum of  $\delta_F$  over



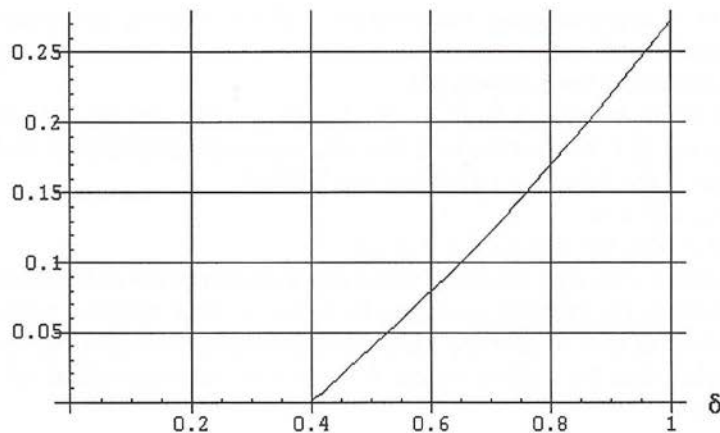


Figure 2. The accuracy function of the solution  $F^o$ .

the set  $\mathcal{F}_X$  is achieved for the solution  $F^2$  and  $r^a(F^o, X) = \frac{w(c^o, F^2) - w(c^o, F^o)}{w(c^o, (F^o \otimes F^2) \cap X)} = \frac{w(c^o, F^2) - w(c^o, F^o)}{c^o(e_4) + c^o(e_9)} = \frac{15 - 13}{2 + 3} = 0.4$ .

In an analogous way values of the stability function and the stability radius of the solution  $F^o$  can be computed. We have

$$\varrho(X, c^o) = \min\{c(e_4), c(e_9), c(e_{11})\} = \min\{2, 3, 1\} = 1.$$

From (13) it follows now that  $s(F^o, X, \varrho) = 0$  for  $\varrho \in [0, 1)$ . This means that  $r^s(F^o, X) = 1$ . The same value of the stability radius can be computed from formula (20). From the fact that  $r^s(F^o, X) = 1$  it follows that the weights of edges incident to the vertex 6 in  $G$  may be perturbed simultaneously (increased or decreased) by at most 1 without destroying the optimality of the tour  $F^o$ .

The combinatorial optimization problem considered in this example is rather small and it is possible to use directly Theorems 1 and 2 to calculate the exact values of the accuracy function, the stability function as well as corresponding radii. For larger instances of difficult combinatorial optimization problems formulae (12), (13) and (19), (20) lead to intractable computations. On the other hand, specially for these problems it would be desired to have practically efficient methods to calculate at least approximate values of the accuracy and stability functions as well as accuracy and stability radii. In the following section such a method is described.

### 3. Envelopes of functions and bounds for radii

A method of approximating the accuracy and the stability functions and the corresponding radii described in this section is based on the so-called subsets of  $k$ -best solutions of the problem (4).

Let  $k$  be an integer,  $k \in \{1, \dots, |\mathcal{F}|\}$ . For  $c \in \mathbf{R}_+^n$  the set  $\mathcal{F}(c, k) \subseteq \mathcal{F}$  is called the *set of  $k$ -best solutions* of the combinatorial optimization problem (4) if and only if the following conditions are fulfilled:

- (i)  $|\mathcal{F}(c, k)| = k$ ,
- (ii)  $M(c, \mathcal{F}(c, k)) \leq m(c, \mathcal{F} \setminus \mathcal{F}(c, k))$ .

In other words,  $\mathcal{F}(c, k)$  is the set of first  $k$  elements from a list of feasible solutions of the problem (4) ordered according to nondecreasing weights. Observe that such a subset is not, in general, uniquely determined. Thus, we will assume in the following that for a given integer  $k$  and  $c = c^o$  some particular set  $\mathcal{F}(c^o, k)$  is considered.

Let

$$L_k(c) = M(c, \mathcal{F}(c, k)) - m(c, \mathcal{F}(c, k)). \quad (21)$$

Observe that for given  $c$  and  $k$  the value  $L_k(c)$  is uniquely determined. This value gives the difference between “the worst” and “the best” solution in any set of  $k$ -best solutions.

The idea of using sets of  $k$ -best solutions in stability analysis was exploited in several papers (see e.g. Piper and Zoltners, 1976, Wilson and Jain, 1988, Libura et al., 1998). This approach is motivated by the fact that calculating  $k$ -best solutions is relatively inexpensive if one compares the time necessary to find a single solution and the time needed to compute some set of  $k$ -best solutions (see e.g. Lawler, 1972, Piper and Zoltners, 1976, Hamacher and Queyranne, 1985/6, Wilson and Jain, 1988, van der Poort et al., 1999). On the other hand, the information provided by the set of  $k$ -best solutions allows to derive useful evaluations of various parameters studied in the sensitivity analysis.

In this section we will follow the approach used in Libura (1999) to calculate lower and upper envelopes of the accuracy function in the case of  $X = E$ . We will state analogous results for arbitrary subset  $X \subseteq E$  and both functions considered. We will also give lower and upper bounds of the accuracy radius and the stability radius with respect to the set  $X$ .

Assume that for some integer  $k \geq 2$  and  $c = c^o$  the set  $\mathcal{F}(c^o, k)$  of  $k$ -best solutions of the problem (4) is known. Let

$$a^k(F^o, X, \delta) = \max_{F \in \mathcal{F}(c^o, k)} a_F(F^o, X, \delta), \quad (22)$$

where, as defined before,

$$a_F(F^o, X, \delta) = \frac{w(c^o, F^o) - w(c^o, F) + \delta \cdot w(c^o, (F^o \otimes F) \cap X)}{w(c^o, F) - \delta \cdot w(c^o, F \cap X)}.$$

Denote for  $q \geq 0$ ,

$$A^k(F^o, X, \delta, q) = \frac{w(c^o, F^o) + \delta \cdot w(c^o, F^o \cap X)}{w(c^o, F^o) + L_k(c^o) - \delta \cdot q} - 1, \quad (23)$$

and let

$$q_{min}^a = \min\{q_1, q_2, q_3\}, \quad (24)$$

where

$$q_1 = w(c^o, F^o) + L_k(c^o), \quad q_2 = w(c^o, X), \quad q_3 = \max_{F \in \mathcal{F} \setminus \mathcal{F}(c^o, k)} w(c^o, F \cap X).$$

Observe that in the definition of  $q_{min}$  two first components, i.e.,  $q_1$  and  $q_2$ , are easy to compute. On the other hand, the evaluation of  $q_3$  may lead, in general, to a difficult optimization problem. However, it is important to stress that in the following we will not need the exact value  $q_{min}^a$  but only an upper bound of this value, which can be easily determined from  $q_1$  and  $q_2$ . Nevertheless, we would be interested in using possibly small upper bound of  $q_{min}^a$ , because this leads to a tighter evaluation of the accuracy function and the accuracy radius. Therefore, in some cases, calculating the value of  $q_3$  or some its upper bound may be considered (see e.g. the example in this section).

The following theorem gives lower and upper bounds for the accuracy function based on  $k$ -best solutions.

**Theorem 3** For  $X \subseteq E$ ,  $\delta \in [0, 1)$  and arbitrary  $q \in [q_{min}^a, w(c^o, F^o) + L_k(c^o)]$  the following inequalities hold:

$$a^k(F^o, X, \delta) \leq a(F^o, X, \delta) \leq \max\{a^k(F^o, X, \delta), A^k(F^o, X, \delta, q)\}. \quad (25)$$

**Proof.** The inequality  $a^k(F^o, X, \delta) \leq a(F^o, X, \delta)$  is a direct consequence of the fact that for any  $X \subseteq E$  and  $\delta \in [0, 1)$  the optimization problem (22) defining  $a^k(F^o, X, \delta)$  is a restriction of the problem (12), which gives the value of  $a(F^o, X, \delta)$ . Indeed, in both problems the objective functions are the same and for the sets of feasible solutions the relation  $\mathcal{F}(c^o, k) \subseteq \mathcal{F}$  is fulfilled.

To prove the upper bound of  $a(F^o, X, \delta)$  in (25) observe that

$$a(F^o, X, \delta) = \max \left\{ \max_{F \in \mathcal{F}(c^o, k)} a_F(F^o, X, \delta), \max_{F \in \mathcal{F} \setminus \mathcal{F}(c^o, k)} a_F(F^o, X, \delta) \right\}. \quad (26)$$

Moreover,

$$\begin{aligned} a_F(F^o, X, \delta) &= \frac{w(c^o, F^o) - w(c^o, F) + \delta \cdot w(c^o, (F^o \otimes F) \cap X)}{w(c^o, F) - \delta \cdot w(c^o, F \cap X)} \\ &= \frac{w(c^o, F^o) - w(c^o, F) + \delta \cdot w(c^o, F^o \cap X) + \delta \cdot w(c^o, F \cap X)}{w(c^o, F) - \delta \cdot w(c^o, F \cap X)} \\ &\quad - \frac{2\delta \cdot w(c^o, F \cap F^o \cap X)}{w(c^o, F) - \delta \cdot w(c^o, F \cap X)} \end{aligned}$$

$$\begin{aligned}
&= \frac{w(c^\circ, F^\circ) + \delta \cdot w(c^\circ, F^\circ \cap X)}{w(c^\circ, F) - \delta w(c^\circ, F \cap X)} - 1 - \frac{2\delta \cdot w(c^\circ, F \cap F^\circ \cap X)}{w(c^\circ, F) - \delta \cdot w(c^\circ, F \cap X)} \\
&\leq \frac{w(c^\circ, F^\circ) + \delta \cdot w(c^\circ, F^\circ \cap X)}{w(c^\circ, F) - \delta \cdot w(c^\circ, F \cap X)} - 1.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\max_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} a_F(F^\circ, X, \delta) &\leq \max_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} \frac{w(c^\circ, F^\circ) + \delta \cdot w(c^\circ, F^\circ \cap X)}{w(c^\circ, F) - \delta w(c^\circ, F \cap X)} - 1 \\
&= \frac{w(c^\circ, F^\circ) + \delta \cdot w(c^\circ, F^\circ \cap X)}{\min_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} [w(c^\circ, F) - \delta \cdot w(c^\circ, F \cap X)]} - 1 \leq A^k(F^\circ, X, \delta, q_{min}^a).
\end{aligned}$$

To prove the last inequality observe that for  $i = 1, 2, 3$ ,

$$\min_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} [w(c^\circ, F) - \delta w(c^\circ, F \cap X)] \geq w(c^\circ, F^\circ) + L_k(c^\circ) - \delta \cdot q_i. \quad (27)$$

Indeed, we have

$$\begin{aligned}
\min_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} [w(c^\circ, F) - \delta \cdot w(c^\circ, F \cap X)] &\geq (1 - \delta) \min_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} w(c^\circ, F) \\
&\geq (1 - \delta)(w(c^\circ, F^\circ) + L_k(c^\circ)) = w(c^\circ, F^\circ) + L_k(c^\circ) - \delta \cdot q_1.
\end{aligned}$$

Also

$$\begin{aligned}
\min_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} [w(c^\circ, F) - \delta \cdot w(c^\circ, F \cap X)] &\geq \min_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} [w(c^\circ, F) - \delta \cdot w(c^\circ, X)] \\
&\geq w(c^\circ, F^\circ) + L_k(c^\circ) - \delta \cdot w(c^\circ, X) = w(c^\circ, F^\circ) + L_k(c^\circ) - \delta \cdot q_2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\min_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} [w(c^\circ, F) - \delta \cdot w(c^\circ, F \cap X)] \\
&\geq \min_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} w(c^\circ, F) - \delta \cdot \max_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} w(c^\circ, F \cap X) \\
&\geq w(c^\circ, F^\circ) + L_k(c^\circ) - \delta \cdot q_3.
\end{aligned}$$

Obviously, from

$$\max_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} a_F(F^\circ, X, \delta) \leq A^k(F^\circ, X, \delta, q_{min}^a)$$

it follows that

$$\max_{F \in \mathcal{F} \setminus \mathcal{F}(c^\circ, k)} a_F(F^\circ, X, \delta) \leq A^k(F^\circ, X, \delta, q)$$

for any  $q \in [q_{min}^a, w(c^\circ, F^\circ) + L_k(c^\circ)]$ , and, finally, from (26) we obtain the upper bound of  $a(F^\circ, X, \delta)$  in (25). ■

Theorem 3 easily specifies for the case of  $X = E$ . Observe that when  $X = E$ , then  $q_{min}^a = q_1$ , because  $q_2 \geq q_1$  and  $q_3 \geq q_1$ . We have thus the following corollary (see Libura, 1999):

**Corollary 3** For an optimal solution  $F^o$  and  $\delta \in [0, 1)$ ,

$$a^k(F^o, E, \delta) \leq a(F^o, E, \delta) \leq \max\{a^k(F^o, E, \delta), A^k(F^o, E, \delta, q_1)\},$$

where

$$a^k(F^o, E, \delta) = \max_{F \in \mathcal{F}(c^o, k)} \frac{w(c^o, F^o) - w(c^o, F) + \delta \cdot w(c^o, F^o \otimes F)}{(1 - \delta)w(c^o, F)}$$

and

$$A^k(F^o, E, \delta, q_1) = \frac{(1 + \delta)}{(1 - \delta)} \cdot \frac{w(c^o, F^o)}{w(c^o, F^o) + L_k(c^o)} - 1.$$

In a similar way bounds for the stability function based on  $k$ -best solutions may be derived. Let

$$s^k(F^o, X, \varrho) = \max_{F \in \mathcal{F}(c^o, k)} s_F(F^o, X, \varrho), \quad (28)$$

where

$$s_F(F^o, X, \varrho) = \frac{w(c^o, F^o) - w(c^o, F) + \varrho \cdot |(F^o \otimes F) \cap X|}{w(c^o, F) - \varrho \cdot |F \cap X|}.$$

Define for  $q \geq 0$ ,

$$S^k(F^o, X, \varrho, q) = \frac{w(c^o, F^o) + \varrho \cdot |F^o \cap X|}{w(c^o, F^o) + L_k(c^o) - \varrho \cdot q} - 1 \quad (29)$$

and let

$$q_{min}^s = \max_{F \in \mathcal{F} \setminus \mathcal{F}(c^o, k)} |F \cap X|.$$

The following theorem holds:

**Theorem 4** For an optimal solution  $F^o$ ,  $X \subseteq E$ ,  $\varrho \in [0, \varrho(X, c^o))$  and arbitrary  $q \in [q_{min}^s, \frac{w(c^o, F^o) + L_k(c^o)}{\varrho(X, c^o)})$  the following inequalities hold:

$$s^k(F^o, X, \varrho) \leq s(F^o, X, \varrho) \leq \max\{s^k(F^o, X, \varrho), S^k(F^o, X, \varrho, q)\}. \quad (30)$$

**Proof.** The proof is analogous as in Theorem 3, so we will omit some details. Observe that

$$s(F^o, X, \varrho) = \max \left\{ \max_{F \in \mathcal{F}(c^o, k)} s_F(F^o, X, \varrho), \max_{F \in \mathcal{F} \setminus \mathcal{F}(c^o, k)} s_F(F^o, X, \varrho) \right\}. \quad (31)$$

In a similar way as for  $a_F(F^o, X, \delta)$  it can be shown that

$$s_F(F^o, X, \varrho) \leq \frac{w(c^o, F^o) + \varrho \cdot |F^o \cap X|}{w(c^o, F) - \varrho \cdot |F \cap X|} - 1.$$

Thus,

$$\max_{F \in \mathcal{F} \setminus \mathcal{F}(c^o, k)} s_F(F^o, X, \varrho) \leq \frac{w(c^o, F^o) + \varrho \cdot |F^o \cap X|}{\min_{F \in \mathcal{F} \setminus \mathcal{F}(c^o, k)} (w(c^o, F) - \varrho \cdot |F \cap X|)} - 1.$$

But

$$\min_{F \in \mathcal{F} \setminus \mathcal{F}(c^o, k)} (w(c^o, F) - \varrho \cdot |F \cap X|) \leq w(c^o, F^o) + L_k(c^o) - q_{min}^s,$$

which leads to inequality

$$\max_{F \in \mathcal{F} \setminus \mathcal{F}(c^o, k)} s_F(F^o, X, \varrho) \leq S^k(F^o, X, \varrho, q_{min}^s).$$

Finally, taking into account that (28) is a restriction of (13) and using (31) we obtain (30). ■

Theorems 3 and 4 can be used now to derive simple lower and upper bounds of the accuracy radius  $r^a(F^o, X)$  and the stability radius  $r^s(F^o, X)$ .

Recall that

$$\mathcal{F}_X = \{F \in \mathcal{F} : w(c^o, (F^o \otimes F) \cap X) \neq 0\}$$

and let

$$R_k^a = \begin{cases} \min \left\{ 1, \min_{F \in \mathcal{F}_X \cap \mathcal{F}(c^o, k)} \frac{w(c^o, F) - w(c^o, F^o)}{w(c^o, (F^o \otimes F) \cap X)} \right\} & \text{if } \mathcal{F}_X \cap \mathcal{F}(c^o, k) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases} \quad (32)$$

Denote for  $q \geq 0$ ,

$$r_k^a(q) = \frac{L_k(c^o)}{w(c^o, F^o \cap X) + q}. \quad (33)$$

**Theorem 5** *If  $X \subseteq E$  and  $q \geq q_{min}^a$ , then*

$$\min \{r_k^a(q), R_k^a\} \leq r^a(F^o, X) \leq R_k^a. \quad (34)$$

**Proof.** The bounds for the accuracy radius correspond to the envelopes of the accuracy function formulated in Theorem 3. The upper bound in (34) is determined by the inequality

$$a^k(F^o, X, \delta) \leq a(F^o, X, \delta). \quad (35)$$

Observe, as in the proof of Theorem 2, that the maximum value of  $\delta$ , for which  $a_F(F^o, X, \delta) \leq 0$  is equal to  $\delta_F = \frac{w(c^o, F) - w(c^o, F^o)}{w(c^o, (F^o \otimes F) \cap X)}$  for any  $F \in \mathcal{F}_X$ . If  $\mathcal{F}_X \cap \mathcal{F}(c^o, k) = \emptyset$  or  $\delta_F \geq 1$  for any  $F \in \mathcal{F}_X \cap \mathcal{F}(c^o, k)$ , then from (35) we have the upper bound  $r^a(F^o, X) \leq 1$ . Otherwise, from (35) it follows that the maximum value of  $\delta$ , for which  $a(F^o, X, \delta) = 0$ , does not exceed  $\min_{F \in \mathcal{F}_X \cap \mathcal{F}(c^o, k)} \delta_F$ , which proves that  $r^a(F^o, X) \leq R_k^a$ .

The lower bound in (34) is determined by the inequality

$$a(F^o, X, \delta) \leq \max\{a^k(F^o, X, \delta), A^k(F^o, X, \delta, q)\}, \quad (36)$$

which, according to Theorem 3, holds for any  $q \in [q_{min}^a, w(c^o, F^o) + L_k(c^o)]$ . From (36) it follows that  $r^a(F^o, X)$  is not less than the minimum of two values:  $r'$  – equal to the largest value of  $\delta \in [0, 1)$  for which  $a^k(F^o, X, \delta) \leq 0$ , and  $r''$  – equal to the largest value of  $\delta$  for which  $A^k(F^o, X, \delta, q) \leq 0$ ,  $q \in [q_{min}^a, w(c^o, F^o) + L_k(c^o)]$ . From the proof of the upper bound of  $r^a(F^o, X)$  we have  $r' = R_k^a$ . To determine  $r''$  observe that from (23) it follows that the largest value of  $\delta$  for which  $A^k(F^o, X, \delta, q) \leq 0$  is equal to  $r_k^a(q) = \frac{L_k(c^o)}{w(c^o, F^o \cap X) + q}$ , where  $q$  is taken arbitrarily from the interval  $[q_{min}^a, w(c^o, F^o) + L_k(c^o)]$ . The best lower bound of the accuracy radius is therefore obtained for  $q = q_{min}^a$ , but the inequality

$$r^a(F^o, X) \geq \min\{r_k^a(q), R_k^a\}$$

holds, obviously, for any  $q \geq q_{min}^a$ . ■

Consider now the case  $X = E$ . We have then  $q_{min}^a = q_1 = w(c^o, F^o) + L_k(c^o)$ . Let  $\mathcal{F}^k = \{F \in \mathcal{F}(c^o, k) : w(c^o, F^o \otimes F) \neq 0\}$  and define

$$R_k = \begin{cases} \min_{F \in \mathcal{F}^k} \frac{w(c^o, F) - w(c^o, F^o)}{w(c^o, F^o \otimes F)} & \text{if } \mathcal{F}^k \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Using this notation we obtain from (34) the following bounds for the accuracy radius  $r^a(F^o, E)$ :

**Corollary 4** For an optimal solution  $F^o$ ,

$$\min \left\{ \frac{L_k(c^o)}{2w(c^o, F^o) + L_k(c^o)}, R_k \right\} \leq r^a(F^o, E) \leq R_k. \quad (37)$$

Analogous bounds may be derived for the stability radius. Let

$$R_k^s = \min \left\{ \varrho(X, c^o), \min_{F \in \mathcal{F}_X^k \cap \mathcal{F}(c^o, k)} \frac{w(c^o, F) - w(c^o, F^o)}{|(F^o \otimes F) \cap X|} \right\}.$$

Denote for  $q \geq 0$ ,

$$r_k^s(q) = \frac{L_k(c^o)}{|F^o \cap X| + q}.$$

**Theorem 6** If  $X \subseteq E$  and  $q \geq q_{min}^s$ , then

$$\min\{r_k^s(q), R_k^s\} \leq r^s(F^o, X) \leq R_k^s. \quad (38)$$

**Proof.** The lower bound for the stability radius  $r^s(F^o, X)$  corresponds to the upper bound of the stability function in (30). The upper bound is a consequence of the lower bound for  $s(F^o, X)$  in (30). The proof of these facts is quite analogous as the proof of Theorem 4 and therefore the details omitted. ■

From (38) the bounds on the stability radius of the solution  $F^o$  in the case  $X = E$  can be easily obtained.

Let

$$m = \max\{|F| : F \in \mathcal{F}\}.$$

Obviously, the inequality  $q_{min}^s \leq m$  holds. Denote

$$R'_k = \min \left\{ \varrho(E, c^o), \min_{F \in \mathcal{F}'_E \cap \mathcal{F}(c^o, k)} \frac{w(c^o, F) - w(c^o, F^o)}{|F^o \otimes F|} \right\}.$$

Using this notation we obtain from (38) for  $X = E$  the following corollary:

**Corollary 5** *If  $F^o$  is an optimal solution and  $q \geq \min\{m, n\}$ , then*

$$\min \left\{ \frac{L_k(c^o)}{|F^o| + q}, R'_k \right\} \leq r^s(F^o, E) \leq R'_k. \quad (39)$$

**Example** (continued)

Consider again the symmetric traveling salesman problem defined on the graph  $G$  from Fig. 1. On that small example it is easy to demonstrate the effect of applying various values of  $k$  in approximating the accuracy function and the accuracy radius of  $F^o$  using  $k$ -best solutions of the problem. Observe also that if, as before,  $X = \{e_4, e_9, e_{11}\}$ , i.e.  $X$  is the set of edges incident to the same vertex 6, then it is easy to calculate an upper bound of the value  $q_3$  for any  $k$ . Indeed, for any feasible solution of the traveling salesman problem exactly two edges incident to any vertex appear in the solution. Thus, we obtain an upper bound of  $q_3$  by taking two edges from  $X$  with largest weights, which leads to inequality  $q_3 \leq 5$ . Observe also that  $q_2 = w(c^o, X) = 6$  does not depend on  $k$ . On the other hand, the value of  $q_1 = w(c^o, F^o) + L_k(c^o)$  depends on  $k$ , but for any value of  $k$  is larger than  $q_2$  and available upper bound of  $q_3$ . Thus, for any  $k$  the best choice for  $q$  is the upper bound of  $q_3$  equal to 5.

Figs. 3 and 4 show lower and upper bounds of the accuracy function for  $q = 5$  and  $k = 4, 5$ , respectively, obtained from Theorem 3. The shadowed regions indicate the gap between these bounds. From these figures it is also easy to read the bounds for the accuracy radius. Observe that for  $k = 5$  these bounds are tight and we obtain the exact value of  $r^a(F^o, X)$ . For  $k = 4$  the lower bound is smaller than the upper bound and we have only the interval which contains the value of the accuracy radius.

The same bounds for the accuracy radius can be obtained directly from Theorem 4 without calculating the accuracy function. From (32) it follows that for  $k = 4, 5$ ,

$$R_4^a = R_5^a = 0.4.$$



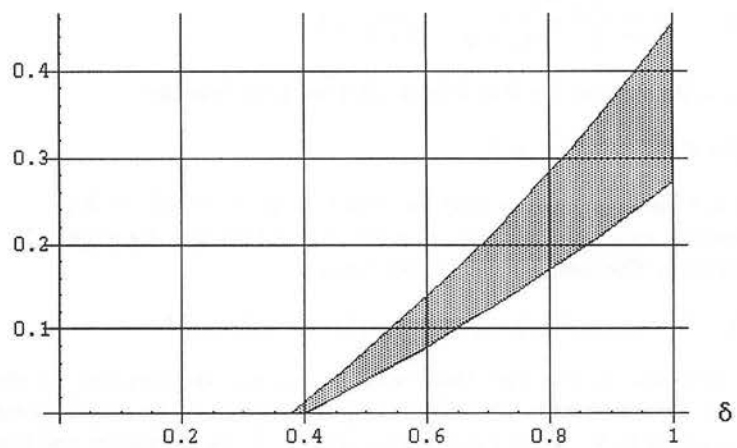


Figure 3. Bounds for the accuracy function  $a(F^o, X, \delta)$  for  $k=4$ .

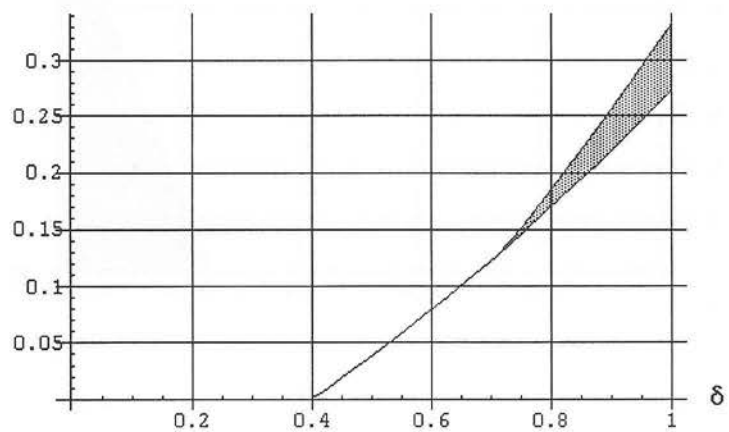


Figure 4. Bounds for the accuracy function  $a(F^o, X, \delta)$  for  $k=5$ .

From (33) we have for  $q = 5$ ,

$$r_4^a(q) = \frac{L_4(c^o)}{w(c^o, F^o \cap X) + q} = \frac{3}{3 + 5} = 0.375,$$

and  $r_5^a(q) = 0.5$ . Thus, for  $k = 4$  from (34) we have bounds

$$0.375 \leq r^a(F^o, X) \leq 0.4,$$

but for  $k = 5$  we obtain from (34) the exact value  $r^a(F^o, X) = 0.4$ .

In a similar way the envelopes of  $s(F^o, X)$  and bounds for  $r^s(F^o, X)$  can be calculated from Theorems 5 and 6. We have

$$\varrho(X, c^o) = \min\{c^o(e_4), c^o(e_9), c^o(e_{11})\} = \min\{2, 3, 1\} = 1.$$

Observe that due to the fact that edges  $e_4, e_9, e_{11}$  are incident to the same vertex of  $G$ , the cardinality  $|F \cap X|$  for any solution  $F$  of the traveling salesman problem is equal to 2, which implies that  $q_{min}^s = 2$ . Fig. 5 shows the lower and the upper bounds for the stability function obtained from (30) for  $k = 5$  and  $q = 2$ . As before, the shadowed region indicates the gap between these bounds.

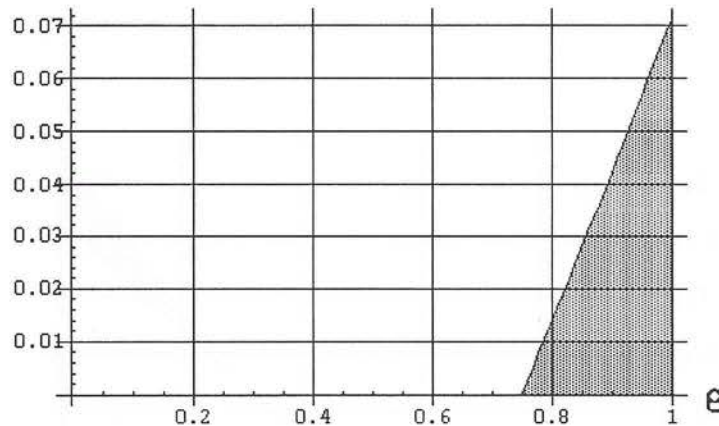


Figure 5. Bounds for the stability function  $s(F^o, X, \varrho)$ .

To evaluate the stability radius of the solution  $F^o$  with respect to  $X$  observe that

$$R_4^s = R_5^s = 1.$$

For  $q = 2$  we have

$$r_4^s(q) = \frac{L_4(c^o)}{|F^o \cap X| + q} = \frac{3}{2 + 2} = 0.75,$$

and  $r_5^s(q) = 1$ . Thus, for  $k = 4$  we have bounds

$$0.75 \leq r^s(F^o, X) \leq 1,$$

but for  $k = 5$  we obtain the exact value  $r^s(F^o, X) = 1$ . This means that the weights of edges  $e_4, e_9, e_{11}$  may be perturbed simultaneously and independently by at most 1 without destroying the optimality of the tour  $F^o$ .

#### 4. Conclusions

The accuracy and the stability functions describe the quality of solution obtained for some original vector of weights in the situation when these weights are perturbed by some amount or only some estimations of actual weights are available. The accuracy radius and the stability radius give the maximum perturbations which still preserve the optimality of given solution. For difficult combinatorial optimization problems calculating the exact values of these functions may lead to intractable computations. Observe that finding the accuracy radius or the stability radius in the case when only single weight is allowed to vary, i.e., when  $|X| = 1$ , is equivalent to determining the so-called tolerance of this weight. But the latter problem is known to be NP-hard for any NP-hard combinatorial optimization problem (see e.g. Ramaswamy and Chakravarti, 1995, van Hoesel and Wagelmans, 1999). Thus, for this type of problems it is of special interest to have methods which give some approximate values of considered functions and corresponding radii. In this paper one such method, based on the notion of  $k$ -best solutions, is described. Given a set of  $k$ -best solutions of the problem one can determine in an easy way envelopes of the accuracy and stability functions as well as bounds for the accuracy and the stability radii. The quality of such approximations depends on the parameter  $k$  and grows with  $k$ , but experiments performed in Libura et al. (1998) on similar approximation problems for the traveling salesman problem suggest that for practical purposes it is not worth to increase the number  $k$  too much, because the improvement of approximation decreases substantially with  $k$ .

In this paper as a measure of quality of a given solution the relative error of this solution is considered. In some cases studying the absolute error of the solution and its dependence on perturbations may appear more adequate. This case was considered for the minisum facility location problem in Labbé et al. (1991).

In this paper we concentrate on describing the quality of an optimal solution of the problem in the case of weight perturbation. All results concerning the evaluation of the accuracy and stability functions might be stated quite analogously for an arbitrary feasible solution of the problem.

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The following is a list of the names of the persons who were present at the meeting held on the 1st day of January, 1911, at the residence of Mr. J. H. Smith, in the city of New York.